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In December 2010 the János Bolyai International Mathematical Prize of the Hungarian Academy of Sciences was awarded to Professor Yuri Ivanovich Manin. In this note we wish to present Professor Manin to the readers of Acta Mathematica Hungarica. The first part is a general overview of his scientific achievements, in the second part we provide some more details about his broad mathematical work.

I.

Yuri Ivanovich Manin is widely regarded as one of the outstanding mathematicians of the 20th century. His work spans such diverse branches of mathematics as algebraic geometry, number theory and mathematical physics. But mathematics is not his only interest. He has been interested and published research or expository papers in literature, linguistics, mythology, semiotics, physics, philosophy of science and history of culture as well. He is one of the few mathematicians who determined essentially the Russian science in the second half of the 20th century.

Yuri Manin was born in 1937 in the town of Simferopol, in Crimea. His father died in the war, his mother was a teacher of literature. His mathematical abilities became apparent already during his school-years, when he slightly improved Vinogradov's estimate of the number of lattice points inside a sphere. Between 1953-58 he studied at the faculty of Mechanics and Mathematics of the Moscow State University, the most prestigious mathematical school of in USSR. His class and the following class included several other talented students: Anosov, Golod, Arnol'd, Kirillov, Novikov and Tyurin. Starting from the second year he became an active member of the seminar led by A. O. Gel'fond and I. R. Shafarevich targeting the work of Hasse and Weil on the ζ-function on algebraic curves over a finite field.

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At this time his first publication appeared containing an elementary proof of Hasse's theorem. After graduation he continued his postgraduate studies at the Steklov Institute of Mathematics under the supervision of I. R. Shafarevich. During these years an algebraic geometry seminar led by Shafarevich began to operate, with the active participation of Manin. These two seminars determined his major mathematical interests and passions: the interface of algebraic geometry and algebraic number theory.

He enriched these fields by numerous fundamental contributions, including the solutions of major problems and the development of techniques that opened new possibilities for research. The top of the iceberg contains two outstanding results in algebraic geometry. The first one was the proof of the analogue of Mordell's Conjecture (now Faltings' Theorem) for algebraic curves over function fields. In Faltings' classical case the statement is the following: a curve of genus greater than 1 has at most a finite number of rational points. Over a function field, in Manin's version, the curve depends on parameters. In this proof a new mathematical object played the key role. Later this was named by Grothendieck the Gauss–Manin connection, and it is a basic ingredient of the study of cohomology in families of algebraic varieties in modern algebraic geometry. The second outstanding achievement was a joint work with his student V. A. Iskovskih: it provided a negative solution of the Lüroth problem in dimension 3. They proved the existence of nonrational unirational 3-folds via deep understanding of the geometry of three-dimensional quartics.

The early research on the set of rational points of bounded height on cubic surfaces was continued and generalized via studying the asymptotics of the distribution of rational points by height on Fano varieties. This generated a sequence of deep conjectures and results, for example the 'conjecture of linear growth', developed and finished by Manin in a joint work with some of his students (Batyrev, Franke, Tschinkel). Moreover, motivated by Mordell's conjecture, Manin and Mumford formulated the so-called Manin–Mumford conjecture which states that any curve, which is different from its Jacobian variety, can only contain a finite number of points that are of finite order in the Jacobian. This problem later solved by M. Raynaud, has developed into the general 'Manin–Mumford theory'.

In number theory, or arithmetic algebraic geometry, Manin developed the so-called Manin–Brauer obstruction to the solvability of Diophantine equations. The Manin obstruction associated with a geometric object measures the failure of the Hasse principle for it; that is, if the value of the obstruction is non-trivial, then the object might have points over all local fields but not over the a global field. For torsors of abelian varieties the Manin obstruction characterizes completely the failure of the local-to-global principle (provided that the Tate–Shafarevich group is finite).

He also obtained fundamental results in the theory of modular forms and modular symbols, and the theory of p-adic L-functions, classification of isogeny classes of formal p-divisible groups; he proved the Weil conjecture for unirational projective 3-folds.

Manin obtained a series of outstanding results in mathematical physics as well, including Yang–Mills theory, string theory, quantum groups, quantum information theory, and mirror symmetry. These papers show the strong symbiosis of mathematics and physics and how they strongly influence each other. For example, an article of Atiyah, Drinfel'd, Hitchin and Manin provided a complete description of instantons by algebro-geometrical methods emphasizing the potential power of algebro-geometrical tools in theoretical physics. Symmetrically, ideas from physics solved crucial open problems in algebraic geometry, see for example the papers of Kontsevich and Manin about quantum cohomology of algebraic varieties. His work with Kontsevich on Gromov–Witten invariants and work on Frobenius manifolds (later on 'F-manifolds' developed with C. Hertling) created new areas of mathematics with strong mathematical machinery and several applications.

He has also written famous papers on formal groups, noncommutative algebraic geometry and mathematical logic.

Professor Manin is the author and coauthor of 11 monographs and about 235 articles in algebraic geometry, number theory, mathematical physics, history of culture and psycholinguistics. In some of his books he created and developed new theories (like in the first one in the next list). Some titles of his books and monographs emphasizing the diversity of subjects:

Cubic forms: algebra, geometry, arithmetic published in 1972,
A course in mathematical logic (1977),
Computable and noncomputable (1980),
Linear algebra and geometry with A.I. Kostrikin published in 1980,
Gauge fields and complex geometry (1984),
Methods in homological algebra and *Homological Algebra* with Sergei Gelfand (1988-89),
Quantum groups and noncommutative geometry (1988),
Elementary particles with I. Yu. Kobzarev (1989),
Introduction in Number Theory with A.A. Panchishkin (1990),
Topics in noncommutative geometry (1991),
Frobenius manifolds, quantum cohomology and moduli spaces (1999).

The famous book *Mathematics and physics* or the selected essays *Mathematics as Metaphor* provide a deep insight in his philosophy of science.

Manin's pedagogical activity started in 1957 at Moscow State University, and he remained there until the early 90's. At parallel, he was principal researcher at Steklov Mathematical Institute. In the period 1968–86 he was not allowed to travel abroad, but starting from 88 he was visiting professor at several Universities including Berkeley, Harvard, Columbia, MIT, IHES. He accepted a professorship at Northwestern University in the United States, and the position in the Board of Directors of the Max Planck Institute in Bonn in 1993. He became Professor emeritus at the Max Planck Institute in 2005.

During these years he was continuously surrounded by a large number of students, the most talented ones wished to be guided under his supervision. He was advisor of 49 students, some of them became celebrated mathematicians. Just a few of them: Kapranov, Beilinson, Zarhin, Danilov, Iskovskih, Shokurov, Drinfeld, Wodzicki, Tsygan, Tschinkel. Manin was a very popular professor with a lot of energy with vivid presentations and real pedagogical vein.

Professor Manin was six times invited speaker at international congresses, he was invited plenary speaker at European Congress of Mathematics. He received several international honors. He was awarded several prices:

Highest USSR National Prize (the so-called Lenin Prize) in 1967 for work in algebraic geometry,

Brouwer Gold Medal in Number Theory from the Dutch Royal Society and Mathematical Society in 1987,

Frederic Esser Nemmers Prize in Mathematics from Northwestern University in 1994,

Rolf Schock Prize in Mathematics of the Swedish Royal Academy of Sciences in 1999,

King Faisal International Prize for Mathematics from Saudi Arabia in 2002,

Georg Cantor Medal of the German Mathematical Society in 2002,

Order Pour le Mérite and *Great Cross of Merit with Star* from Germany in 2007 and 2008; and the

János Bolyai International Mathematical Prize of the Hungarian Academy of Sciences in 2010.

He is elected member of several scientific academies: Academy of Sciences, Russia; Academy of Natural Sciences, Russia; Royal Academy of Sciences, the Netherlands; Academia Europaea, Max-Planck-Society for Scientific Research, Germany; Göttingen Academy of Sciences; Pontifical Academy of Sciences, Vatican; German Academy of Sciences; American Academy of Arts and Sciences; Académia des Sciences de l'Institut de France.

He is recipient of several honorary degrees of famous universities: Sorbonne, Oslo, Warwick.

II.

Since the scientific activity of Professor Manin covers many areas, and in all of them his impact was so huge, in such a short presentation necessarily we have to make some selection. In this note, we concentrate on his mathematical achievements, but this also is so wide that I decided to discuss only few of them. The choices are subjective and very selective, nevertheless even this selection guides the reader through many areas of mathematics. In this presentation we also wish to give a little historical background of the corresponding mathematical achievements, and mention some further developments showing their fundamental role and impact in the evolution of mathematical ideas.

1. The proof of Mordell conjecture over function fields

The origin of Mordell conjecture goes back to the search of non-zero rational solutions of a homogeneous equation $f = 0$ in three variables with rational coefficients. Usually, the complex algebraic curve C, the zero set of f in the complex projective plane $\mathbb{CP}^2$ is also associated with such an equation. If C is smooth then it is a Riemann surface of genus $g \geqq 0$, and its geometry/topology has a qualitative effect on the previous Diophantine question as well.

For example, if $g = 0$, then f is either linear or quadratic. A linear equation, with rational coefficients has evidently infinitely many rational solutions. On the other hand, a conic either has no solution (like $x^2 + y^2 = 3z^2$), or infinitely many (like $x^2 + y^2 = z^2$).

The $g = 1$ case is incomparably more interesting. It corresponds to smooth elliptic, degree three curves E of $\mathbb{CP}^2$. The complex points of such a curve form an abelian group. In order to define this, one first fixes an arbitrary point O for the neutral (zero) element. For example, in the case of $y^2z = x^3 - xz^2$ (the projective closure of $y^2 = x^3 - x$) we can take for O the inflection point at infinity $E \cap \{z = 0\}$. Then, in the group law of E, $P \oplus Q \oplus R = O$ if and only if the points P, Q and R are the intersection points of E with a line.

If O is a rational point (that is, all its coordinates are rational numbers), then the set of all rational points $E(\mathbb{Q})$ form a subgroup. For example, for the previous equation $y^2z = x^3 - xz^2$,

$$E(\mathbb{Q}) = \big\{ O; (0,0,1); (1,0,1); (-1,0,1) \big\} \approx \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Mordell proved in 1922 that $E(\mathbb{Q})$ is a finitely generated abelian group. In general, the rank of $E(\mathbb{Q})$ is not necessarily zero, for example, if C is the closure of $y^2 + y = x^3 - x$ then (according to Tate) $E(\mathbb{Q}) = \mathbb{Z}$. It is interesting to mention that the torsion part is seriously obstructed. Indeed, in 1977 Mazur proved that $E(\mathbb{Q})_{\mathrm{tors}}$ is isomorphic to one of the following 15 groups: $\mathbb{Z}/n\mathbb{Z}$ for $n = 1, 2, \dots, 10$, or 12, and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z}$ for $n = 1, \dots, 4$.

One can try to generalize the result of Mordell in two directions.

First, one can consider a larger field of 'coefficients' over which the equation of the curve is given. For example, for elliptic curves defined over number fields (finite field extensions of $\mathbb{Q}$) one has the following generalizations. By a theorem of Mordell–Weil (1922/28) if E is an elliptic curve (or even an abelian variety) defined over a number field, then $E(M)$ (the set of solutions with all coordinates in M) is a finitely generated abelian group. For example, if $M = \mathbb{Q}(i)$ $\big(i = \sqrt{-1}\big)$, and E is the closure of $y^2 = x^3 - x$ as above, then

$$E\big(\mathbb{Q}(i)\big) = \big\{ O; (0,0,1); (1,0,1); (-1,0,1);$$

$$(i, -1+i, 1); (i, 1-i, 1); (-i, 1+i, 1), (-i, -1-i, 1)\big\} \approx \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}.$$

The torsion part $E(M)_{\text{tors}}$ is again obstructed: in 1990 Kamienny listed 26 groups and proved that if M is a quadratic number field then $E(M)_{\text{tors}}$ is isomorphic with one of them. The Torsion Conjecture states that if M is a number field then there is a finite list of possibilities for $E(M)_{\text{tors}}$, and its 'strong' version claims that this list depends only on $[M : \mathbb{Q}]$. The Strong Uniform Boundedness Conjecture was proved by Merel in 1994 showing that the order of $E(M)_{\text{tors}}$ can be bounded by a function of $[M : \mathbb{Q}]$. The proof relies on three fundamental ingredients: results of Mazur and Kamienny, the innovative winding quotient of Merel, and the use of Manin's presentation of the homology group of modular curves.

One the other hand, Mordell conjectured that any smooth curve defined over a number field and of genus $g > 1$ has only finitely many rational points. This was proved by G. Faltings in 1983, and now is known as Faltings' theorem. Faltings' proof relied on the known reduction to the Tate Conjecture, and several deep tools of algebraic geometry.

In fact, Faltings proved the more general Shafarevich Conjecture. This predicted that there are only finitely many isomorphism classes of curves of genus greater than zero over a number field with specified good reduction. This fact will stay valid for abelian varieties too. The abelian varieties are those projective algebraic varieties which have an algebraic groups structure too. The reduction of the Mordell Conjecture to the Shafarevich Conjecture was due to Parshin in 1971. An immediate application of Faltings' theorem is to Fermat's Last Theorem, showing that for any $n > 4$ there are at most finitely many primitive solutions to $x^n + y^n = z^n$.

Manin in 1963 [1] considered and proved Mordell Conjecture in a more general situation, that is for curves defined over fields of functions. This means that the curve depends on parameters, one has to consider families of curves. Manin in his proof introduced a new technical tool, which now in modern algebraic geometry is called the *Gauss–Manin connection.*

The Gauss–Manin connection is a connection on a vector bundle associated with a topologically locally trivial family of algebraic varieties. The base space of the vector bundle is the set of parameters defining the family of varieties, and the fibers usually are the de Rham cohomology groups of the fibers. Flat sections of the bundle are described by certain differential equations. The connection, or the flat sections, allow one to move a cohomology class from a fiber of the family to any nearby fiber. These differential equations have their generalization in the theory of D-modules.

A different proof of the Mordell Conjecture for function fields was given by Grauert in 1965, two proofs by Parshin (1968 and 1990) and in 1990 Coleman found and corrected a gap in Manin's proof.

The Mordell Conjecture led Manin to the following question (asked independently by Manin and Mumford, called the *Manin–Mumford Conjecture*): Consider a curve defined over a number field, and fix an embedding of the curve in its Jacobian. Recall that for any curve C of genus $g \geqq 1$, its Jacobian J is an abelian variety of dimension g; J is covered by C^g and any point in J comes from a g-tuple of points of C.

The Manin–Mumford Conjecture says that C, embedded in its Jacobian variety J can only contain a finite number of points that are of finite order in J, provided that $g > 1$. This conjecture was verified by M. Raynaud in 1983. Various other proofs and generalizations appeared by Raynaud, Serre, Coleman; recently in 2001 it was reproved by Hrushovski using model theory, and by Pink–Roessler and Roessler using algebraic geometry.

Long before either of the Mordell Conjectures or Manin–Mumford Conjecture was settled, it was Serge Lang 1965 who realized that these two statements are special cases of a more general conjecture, which is usually called the Mordell–Lang conjecture; it was proved by McQuillan in 1995.

Manifolds with many rational points [4,5]**.** Mordell Conjecture is a particular case of the general principle which is expected to be valid in Diophantine geometry: if the canonical line bundle is ample (a fact which is guaranteed by varieties of general type, or in the presence of hyperbolicity or negative curvature), the manifold must have few rational points. Recall that the canonical line bundle is automatically associated with any smooth manifold, it is provided by the top-dimensional forms.

On the contrary, when the dual of the canonical line bundle is ample (a fact which is true for Fano manifolds, or in the presence of ellipticity or positive curvature), one expects that generally it has many rational points.

In the end of the 80's, Manin launched a research program, grouped around the *Manin's Linear Growth Conjecture,* which targeted the quantitative study of manifolds with many rational points. The main notions, available tools and conjectures were expounded in his joint article with V. Batyrev in 1990. Two general new phenomena were discovered: (a) the existence of the 'accumulating subvarieties' concentrating anomalously many points, and (b) linear growth of the number of points of bounded anticanonical height on the complement to all accumulation subvarieties.

The 'conjecture of linear growth' was proved by Manin together with Franke and Tschinkel for homogeneous Fano varieties, and for certain Del–Pezzo surfaces with Batyrev and Tschinkel.

2. Disproof of Lüroth Conjecture

Rational varieties play a distinguished role in the classification and study of different properties of algebraic varieties. An irreducible variety X defined over an algebraically closed base field k is called rational if there exists a birational map between X and the n-dimensional projective space $\mathbb{P}^n_k$, that is, if some Zariski open sets of X and $\mathbb{P}^n_k$ are isomorphic. In algebraic language this means that the field of rational functions of X is a pure transcendental extension $k(t_1,\dots,t_n)$ of k; or, if some Zariski open set of X can be parametrized by n affine coordinates $t_1,\dots,t_n$.

For example, smooth rational curves over $k=\mathbb{C}$ are Riemann surfaces with $g=0$, and they are isomorphic to $\mathbb{CP}^1$. This external position of these curves becomes even more emphasized if we consider finite algebraic coverings $C_1\to C_2$ of curves with arbitrary genera (or topological ramified coverings of Riemann surfaces). Then one has $g(C_1)\geqq g(C_2)$, and the difference can be expressed in terms of the covering degree and the ramification indices. This means that if $g(C_1)=0$ then necessarily $g(C_2)=0$ too. In other words, the existence of a finite morphism $\mathbb{P}^1\to C$ guarantees the rationality of C. In the algebraic language of rational functions this is formulated as follows: if L is a subfield of a pure transcendental extension $k(t)$ of k, containing k, then L is also pure transcendental.

This is Lüroth theorem valid for curves.

The analogue of Lüroth theorem for surfaces is also true: let L be a subfield of a pure transcendental extension $k(t_1,t_2)$ of k, containing k, such that $k(t_1,t_2)$ is a finite separable extension of L, then L is also pure transcendental extension of k. This is Castelnuovo's theorem 'on the rationality of plane involutions'. Its proof runs over the same scenario as in the case of curves. First, Castelnuovo's Criterion provides a numerical characterization of rationality: The smooth 2-dimensional variety is rational if and only if its arithmetical genus p_a and second plurigenus P_2 vanish. Recall that p_a is the difference of the geometric genus p_g and the irreqularity q. Since p_g, q and P_2 vanish for $\mathbb{P}^2$ and behave monotonously with respect to coverings $X_1\to X_2$, one gets that all these invariants vanish for X_2 whenever $X_1=\mathbb{P}^2$; hence X_2 is rational as well.

The Lüroth Conjecture/Problem predicted that this is true for higher dimensional varieties too. In 1971 [2] Iskovskih and Manin provided a counterexample to this conjecture: that is, one can cover a 'complicated' variety by a rational one, hence the expected monoteneity property is broken. The method developed in their article was based on a deep and technically sophisticated study of birational transformations. It created a whole new chapter of algebraic geometry studying birational rigidity of Fano manifolds.

Independently, in 1972 Clemens and Griffiths found other examples too: any smooth cubic hypersurface $X_3 \subset \mathbb{P}^4$ has a 2-sheeted ramified cover which is rational, but X_3 itself is not rational. Other examples were found later by Artin–Mumford, Saltman and Bogomolov.

The examples put in evidence, besides the family of rational varieties, a new family, the so-called 'unirational varieties': X is called unirational if there exists a dominant rational map from $\mathbb{P}^n$ to X.

The examples also show that in the study of higher dimensional varieties, starting from dimension three, the methods and strategy used for curves or surfaces will not work, one has to invent totally new structural models for the classification. As an answer to this, later in 1992 Kollár, Miyaoka and Mori introduced the class of rationally connected varieties, which is now one of the key notions in the Minimal Model Program of the classification of higher dimensional varieties. By this approach, the intrinsic geometry of varieties is captivated in the geometry of the rational curves on the variety and by their deformations.

3. The Brauer–Manin obstruction

For Diophantine equations the most fundamental question is the decidability whether they have any integral solution. A classical test (already used by Gauss) is to try to solve the equations modulo some integer m: if for some m one gets no solution then definitely the system has no integral solution either. In general this is not an 'if and only if' statement: for example the equation

$$\left(x^2 - 13\right)\left(x^2 - 17\right)\left(x^2 - 221\right) \equiv 0 \pmod{m}$$

has solutions for any m, but obviously has no rational solution. Nevertheless, for low-degree forms, the above test 'almost' works. For example, the equation $a_1x_1 + \cdots + a_nx_n = b$ with integral coefficients has an integral solution if and only if it has a solution modulo m for every m. The case of homogeneous polynomial $f(x_1, \ldots, x_n)$ with integral coefficients of degree two is also very special. But, note that in this case one has another natural obstruction to get non-zero solutions of $f(x) = 0$. Namely, if f is definite, e.g. $f = \sum x_i^2$, then we cannot get any non-zero solution even over the real numbers. For degree two forms these two obstructions provide an 'if and only if' criterion: this is formulated by the Minkowski–Hasse theorem which says that a quadratic equation has an integral solution if and only if it has a solution over the real numbers and also over all the p-adic fields for every prime p. Or, f should not be a definite form, and for every prime p

and positive integer k the congruence $f(x_1, \dots, x_n) \equiv 0 \pmod{p^k}$ has a solution such that at least one of the coordinates is not divisible by p. This is the 'Hasse's local-to-global principle'. (Invariants of the form over the p-adic field, and the signature over the reals, called 'local' invariants of the quadratic form, constitute a complete set of obstructions for its solvability. The Hasse principle, in general, says that if a variety defined over a field k has a point over every completion of k, then it has over k as well.)

The first example of this subsection shows that the Minkowski–Hasse theorem cannot be extended for forms of degree six. In fact

$$\left(x^2+3y^2-17z^2\right)\left(x^2+5y^2-7z^2\right)$$

is also a counterexample of degree four. One can argue that this form is not irreducible, but even degree three irreducible equations can be found which contradict the Minkowski–Hasse theorem (e.g. $3x^3+4y^3+5z^3=0$ found by Selmer in 1951).

In his talk at the International Congress of Mathematicians in 1971, Manin proposed a different obstruction [3]. It can be non-trivial even if the first (Minkowski–Hasse) obstruction vanishes, that is, even if the equation has points over all local fields, the Manin obstruction might rule out the possibility of the existence points over the global field. This was for a long time the only known general new obstruction in the theory of Diophantine equations. Considerable amount of work were dedicated to the proof of the solvability of various classes of equations in the case of vanishing of this obstruction.

The obstruction relies on the construction of the Brauer group. Once a field K is fixed, its Brauer group $\mathrm{Br}\,(K)$ is the abelian group of Morita equivalence classes of central simple algebras of finite rank over K, where the group law is induced by the the tensor product of algebras. The Brauer group is one of the principal invariants available for measuring the degree of complexity of the field K. If K is a finite field or a field of transcendence degree 1 over an algebraically closed field, then $\mathrm{Br}\,(K)=0$. Hence $\mathrm{Br}\,(\mathbb{C})=0$. On the other hand, $\mathrm{Br}\,(\mathbb{R})=\mathbb{Z}/2\mathbb{Z}$, and 'usually' it has infinite order; e.g. $\mathrm{Br}\,(K)=\mathbb{Q}/\mathbb{Z}$ for any non-archimedean local field K.

More generally, one defines for algebraic schemes the Grothendieck–Brauer group in terms of Grothendieck–Azumaya algebras. The rather technical Manin obstruction was defined in their language. Recently it was verified that all these groups can be determined as cohomology groups as well, entering in cohomology exact sequences.

For torsors of abelian varieties, under the assumption that the Tate–Shafarevich group is finite, the Manin obstruction is an absolute invariant (that is measures completely the failure of the local-to-global principle). There are however examples, due to Skorobogatov (2001) of varieties with trivial Manin obstruction which have points everywhere locally without global points.

4. Modular forms and zeta functions [6–8]

Modular forms are central objects of number theory. Their Fourier coefficients are interesting number-theoretical functions like numbers of solutions of Diophantine equations and congruences. Their Mellin transforms are the L-series encoding the arithmetics of the representations of the Galois group in the commutative class field theory. From geometric point of view, they are differential forms on various moduli spaces.

Historically, the theory of modular forms was developed in several periods. The first period, in the first part of the nineteenth century is related with the theory of elliptic functions. Then, in the second part of the nineteenth century, by Felix Klein and others, it was used in connection with one variable automorphic forms. The next period is related with the work of Hecke about 1925, then in the 60's by the formulation of the Taniyama–Shimura–Weil conjecture their deep significance in number theory became even more clear. This conjecture establishes a connection between elliptic curves defined over $\mathbb{Q}$ and modular forms. It attracted considerable interest in the 1980's when G. Frey suggested (and Ribet proved) that it implies Fermat's Last Theorem. It was proved in 1995 by A. Wiles for all semistable elliptic curves over the rationals, with help of R. Taylor; now it is known as 'Modularity Theorem'.

In a series of articles published in the 70s, Manin combined p-adic analysis, modular forms and number theory. He established the fundamentals of the theory of modular symbols and p-adic L-series related to the modular forms. His plenary talk at the ICM in 1978 was also dedicated to this theory.

5. Mathematical physics

A considerably large part of Manin's research is devoted to mathematical physics and to the application of algebraic geometry to mathematical physics. Here we will give a short overview.

In Manin's approach and work in mathematical physics, the strongest mathematical tools are provided by algebraic geometric methods. They play far more prominent role than representation theory or functional analysis that constituted the mathematical tools in the earlier phase.

Quantum cohomology. The theory of quantum strings propagating in a space-time with non-trivial topology led in the 90's to the development of a new mathematical theory, the 'quantum cohomology'. The first notions of this geometric theory are due to the physicists E. Witten and C. Vafa. Its mathematical foundations in the framework of algebraic geometry (as opposed to symplectic geometry) were laid down in the articles by M. Kontsevich and Manin in 1994, and extended in the articles of Manin and Behrend. Manin's monograph [12] summarizes the first decade of the development of this rich theory.

In quantum cohomology one considers a deformation of the classical cohomology ring: instead of counting only the intersection points of the cycles, one counts with weights points connected by algebraic curves passing through them. In this description the intersection theory of algebraic geometry provides the basic tool.

Quantum groups [11]. The main facts of the theory of quantum groups were discovered by Manin's student V. Drinfeld, a fact which (together with his work on Langlands Conjecture for GL_2) earned him the Fields Medal. Manin developed a different approach, he considers these groups as symmetry objects. The simplest quantum group $GL_q(2)$ is the symmetry group of the 'Manin plane' with coordinates x, y satisfying $xy = qyx$, exactly in the same way as the usual linear group $GL(2)$ is the automorphism group of the usual plane.

Instantons. Manin with Drinfeld classified the self-dual solutions of Yang–Mills equations (these are called 'instantons'). This solution, found simultaneously and independently by M. Atiyah and N. Hitchin, was published in a famous article signed by all four authors and became known as the AHDM construction [**?**]. The theory of instantons, developed further by S. Donaldson, plays a central role in the low-dimensional differential geometry. ¡ref:11?¿

Computation of the Polyakov measure in the theory of bosonic strings [9]. After the theory of instantons, the article [9] of Manin was the first breakthrough of algebraic geometry in string theory: moduli spaces of algebraic curves entered in mathematical physics.

III.

We end this presentation with a sentence borrowed from an issue of Moscow Math. Journal dedicated to Manin's 65th birthday:

> *The example he set for those around him was not that of a monomaniac mathematician, but of a deep scholar with wide interest, for whom penetration into the mystery of knowledge is much more important than professional success.*

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