NOTE ON CONSECUTIVE ABUNDANT NUMBERS

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A positive integer N is called an abundant number if

$$\sigma(N) \geqslant 2N,$$

where $\sigma(N)$ denotes the sum of the divisors of N including 1 and N. Abundant numbers have been recently investigated by Behrend, Chowla, Davenport, myself, and others; it has been proved, for example, that they have a density greater than 0. I prove now the following

THEOREM. *We can find two constants c_1, c_2 such that, for all sufficiently large n, there exist $c_1 \log\log\log n$ consecutive integers all abundant and less than n, but not $c_2 \log\log\log n$ consecutive integers all abundant and less than n.*

* Received and read 13 December, 1934.

I prove the first part by using an idea due* to Chowla and Pillai. Write

$$A = \prod_{p < \frac{1}{2}\log n} p,$$

where the p denote prime numbers, so that, for sufficiently large n,

$$A < n,$$

as is obvious either from the prime number theorem or from elementary reasoning. Hence, if c_3 is a suitable constant, and the c's have this meaning throughout the paper,

$$\frac{\sigma(A)}{A} = \prod_{p < \frac{1}{2}\log n} \left(1 + \frac{1}{p}\right) > c_3 \log\log n.$$

Write $$a_1 = 2.3, \quad a_2 = 5.7 \ldots p_1, \quad a_3 = p_2 \ldots p_3, \quad \ldots,$$

where a_1 is an abundant number, p_1 denotes the smallest prime such that a_2, the product of the primes from 5 to p_1, is an abundant number (clearly p_1 exists), p_2 is the prime following p_1 and p_3 is the smallest prime such that a_3, the product of the primes between p_2 and p_3, is an abundant number, and so on. For each of the a's, $\sigma(a)/a < 3$, thus

$$\frac{\sigma(a_2)}{a_2} = \left\{\frac{\sigma(a_2)/p_1}{a_2/p_1}\right\} \left(1 + \frac{1}{p_1}\right),$$

where the first factor on the right is less than 2. Hence, if x denotes the number of the a's not exceeding A,

$$x > c_4 \log\log\log n.$$

For $$\frac{\sigma(a_1)}{a_1} \frac{\sigma(a_2)}{a_2} \ldots \frac{\sigma(a_{x+1})}{a_{x+1}} \geqslant \frac{\sigma(A)}{A} > c_3 \log\log n,$$

and so $$3^{x+1} > c_3 \log\log n,$$

and the result follows.

Now consider the simultaneous congruences

$$y \equiv r-1 \pmod{a_r} \quad (r = 1, 2, \ldots, x).$$

These obviously have a solution with $0 < y < A < n$. Since any multiple of an abundant number is also an abundant number,

$$y, \; y-1, \; y-2, \; \ldots, \; y-x+1$$

are all abundant numbers. This proves the first part.

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* "On the error terms in some asymptotic formulae in the theory of numbers", *Journal London Math. Soc.*, 5 (1930), 95–101.

Suppose now that, for n sufficiently large, there exist consecutive integers $m, m-1, \ldots, m-k+1$ all abundant for some $m \leqslant n$ and for $k > c_5 \log\log\log n$ for every constant c_5. Let q be the first prime such that $qc_6 > 4 \log q$, where

$$\prod_{p<q} \left(1-\frac{1}{p}\right) > \frac{c_6}{\log q},$$

the product being extended to the primes p. Denote by $b_1, b_2, \ldots, b_z$ the integers between m and $m-k+1$ not divisible by a prime $q_1, q_2, \ldots \leqslant q$. Then, by the sieve of Eratosthenes, *i.e.* excluding multiples of $q_1, q_2, \ldots$, and the inequality above,

$$z > \frac{c_6 k}{\log q}.$$

Since
$$\frac{\sigma(b)}{b} < \prod_{p\mid b} \left(1+\frac{1}{p-1}\right),$$

we have
$$2^z \leqslant \prod_{r=1}^{z} \frac{\sigma(b_r)}{b_r} < \prod_{p>q} \left(1+\frac{1}{p-1}\right)^{[k/p]+1},$$

since at most $[k/p]+1$ of the b are divisible by a prime p. For the primes p up to $q < p \leqslant k$, we write

$$\left[\frac{k}{p}\right]+1 \leqslant \frac{2k}{p},$$

$$\prod_{q<p\leqslant k} \left(1+\frac{1}{p-1}\right)^{2k/p} < \prod_{q<p\leqslant k} \left(1+\frac{1}{p(p-1)}\right)^{2k} < e^{2k \sum_{q<p\leqslant k} 1/\{p(p-1)\}} < e^{2k/q}.$$

For the primes $p > k$, we note that each integer less than n has less than $\log n/\log 2$ different prime factors, and so the product $b_1 b_2 \ldots b_z$ has less than $z \log n/\log 2$ different prime factors.

Since the number of primes not greater than $4z \log^2 n$ is greater than $z \log n/\log 2$ for sufficiently large n,

$$\prod_{\substack{p>k \\ p\mid b_1\ldots b_z}} \left(1+\frac{1}{p-1}\right) < \prod_{p<4z\log^2 n} \left(1+\frac{1}{p-1}\right) < c_7(2\log\log n+\log z).$$

Hence
$$2^z < c_7\, e^{2k/q}(2\log\log n+\log z),$$

and so
$$2^{c_6 k/\log q} < c_7\, e^{2k/q}(2\log\log n+\log z).$$

But
$$c_6 q > 4\log q,$$

and so, since $z < k$,

$$2^{c_6 k/2\log q} < c_7(2\log\log n+\log k) < 2c_7^2 \log k \log\log n$$

for $a+b < ab$ if $a > 2,\ b > 2$.

But $2^{c_6 k/4 \log q} > c_7 \log k$ for sufficiently large k, and so we should have $2^{c_6 k/4 \log q} < 2c_7 \log\log n$, which is not true if

$$k > \frac{4 \log q \log\log\log n}{c_6 \log 2}.$$

This proves the theorem.

By the same method we can prove that, for every $\epsilon > 0$, a constant $c_8 = c_8(\epsilon)$ exists such that, if $n > n(\epsilon)$, then among $c_8 \log\log\log n$ consecutive integers less than n, there is at least one, say m, such that $\sigma(m)/m < 1+\epsilon$. We can also prove by a longer method that, if

$$\frac{f(n)}{\log\log\log n} \to \infty \quad \text{as} \quad n \to \infty,$$

then the abundant numbers have the same density in the interval $n, n+f(n)$ as in the interval 1, n.

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