EXTRAIT
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NOTE ON SOME ADDITIVE PROPERTIES OF INTEGERS

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1. It is well known that for suitable n's both the equations $n=x^2-y^2$ and $n=x^2+y^2$ have more than $n^{\frac{c_1}{\log\log n}}$ solutions. I show that for suitable n's the number of solutions of the equations $n=p^2-q^2$ resp. $n=p^2+q^2$ (p, q primes) is greater than $n^{\frac{c_2}{\log\log n}}$.

I sketch the proof for $n=p^2-q^2$.

Let $A=2\cdot 3\cdots p_r$, the product of consecutive primes, be sufficiently large. By elementary method we prove that the number of solutions ot the congruence $p^2-q^2\equiv 0 \pmod{A}$ with $0<q<p<A$ is greater than $A^{1+\frac{1}{4\log\log A}}$. But the integers of the form p^2-q^2 with $0<q<p<A$ lie all between 0 and A^2, hence there exists a multiple of A say $n(<A^2)$ such that the number of solutions of the equation $n=p^2-q^2$ is greater than $A^{\frac{1}{4\log\log A}}>n^{\frac{1}{8\log\log n}}$.

The proof for $n=p^2+q^2$ is much more complicated but also elementary. It requires Brun's method.

2. Schnirelmann proved that there exists a constant c_3 such that every integer is the sum of c_3 or less primes. Some time ago Heilbronn-Landau-Scherk proved that $c_3\leqq 71$. By Brun's method I proved that there exists a constant c_4 such that any integer is the sum of c_4 or less positive and negative squares of primes. The same result holds for any powers of primes. It can be proved also that the density of integers of the form $p^2+q^2-r^2-s^2$ is positive.

3. Now I sketch some new results of N. P. Romanoff (Tomsk).

Let us denote by $f(x_1, x_2, \cdots, x_{k_1};\, y_1, y_2, \cdots y_{k_2} | n)$ the number of integers not exceeding n belonging to the sequence x_i, y_j and x_i+y_j. It is an old and most important problem of the additive theory of numbers to determine the value of f for given x_i and y_j. But this can be solved only for special sequences. Romanoff deduced 4 formulas for the mean value of f for general sequences of integers.

First mean-value-theorem:

$$\sum_{1 \leqq x_1 < x_2 < \cdots < x_{k_1} \leqq n} f(x_1, x_2, \cdots, x_{k_1}; y_1, y_2, \cdots, y_{k_2} | n)$$

$$= n\, C^n_{k_1} - \sum_{z=1}^{n-k_2} C^{n-1-\bar{y}_z+z}_{k}$$

where $C^n_k = \binom{n}{k}$ and $\bar{y}_z$ denotes the complementary sequence of y_z.

Second mean-value-theorem:

$$\sum_{x_1=1}^{n} \sum_{x_2=1}^{n} \cdots \sum_{x_{k_1}=1}^{n} f(x_1, x_2, \cdots, x_{k_1}; y_1, y_2, \cdots, y_{k_2} | n)$$

$$= n^{k+1} - \sum_{z=1}^{n-k_2} (n-1-\bar{y}_z+z)^{k_1}.$$

Third mean-value-theorem:

$$\sum_{1 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_{k_1} \leqq n} \; \sum_{1 < y_1 < y_2 < \cdots < y_{k_2} \leqq n} f(x_1, x_2, \cdots x_{k_1}; y_1, y_2, \cdots y_{k_h} | n)$$

$$= n\, C^n_{k_1} C^n_{k_2} - C^n_{k_1+1} C^n_{k_2+1} + C^n_{k_1+1} C^{n-k_1+1}_{k_2}$$

Fourth mean-value-theorem:

$$\sum_{1 \leqq x_1 < x_2 < \cdots < x_k \leqq n} f(x_1, x_2, \cdots, x_{k_1}; x_1, x_2, \cdots x_k | n)$$

$$= n\, C^n_k - 2\, C^n_{k+2} - C^n_{k+1} + 2^{k+3} C^{\left[\frac{n}{2}\right]}_{k+2} + 2^{k+1}(1+2\varepsilon_n)\, C^{\left[\frac{n}{2}\right]}_{k+1}$$

with $\varepsilon_n = 0$ if n even, and $\varepsilon_n = 1$ if n odd.

The proof depends on elementary combinatoric methods.