ON THE DENSITY OF THE ABUNDANT NUMBERS

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PAUL ERDÖS*.

1. The object of this paper is to give a proof that the quotient $A(n)/n$ tends to a limit as $n \to \infty$, where $A(n)$ denotes the number of abundant numbers† not exceeding n. It was proved by Behrend‡ that, for all sufficiently large n, this quotient lies between ·241 and ·314. The fact that it tends to a limit as $n \to \infty$ has been proved by Davenport§, and he states that similar proofs have been found independently by Behrend and Chowla.

* Received 4 April, 1934; read 26 April, 1934.

† An abundant number is a number m for which $\sigma(m) \geqslant 2m$, where $\sigma(m)$ is the sum of the divisors of m, including 1 and m.

‡ *Berliner Sitzungsberichte* (1932), 322–328; (1933), 280–293.

§ *Ibid.* (1934), 830–837.

The proof which I give here differs entirely from that of Davenport, and requires only elementary considerations.

First we make the following observation. Let $a_1 < a_2 < a_3 \ldots$ be an infinite sequence of positive integers, and let $A(n)$ denote the number of numbers not exceeding n which are divisible by at least one a_k. Then, if $\Sigma\,(1/a_k)$ converges, the quotient $A(n)/n$ tends to a limit as $n \to \infty$. For let $A_k(n)$ denote the number of numbers not exceeding n which are divisible by a_k but not by any of $a_1, a_2, \ldots, a_{k-1}$. Then we have

$$(1) \qquad \frac{A(n)}{n} = \sum_{k=1}^{\infty} \frac{A_k(n)}{n}.$$

Now, trivially, $$0 \leqslant A_k(n) \leqslant \left[\frac{n}{a_k}\right] \leqslant \frac{n}{a_k}.$$

Hence the series on the right of (1) converges uniformly in n, in virtue of the convergence of $\Sigma\,(1/a_k)$. Also it is clear that, for each fixed k, $A_k(n)/n$ tends to a limit A_k as $n \to \infty$, and ΣA_k converges. Hence $\lim_{n\to\infty} A(n)/n$ exists and has the value $\sum_1^{\infty} A_k$.

We now apply this result to the abundant numbers. Since any multiple of an abundant number is abundant, we obtain all abundant numbers by taking all multiples of all primitive abundant numbers, where a primitive abundant number is defined as an abundant number of which no proper divisor is abundant.

We shall prove in this paper that the number of primitive abundant numbers not exceeding n is $o(n/\log^2 n)$. From this it follows that the sum of the reciprocals of the primitive abundant numbers converges, and hence that the quotient $A(n)/n$ tends to a limit as $n \to \infty$.

2. Lemma 1. *The number of integers $m \leqslant n$ which do not satisfy all of the following three conditions*

(1) *if $p^\alpha \mid m$ and $\alpha > 1$, then $p^\alpha < (\log n)^{10}$;*

(2) *the number of different prime factors of m is less than 10ν, where $\nu = \log\log n$;*

(3) *the greatest prime factor of m is greater than $n^{1/(20\nu)}$;*

is $o(n/\log^2 n)$.

The number of integers $m \leqslant n$ which do not satisfy (1) is less than

$$\sum_{\substack{p^\alpha > (\log n)^{10} \\ \alpha \geqslant 1}} \frac{n}{p^\alpha} < \sum_{p \geqslant (\log n)^5} \frac{2n}{p^2} + \sum_{p < (\log n)^5} \frac{2n}{p^{\alpha(p)}},$$

where $a(p)$ is the least integer a such that $p^a \geqslant (\log n)^{10}$. Hence the number in question is less than

$$\sum_{k \geqslant (\log n)^5} \frac{2n}{k^2} + (\log n)^5 \frac{2n}{(\log n)^{10}} = O\left(\frac{n}{(\log n)^5}\right).$$

If m is an integer which does not satisfy (2), then

$$d(m) \geqslant 2^{10\nu} = (\log n)^{10 \log 2} > (\log n)^4.$$

Since
$$\sum_1^n d(m) = O(n \log n),$$

it follows that the number of numbers $m \leqslant n$ which do not satisfy (2) is $O(n/\log^3 n)$.

As regards the integers not satisfying (3), we may suppose without loss of generality that they do satisfy (2): thus each of them is a product of powers of not more than 10ν primes each less than or equal to $n^{1/(20\nu)}$. The number of such prime powers does not exceed

$$n^{1/(20\nu)} \frac{\log n}{\log 2},$$

and so the number of possible combinations of them taken not more than 10ν at a time does not exceed

$$\left(n^{1/(20\nu)} \frac{\log n}{\log 2}\right)^{10\nu} = o\left(\frac{n}{(\log n)^2}\right).$$

LEMMA 2. *A primitive abundant number not exceeding n, which satisfies the three conditions of Lemma 1, necessarily has a prime divisor between $(\log n)^{10}$ and $n^{1/(40\nu)}$, if n is sufficiently great.*

For suppose that $m = ab$ is such a primitive abundant number, where all prime factors of a are less than $(\log n)^{10}$, and all prime factors of b are greater than $n^{1/(40\nu)}$. Then $\sigma(m)/m \geqslant 2$, but $\sigma(a)/a < 2$, and so

$$\frac{\sigma(a)}{a} \leqslant 2 - \frac{1}{a} < 2 - \frac{1}{(\log n)^{100\nu}},$$

by (1) and (2) of Lemma 1. Also, by (2) of Lemma 1,

$$\frac{\sigma(b)}{b} < \prod_{p \mid b} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots\right)$$
$$< \prod_{p \mid b} \left(1 + \frac{2}{p}\right) < \left(1 + \frac{2}{n^{1/(40\nu)}}\right)^{10\nu} < 1 + \frac{40\nu}{n^{1/(40\nu)}},$$

provided that n is sufficiently great. Hence

$$\frac{\sigma(m)}{m}=\frac{\sigma(a)}{a}\frac{\sigma(b)}{b}<2$$

for sufficiently large n, which is contrary to hypothesis.

LEMMA 3. *If m is a primitive abundant number not exceeding n, which satisfies the three conditions of Lemma 1, and n is sufficiently great, then*

$$2\leqslant\frac{\sigma(m)}{m}<2+\frac{2}{n^{1/(20\nu)}}.$$

For let p be the greatest prime factor of m. By (1) and (3) of Lemma 1, $p^2\nmid m$, and $p>n^{1/(20\nu)}$. Hence, writing $m=pm'$,

$$\frac{\sigma(m)}{m}=\frac{\sigma(m')}{m'}\left(1+\frac{1}{p}\right)<2\left(1+\frac{1}{p}\right)<2+\frac{2}{n^{1/(20\nu)}}.$$

3. THEOREM. *The number of primitive abundant numbers not exceeding n is $o(n/\log^2 n)$.*

It is sufficient to prove that the number of integers not exceeding n, which satisfy the three conditions of Lemma 1, and which also satisfy the conclusions of Lemmas 2 and 3, is $o(n/\log^2 n)$. Denote these (different) integers by $b_1, b_2, \ldots, b_k$. Each b_i has a simple prime factor p_i between $(\log n)^{10}$ and $n^{1/(40\nu)}$. Write $b_i=p_i c_i$, so that $c_i<n/(\log n)^{10}$. Then to prove the theorem it will clearly suffice to show that the numbers c_i $(i=1, 2, \ldots, k)$ are all different.

Suppose that this is not so, *i.e.* suppose that $c_\nu=c_\mu$, $\nu\neq\mu$. Then, evidently, $p_\nu\neq p_\mu$ (for, if so, $b_\nu=b_\mu$, which is not the case). Now

$$\frac{\sigma(b_\nu)}{b_\nu}=\frac{\sigma(c_\nu)}{c_\nu}\frac{p_\nu+1}{p_\nu},$$

and similarly with μ for ν. Hence

$$\frac{\sigma(b_\nu)}{b_\nu}\frac{b_\mu}{\sigma(b_\mu)}=\frac{p_\mu(p_\nu+1)}{p_\nu(p_\mu+1)}.$$

The right-hand side is not 1, and we can suppose without loss of generality that it is greater than 1. Then

$$\frac{\sigma(b_\nu)}{b_\nu}\frac{b_\mu}{\sigma(b_\mu)}\geqslant 1+\frac{1}{p_\nu(p_\mu+1)}\geqslant 1+\frac{1}{n^{1/(20\nu)}}.$$

But, by Lemma 3,

$$\frac{\sigma(b_\nu)}{b_\nu}\frac{b_\mu}{\sigma(b_\mu)} < \frac{2+(2/n^{1/(20\nu)})}{2} = 1+\frac{1}{n^{1/(20\nu)}},$$

which is a contradiction. Thus the theorem is established.

4. It will be seen that the method used in this paper leads immediately to a much better result than $o(n/\log^2 n)$ for the number of primitive abundant numbers not exceeding n. I shall prove in a subsequent paper that this number lies between

$$\frac{n}{e^{c_1(\log n \log\log n)^{\frac{1}{2}}}} \quad \text{and} \quad \frac{n}{e^{c_2(\log n \log\log n)^{\frac{1}{2}}}},$$

where c_1 and c_2 are two absolute constants.

Before closing my paper I would express my sincere gratitude to Mr. H. Davenport for having so kindly aided me in my work.

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