Extremal Graph Problems, Degenerate Extremal Problems, and Supersaturated Graphs

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A Survey of Old and New Results

Notation. Given a graph, hypergraph G_n, ..., the upper index always denotes the number of vertices, $e(G)$, $v(G)$ and $\chi(G)$ denote the number of edges, vertices and the chromatic number of G respectively. Given a family $\mathcal{L}$ of graphs, hypergraphs, $\mathbf{ex}(n, \mathcal{L})$ denotes the maximum number of edges (hyperedges) a graph (hypergraph) G_n of order n can have without containing subgraphs (subhypergraphs) from $\mathcal{L}$. The problem of determining $\mathbf{ex}(n, \mathcal{L})$ is called a *Turán-type extremal problem.* The graphs attaining the maximum will be called *extremal* and their family will be denoted by $\mathbf{EX}(n, \mathcal{L})$.

1. Introduction

Let us restrict our consideration to ordinary graphs without loops and multiple edges. In 1940, P. Turán posed and solved the extremal problem of K_{p+1}, the complete graph on $p+1$ vertices [39, 40]:

TURÁN THEOREM. *If $T_{n,p}$ denotes the complete p–partite graph of order n having the maximum number of edges (or, in other words, the graph obtained by partitioning n vertices into p classes as equally as possible, and then joining two vertices iff they belong to different classes), then*

(a) If $T_{n,p}$ contains no K_{p+1}, and

(b) all the other graphs G_n of order n not containing K_{p+1} have less than $e(T_{n,p})$ edges. Using a theorem of Erdős and Stone [24], Erdős and Simonovits [18] derived that if $\mathcal{L}$ is an arbitrary family of forbidden graphs, and

$$p(\mathcal{L}) = \min_{L\in\mathcal{L}} \chi(L) - 1, \tag{1}$$

then

$$\mathbf{ex}(n,\mathcal{L}) = \left(1-\frac{1}{p}\right)\binom{n}{2} + o(n^2). \tag{2}$$

They also conjectured, and later proved independently [13, 14, 30]

Theorem 1. *Given a family $\mathcal{L}$ of forbidden graphs with minimum chromatic number $p+1$ and an $L\in\mathcal{L}$ with chromatic number $p+1$, then for $c = 2-1/v(L)$,*

$$\mathbf{ex}(n,\mathcal{L}) = \left(1-\frac{1}{p}\right)\binom{n}{2} \ + O(n^c), \tag{2*}$$

and every extremal graph S_n can be obtained from a $T_{n,p}$ by deleting and adding $O(n^{2-c})$ edges. Further, the minimum degree

$$\underline{d}(S_n) = \left(1-\frac{1}{p}\right)n + O(n^{c-1}).$$

This means that the extremal numbers and the extremal structure depend very loosely on $\mathcal{L}$, they are asymptotically determined by the minimum chromatic number $p+1$. Further, (2) implies that $\mathbf{ex}(n,\mathcal{L}) = o(n^2)$ iff $p=1$, that is, iff $\mathcal{L}$ contains at least one bipartite graph. This case will be called *degenerate*.

Recently, writing a survey on extremal graph theory [36], I came to realize that one of the most intriguing, most important and rather underdeveloped areas of extremal graph theory is the theory of degenerate extremal graph problems. In this case the structure of extremal graphs tends to become very complicated.

Of course, there are very many interesting results in this field, see e.g. [1, 2, 6, 7, 12, 15, 17, 20, 26, 29, 32]. Still, this underdevelopment was one of the reasons why the author with some other coauthors started investigating degenerate extremal graph problems more systematically. Some of the newer results achieved by the author and others can be found in [25, 21, 22, 32]. Here I would like to give a survey which is almost completely devoted to degenerate extremal graph problems, degenerate extremal hypergraphs problems and the corresponding *theory of supersaturated graphs*. Some repetition compared to [36] is unavoidable, but I have tried to minimize it. Anyone wishing to find further literature on extremal graph theory is recommended to read, among others, Bollobás' excellent book [5], or my survey [36].

Before restricting our consideration to degenerate extremal problems, let us pose the following question:

2. Which General Questions Should be Asked in Extremal Graph Theory?

Obviously, such a question can be answered in many different ways, and most of the answers are biased in some way or other. Perhaps one fair answer could be:

(a) Try to find general questions, related to the *general* theory of extremal graphs, and

(b) try to ask specific questions, which cannot be answered by the general theory, which attack completely new areas, new phenomena, and therefore lead to completely new (and meaningful!) theories.

It is meaningless to speak too much of (b) in generalities. The reader, however, should remember that the whole of extremal graph theory (and many other theories) developed this way, inductively: through solving particular problems, proving conjectures which often seemed too particular at first glance, and still they often led to many other interesting and general results, rich theories. So let us get back to (a) and try to give a sketch of the general problems in extremal graph theory.

The problems in extremal graph theory can be classified according to their OBJECTS and their TYPES. This is illustrated (without much explanation) on the chart on the next page.

Below we shall see that many of the extremal graph theorems can quite well be described by these categories.

3. Degenerate Extremal Graph and Hypergraph Problems

Above we restricted our consideration to ordinary graphs, however, extremal graph problems naturally arise for r–uniform hypergraphs as well. Further, extremal graph theorems are known for multigraphs, multidigraphs, multi-hypergraphs as well, see e.g. [7, 8, 12, 29, 32]. However, in the "multi-" cases we have to fix an upper bound on the multiplicity of edges to get finite maxima. For the sake of simplicity, in this survey we shall restrict ourselves to ordinary graphs and uniform hypergraphs. The definitions of $\mathbf{ex}(n, \mathcal{L})$ and $\mathbf{EX}(n, \mathcal{L})$ are obvious.

Definition 1. Let us consider r–uniform hypergraphs and let $\mathcal{L}$ be a family of forbidden hypergraphs. The extremal hypergraph problem of $\mathcal{L}$ will be called *degenerate* if $\mathbf{ex}(n, \mathcal{L}) = o(n^r)$.

TWO EXAMPLES.

Theorem 2 (Kővári–T. Sós–Turán, [26])**.** *Let $K_{p,q}$ denote the complete bipartite graph with p and q vertices in its color-classes. Then*

$$\mathbf{ex}(n, K_{p,q}) \le \frac{1}{2}\sqrt[p]{q-1}\,n^{2-1/p} + O(n). \tag{3}$$

OBJECTS	TYPES OF QUESTIONS
ordinary graphs multigraphs digraphs hypergraphs .... graphs with partitioned vertex set (subgraphs of $K(n_1, \ldots, n_r)$)	non-degenerate extremal problems degenerate extremal problems – lower bounds using random graphs – lower bounds by finite geometrical constructions – upper bounds by counting arguments -upper bounds by recursions supersaturated graphs compactness results .... Perturbation problems – degree perturbation – chromatic perturbation – Turán-Ramsey theorems (Ramsey perturbation)
APPLICATIONS	(α) application of other principles in extremal graph theory – algebraic methods, eigenvalue methods – random graph "constructions" (see above) (β) Application of extremal graph theory in combinatorial geometry, analysis, probability theory. (γ) continuous versions of extremal graph theorems.

This was generalized by Erdős [12]: #

Theorem 3. *Restrict ourselves to r–uniform hypergraphs. Let $K_r^{(r)}(m, \ldots, m)$ denote the r–uniform hypergraph obtained by fixing rm vertices x_{ij} ($i = 1, \ldots, r,\ j = 1, \ldots, m$) and taking all the m^r "transversals $(x_{1,j_1}, x_{2,j_2}, \ldots, x_{r,j_r})$ as hyperedges. Then*

$$\mathbf{ex}(n, K_r^{(r)}(m, \ldots, m)) = O(n^{r-m^{-(r-1)}}).$$

Remark 1. Theorem 3 is a very good example of what was stated of questions about type (b) in §1. Its proof is neither trivial nor too complicated, and one not too much acquainted with extremal graph theory will find it too particular. However, on the one hand it is a direct and elegant generalization of Theorem 2, (apart from the value of the constant). and on the other hand, it is widely applicable in extremal graph problems. One of its important applications is

Proposition 1. *Restrict our considerations to r–uniform hypergraphs. The problem of $\mathcal{L}$ is degenerate iff $\mathcal{L}$ contains an L^* the vertices of which can be colored by r colors so that each r–edge gets r different colors.*

Proof. The condition of the proposition is equivalent to the requirement that L^* be contained in some $K_r^{(r)}(m,\dots,m) = L'$. By (4) $\mathbf{ex}(n, L') = o(n^r)$. If $L \subseteq L'$, then $\mathbf{ex}(n, L) = o(n^r)$, and consequently, $\mathbf{ex}(n, \mathcal{L}) = o(n^r)$. This proves one half of Proposition 1. The other half follows easily from the fact that if no $L \in \mathcal{L}$ is contained in any $K_r^{(r)}(m,\dots,m)$, then $S_n := K_r^{(r)}(m,\dots,m)$ for $m = n/r$ contains no prohibited subhypergraphs and has cn^r edges. Thus $\mathcal{L}$ is not degenerate. Further applications of (4) will be given later.

If we wish to give some other examples of degenerate extremal problems - as we do - then we should start with the simplest ones. The extremal problem for paths was solved by Erdős and Gallai [15], the problem for even cycles by Erdős (unpublished), and a generalization of it was given by Bondy and Simonovits [2]. Here we mention only the original result on $\mathbf{ex}(n, C_{2t})$, (where C_{2t} denotes the cycle of $2t$ vertices).

Theorem 4. $\mathbf{ex}(n, C_{2t}) = O(n^{1+1/t})$.

Turán asked the following question: if we fix one of the regular polyhedra, (tetrahedron, cube, octahedrcn, dodecahedron, icosahedron), and denote the graph formed by its vertices and edges by L, how large is $\mathbf{ex}(n, L)$? His theorem answers the case of the tetrahedron, the others were settled by Simonovits (dodecahedron, icosahedron, [33],[34], and by Erdős and Simonovits, (octahedron, cube), see [20],[19]. Again, rather surprisingly, one of the simplest polyhedra, the cube turned out to be the most difficult case.

Theorem 5. *If Q denotes the graph determined by the vertices and edges of a cube, and Q^* denotes the graph obtained by joining two opposite vertices of the cube, then*

$$\mathbf{ex}(n,Q) \le \mathbf{ex}(n,Q^*) = O(n^{8/5}).$$

We conjecture that these results are sharp [1]. Unfortunately, no nontrivial lower bound is known for the cube problem. (In this sense the word "settled" does not apply to the cube.)

4. Why are Degenerate Extremal Graph Problems Important?

Since the main topic of this survey is the theory of degenerate extremal graph problems, it is quite natural to ask the above question. There are always many possible answers to such questions. One of them could be that in some sense many nondegenerate extremal graph problems can be reduced to degenerate extremal problems. More precisely, if $\mathcal{L}$ is a given family of forbidden graphs, one defines $p = p(\mathcal{L})$ by (l) and then one can find (in many cases) families $\mathcal{M}_1, \dots, \mathcal{M}_p$ of forbidden graphs such that

(a) If S_n is extremal for $\mathcal{L}$, then we can find p extremal graphs $T_i \in \mathbf{EX}(n, \mathcal{M}_i)$ of $n/p + o(n)$ vertices each, such that "S_n is the product of the T_i's", where the product of the T_i's is defined as follows: take vertex disjoint copies of these graphs and join every $x \in V(T_i)$ to every $y \in V(T_j)$ for every $1 \le i < j \le p$. (The product of many graphs will be denoted by $\mathbf{X}\, T_i$, of two graphs G and H by $G \times H$.)

(b) If T_i are now arbitrary graphs not containing subgraphs from $\mathcal{M}_i$, then $\mathbf{X}\, T_i$ contains no $L \in \mathcal{L}$.

(c) The extremal graph problem of $\mathcal{M}_i$ is degenerate.

Here I give only one illustration of such "reduction' theorems, namely the octahedron theorem. However, there are many other similar reduction theorems, e.g. the icosahedron and the dodecahedron theorems [34, 33].

[1] This means the existence of a $c > O$ such that $\mathbf{ex}(n,Q) > cn^{8/5}$ $(n > 8)$.

6 (THE OCTAHEDRON THEOREM)**.** *(Erdős, Simonovits [20]). If L denotes the octahedron graph (=$K_{2,2,2}$), then there exists an n_1 such that for $n > n_1$ every extremal graph $S_n \in \mathbf{EX}(n, L)$ is the product of two other graphs: $S_n = U_1 \times U_2$, where U_1 is extremal for C_4 , U_2 is extremal for P_3, $v(U_i) = n/2 + o(n)$ (where $v(G)$ denotes the number of vertices of G), and if U' and U'' are some other graphs and U' contains no C_4, U'' contains no P_3, then $U' \times U''$ contains no octahedron graph L.*

Generally, theorems stating that the extremal graph for a non-degenerate extremal problem is the product of some extremal graphs of a degenerate problem, will be called REDUCTION THEOREMS. And now we can say that degenerate extremal graph problems are interesting partly in their own right, and partly because nondegenerate extremal graph theorems are often reducible to degenerate extremal graph theorems, see [35].

5. Some General Open Questions on Degenerate Extremal Graph Problems

To be sincere, many of the open problems on degenerate extremal problems seem to be completely hopeless because the extremal graphs conjectured have a fairly regular and yet very complicated structure. Still, it is worth formulating quite general and hopeless conjectures, since they often indicate very well in which direction we should go, or which types of simpler questions should be solved.

Below, we formulate four general questions.

5.1. The Problem Of Rational Exponents

Conjecture 1. *Let $\mathcal{L}$ be a family of bipartite graphs. Does there exist a rational $\alpha \in [1, 2)$ such that $\mathbf{ex}(n, \mathcal{L})/n^\alpha$ converges to a positive limit.*

Here the requirement that the limit should be positive, is important otherwise the conjecture would immediately follow from Kővá'ri- T. Sós-Turán theorem. A weakening of this conjecture is

Conjecture 2. *Given a family $\mathcal{L}$ of bipartite forbidden graphs, there exist constants, c and $c' > O$ and $\alpha \in [1, 2)$ for which*

$$cn^{\alpha} \leq \mathbf{ex}(n, L) \leq c'n^{\alpha}.$$

The main idea behind both conjectures is that while the structure of extremal graphs may be fairly complicated, it must be simple in some other sense. For 3-uniform hypergraphs, Conjecture 2 does not hold, as was shown by Ruzsa and Szemerédi [29].

5.2. The Problem of Strong Compactness

Conjecture 3. *Let $\mathcal{L}$ be a finite family of bipartite graphs. Then there is an L in $\mathcal{L}$ such that*

$$1 \leq \mathbf{ex}(n, L)/\mathbf{ex}(n, \mathcal{L}) = 0(1)$$

Conjecture 4. *Let $\mathcal{L}$ be a finite family of bipartite graphs. Then there is an $L \in \mathcal{L}$ such that*

$$\mathbf{ex}(n, L)/\mathbf{ex}(n, \mathcal{L}) \to 1.$$

Clearly, the second conjecture is much stronger. Both assert that if we prohibit a finite family of bipartite graphs, then there is always one among them, the exclusion of which has almost the same effect as if we excluded the whole of $\mathcal{L}$: one forbidden graph dominates the whole problem. Here the finiteness of $\mathcal{L}$ is very important: if $\mathcal{L}$ is the (infinite) family of all the cycles, then $\mathbf{ex}(n, \mathcal{L}) = n - 1$. However, for any finite subclass $\mathcal{L}$ of $\mathcal{L}$ there exists an $\alpha > 0$ such that $\mathbf{ex}(n, L^*) > cn^{1+\alpha}$.

For non-degenerate extremal graph problems even the stronger Conjecture 4 immediately follows from the Erdős-Simonovits theorem [18]. One final remark should be made: we stated Conjectures 3,4 in affirmative form, however some examples suggest that perhaps they are false. It would be interesting to find counterexamples.

5.3. The Problem of Everywhere Dense Exponents

Conjecture 5. *For every $1 \leq \alpha \leq \beta \leq 2$ there exists a finite family $\mathcal{L}$ of forbidden graphs such that*

$$n^\alpha < \mathbf{ex}(n, \mathcal{L}) < n^\beta,$$

if n is large enough.

Conjecture 6 (Stronger)**.** *For every $1 \leq \alpha < \beta \leq 2$, there exists a finite $\mathcal{L}$ and a $\gamma \in (\alpha, \beta)$ such that $\lim \mathbf{ex}(n, L)/n^\gamma$ exists and is positive.*

(Perhaps for every rational $\gamma \in (1, 2)$ there exists such a finite $\mathcal{L}$.)

5.4. The Problem of Weak Vertices

In some reduction theorems the following property of $\mathcal{L}$ plays important role: "One can delete a vertex v of $\mathcal{L}$ so that $\mathbf{ex}(n, L - v) = o(\mathbf{ex}(n, L))$.' Vertices satisfying this condition will be called **weak**. The problem we would like to state is: "Characterize those bipartite graphs which have weak vertices. "

Clearly, the trees have no weak vertices, C_{2k}, $K_{p,q}$ for $p = 2, 3$ do have (and probably $K_{p,q}$ does have weak vertices even for $p \geq 4$) The graph in Figure 1 (as it is easy to check) has no weak vertices. (For some literature, see [31].)

Figure 1:

PROOF METHODS: Obviously, to solve an extremal problem we need an upper and a lower bound on $\mathbf{ex}(n, \mathcal{L})$. Both may be fairly difficult to find. However, at present, we know much less about how to get lower bounds. That is why we start with this question.

6. How to Get Lower Bounds?

In trying to find a lower bound for a degenerate Turán-type extremal problems we usually need a construction, but sometimes use the method of random graphs. However, the method of random graphs somehow turns out to be rather weak; I do not know of any case of a finite degenerate $\mathcal{L}$ when the method of random graphs gave a sharp lower bound. The explicit constructions are mostly based on FINITE GEOMETRICAL arguments. The method of using finite geometrical constructions seems to be fairly powerful, and many times gives sharp lower bounds. Still, we do not know enough about finite geometries, and perhaps this is why in many cases we only conjecture that an appropriate finite geometrical construction could help, but are unable to find it. I shall not go into detailed discussion of this method, it is described in [5] or [36], or in the original papers, e.g., [1, 6].

7. How to Get Upper Bounds in Degenerate Extremal Problems?

Mostly we use one or two of the following methods (combined):

(a) blowing up
(b) recursion theorems
(c) application of supersaturated theorems

Since we often combine these methods, several of the illustrations given below will be "mixed" ones.

8. Blowing Up

First we sketch the method, then give some applications. In many extremal graph problems, we first apply some regularization theorem to the original graph, asserting that in any graph G_n we can find a subgraph G_m the minimum degree of which is almost its average degree and such that G_m still has many edges. One such theorem is

Theorem 7. *Let G_n be a graph with E edges and average degree $d = 2E/n$. Then there exists a subgraph G_m with $\underline{d}_(G_m) > d/2$, where $\underline{d}(G)$ denotes the minimum degree. (Clearly, if $d = \frac{2E}{m} \to \infty$, then $m \to \infty$, too.)*

This theorem is mostly applied with $e(G_n) = cn^\alpha$ where $\alpha > 0$. Sometimes we need the sharper result of Erdős and Simonovits [20]:

Theorem 8. *For every $\alpha > 0$ and $c_1 > 0$, there exist a $\beta > 0$ and $c_2, c_3 > 0$, such that if $e(G_n) > c_1 n^\alpha$, then G_n contains a subgraph G_m with $m > n^\beta$, and $\underline{d}(G_m) > c_2 m^\alpha$, $\overline{d}(G_m) < c_3 m^\alpha$ [2] where $\overline{d}(G)$ the denotes the maximum degree. If $c_1 \to \infty$, then $c_2, c_3 \to \infty$.*

Both results say that we may restrict our consideration primarily to regular graphs. The next lemma asserts that we may assume that G_n is bipartite:

Lemma 1. *If G is an arbitrary graph and H is a bipartite subgraph of G having the maximum number of edges, then $d_H(x) \geq d_G(x)/2$ for every vertex, where $d_H(x)$ and $d_G(x)$ denote the degrees in the corresponding graphs.*

Assume now that we would like to prove that $\mathbf{ex}(n, \mathcal{L}) = O(n^{1+\alpha})$. Then we fix a graph G_n with at least $c_1 n^{1+\alpha}$ edges, choose a $G_m \subseteq G_n$ according to Theorem 7 and then a bipartite $H_m \subseteq G_m$ with $\underline{d}(H_m) \geq (1/2)c_2 m^\alpha$. After these two preparatory steps, we can carry out the actual "blowing up". We choose an arbitrary $x \in V(H_m)$ and denote by S_j the set of vertices having distance j from x. Then we prove that the condition "$L \not\subseteq H_m$" implies that, for some $r = r_{\mathcal{L}}$ the size of S_j is much larger than the size of S_{j-1}, $j = 1, 2, \ldots, r$. We know that $|S_1| \geq \underline{d}(H_m) \geq c_2 m^\alpha$, and we known also that

[2] Correction!

$|S_r| < m$. These two facts yield that for the considered exponent α , c_2 is bounded, and therefore c_1 is bounded as well. This yields that

$$\mathbf{ex}(n, \mathcal{L}) = O(n^{1+\alpha}).$$

This method was used by Bondy and Simonovits to prove the *even cycle* theorem [2] and also by Faudree and Simonovits to prove a generalization of this result, see below [25].

Bondy and Simonovits conjectured the following generalization Theorem 4:

Definition 2. Let $C_{k,t}$ denote the graph obtained by joining two vertices x, y by k vertex-independent paths of t edges each.

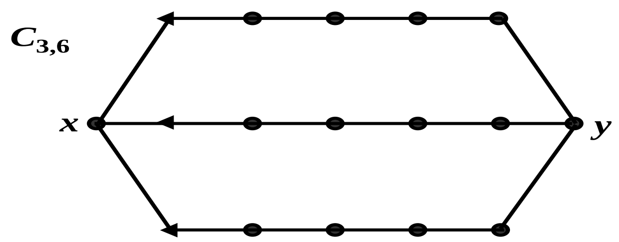

Figure 2: $C_{3,6}$

Theorem 9 (Faudree, Simonovits [25])**.** $\mathbf{ex}(n, C_{k,t}) = O(n^{1+1/t})$.

Clearly, this generalizes Theorem 4. As a matter of fact, Faudree and Simonovits have proved a much more general theorem.

Definition 3. Let L be a bipartite graph with a given 2-colouring χ, say in red and blue. Let x be a new vertex and join it to each blue vertex of L by vertex-independent paths of $t-1$ edges (Figure 3). The resulting graph will be denoted by L_t or $L_t(L, \chi)$.[3]

Now we can formulate the Faudree-Simonovits recursion theorem.

[3]Simonovits, 2010: In the original version the colouring was denoted by c.

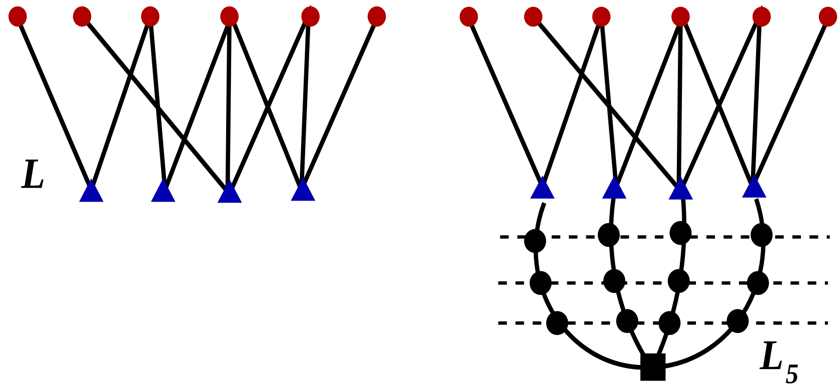

Figure 3: L and $L_t(L,\chi)$

Theorem 10. *If T is a tree or a forest with a given 2-colouring χ, and $L^* = L_t(L,\chi)$, then $\mathbf{ex}(n, L^*) = O(n^{1+1/t})$.*

Clearly, starting with a star $T = K_{1,k}$ we get back to Theorem 9. In proving Theorem 10 we used again the blowing up method. Theorem 10 has many different applications and some generalization. Here, we have no space to go into the details.

9. Recursion Theorems

If we wish to prove general degenerate extremal graph theorems, (or supersaturated theorems, see below), one way to do this is to look for recursion theorems. These assert that if we have bounds for a graph L, then we can obtain other bounds for some more complicated graph L^*. Generally, this L is obtained from L by some given operation. Theorem 10 above is one illustration of such recursion theorems. Below, we shall give another recursion theorem, due to Erdős and Simonovits.

Definition 4. Let L be a bipartite graph properly coloured by red and blue. Let $K_{t,t}$ be coloured by red and blue and join each red vertex of $K_{t,t}$ to each blue one of L, each blue vertex of $K_{t,t}$ to each red one of L, (see Fig. 4). The resulting graph will be denoted by $L(t)$.

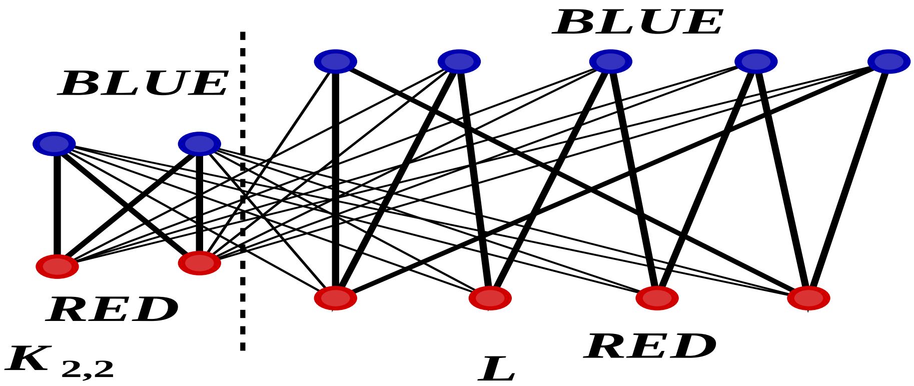

Figure 4:

Theorem 11. *Assume that L is a given bipartite graph with a fixed 2-colouring, and that* $\mathbf{ex}(n, L) = O(n^{2-\alpha})$*. Define β by*

$$\frac{1}{\beta} - \frac{1}{\alpha} = t.$$

Then $\mathbf{ex}(n, L(t)) = O(n^{2-\beta})$*.*

One application of this theorem is when $L = C_6$, hence $\mathbf{ex}(n, L) = O(n^{4/3})$. Since $\beta = 8/5$, and $L(1) = Q^*$ of Theorem 5, Theorem 11 immediately yields Theorem 5 . As a matter of fact, Theorem 11 was obtained by trying to prove Theorem 5. However Theorem 11 has many other interesting consequences, too; for example,

Theorem 12. *Let L be a bipartite graph properly coloured by red and blue. If we may graph delete t red and t blue vertices so that the resulting graph is a tree, then* $\mathbf{ex}(n, L) = O(n^{2-1/(t+1)})$*.*

The proof of Theorem 12 from Theorem 11 is trivial. Theorem 12 contains a result of Erdős (a sharpening of the Kővári-T. Sós-Turán theorem) as a special case. Indeed, let $L = (K_{p+1,p+1}$–an edge). Then Theorem 12 can be applied: we may delete $p-1$ red and $p-1$ blue vertices of L so that the resulting graph is a path P_3. Thus

$$\mathbf{ex}(n, L) = O(n^{2-1/p})$$

There are also recursion theorems of other types. For instance, Erdős and Simonovits [22] proved a result where they obtained upper bounds on $(r+1)$-uniform hypergraph problems from r–uniform hypergraph results: the recursion goes according to the edge-size of the hypergraph. This theorem implies Theorem 3 immediately.

10. Supersaturated Graphs

To describe how supersaturated graph theorems imply degenerate extremal graph results, we first explain what we mean by a supersaturated graph.

Let us consider the case when $\mathcal{L}$ is fixed and G_n has $E > \mathbf{ex}(n, L)$ edges. Clearly, G_n must contain forbidden subgraphs $L \in \mathcal{L}$. As it turns out, in most cases G_n must contain not only one but very many prohibited $L \in \mathcal{L}$. Such graphs will be called supersaturated or $\mathcal{L}$–supersaturated .

There exists a fairly extensive literature on the case when G_n is K_p–supersaturated. Without trying to be complete in any sense, we list the following references: [3, 4, 5, 11, 12, 27, 28]. For the case of L- supersaturated graphs in general, see [21, 22, 23, 37].

Here we are interested primarily in supersaturated graphs where the corresponding extremal problem is degenerate. Nevertheless, let us start with a quite general theorem.

Theorem 13 (Erdős, Simonovits, [22])**.** *Let us consider r–uniform hypergraphs and let $\mathcal{L}$ be a given finite family of forbidden graphs. Assume (for the sake of simplicity) that all the graphs $L \in \mathcal{L}$ have v vertices. Then, for every $c > O$, there exists a $c' > O$ such that, if*

$$e(G_n) > \mathbf{ex}(n, L) + cn^r,$$

then G_n contains at least $c'n^v$ copies of $L \in \mathcal{L}$.

Obviously, this result is sharp in the sense that G_n has at most $O(n^v)$ copies of $L \in \mathcal{L}$. We arrive at much deeper and more difficult problems if we consider the case of "weakly supersaturated graphs": the case when $e(G_n) = \mathbf{ex}(n, \mathcal{L}) + o(n^r)$. For the sake of simplicity, we shall restrict below our considerations to the case of ordinary graphs ($r = 2$). Further, we shall assume that $\mathcal{L}$ consists of bipartite graphs only. The theorems below state (in some sense) that if $e(G_n) > \mathbf{ex}(n, L)$ is fixed, then G_n has the minimum number of copies of $L \in \mathcal{L}$ if G_n is a random graph. Observe first that if G_n is a random graph with E edges, then G_n contains on average

$$\binom{n}{v}\left(\frac{2E}{n^2}\right)^e \approx c_{\mathcal{L}} \cdot \frac{E^e}{n^{2e-v}}$$

copies of $L \in \mathcal{L}$.

Conjecture 7. *Let L be a bipartite forbidden graph such that $\mathbf{ex}(n, L) = O(n^{2-\alpha})$, for some $\alpha \in (0, 1)$. Then there exist two constants c and $c' > 0$ such that if $E = e(G_n) > cn^{2-\alpha}$, then G_n contains at least $c'E^e/n^{2e-v}$ copies of L, where $e = e(L)$, $v = v(L)$.*

A weakening of this conjecture is

Conjecture 8. *For every L, there exists an α and $c, c' > 0$ such that if $e(G_n) > cn^{2-\alpha}$, then G_n contains at least*

$$c' \frac{E^e}{n^{2e-v}}$$

copies of L, where $e = e(L)$ and $v = v(L)$.

In [37], Simonovits proved Conjecture 7 for C_{2t}, with $2 - \alpha = 1 + 1/t$; in [21], Erdős and Simonovits proved it for P_t [4]; in [23] they proved it for many other cases, including the cube graph and $K_{p,q}$. Since [23] is published in this very volume, there is no reason to repeat the results listed either in the introduction of [23] or among its main results. The reader will find there, among other things, a conjecture stronger than Conjecture 7. Further, one of the main results of [23] is a "RECURSION THEOREM ON SUPERSATURATED GRAPHS":

Theorem 14. *If Conjecture 7 holds for L with some constant α, then it also holds for the β defined in Theorem 11 and the $L(t)$ defined in the preceding definition.*

[4] For walks W^k of length k the analogous (or, more precisely, a sharper) inequality follows from some matrix theoretical inequalities.

(Thus, for example, the fact that Conjecture 7 holds for C_6 implies that it also holds for the graph Q^* obtained from the cube graph Q by joining two opposite vertices.)

11. How to Apply Theorems on Supersaturated Graphs to Prove Extremal Graph Theorems

The method of using supersaturated graph results to prove Turán-type extremal theorems is as follows: We would like to prove that $\mathbf{ex}(n, L) = O(n^{2-\alpha})$. Assume that we can find another bipartite graph L' such that

(i) if $e(G_n) > E = cn^{2-\alpha}$ for some large c, then G_n contains at least $N = N(L', E)$ copies of L';
(ii) if G_n contains at least N copies of L', then it contains a copy of L.

From (i) and (ii) we derive that $\mathbf{ex}(n, L) < cn^{2-\alpha}$.

Of course, this very method is often used with a slightly different "ideology": we say that "we count the number of copies of L' in G_n". First we get an upper bound on this, using the fact that $L \not\subseteq G_n$, then we get a lower bound on the number of copies of L' in G_n, using the fact that it has many edges. Comparing the two estimates, we obtain that G_n cannot have too many edges.

Obviously, there is not much difference between the two viewpoints. Still, in this survey we choose the first one, which will turn out to be quite useful.

Let us give some illustrations. (l) The proof of the Kővári-T. Sós-Turán theorem is one good example: we calculate the number of copies of $L' = K_{1,p}$ in a graph. Each vertex of degree d yields $\binom{d}{p}$ such $K_{1,p}$'s. Hence one can easily find a (sharp) lower bound on the number of copies of $K_{1,p}$ in G_n in terms of $e(G_n)$ and n. On the other hand, if the number of these is larger than $(q-1)\binom{n}{p}$, then there exists a p-tuple $x_1, \ldots, x_p$ occurring in at least q stars $(y_j, x_1, \ldots, x_p)$. These vertices x_1 and y_j form a $K_{p,q}$ in G_n.

The cube theorem (Theorem 5), and the more general Theorem 11 (The recursion theorem on $L(t)$) were proved also by using a supersaturated graph argument. There we counted the number of copies of C_4 in G_n. Further, the "supersaturated graph recursion theorem" of [23], that is, Theorem 14, was again proved by a similar method.

One interesting application of this method is to the case when the forbidden graphs are cycles. To describe this application we first formulate two conjectures [21].

Conjecture 9.

$$\mathbf{ex}(n, C_{2t}) = \frac{1}{2} n^{1+1/t} + o(n^{1+1/t}).$$

Conjecture 10. *For any t and $k \geq 2$*

$$\mathbf{ex}(n, C_{2t}, C_{2k-1}) = \left(\frac{n}{2}\right)^{1+1/t} + o(n^{1+1/t}).$$

In both cases the upper bound of the conjecture follows from the Bondy-Simonovits or the Erdős theorem, apart from the value of the multiplicative constants. The background to these conjectures is that one may reasonably believe that the extremal graphs are almost regular, and if we exclude only C_{2t}, then the degrees are around $\sqrt[t]{n}$, while in the second case the extremal graphs are bipartite, with roughly $n/2$ vertices in each colour class and the degrees are only around $\sqrt[t]{n/2}$. Although many results are known on similar problems about even cycles, Conjecture 9 has been specified only for $t = 2$. [5] In [21], Erdős and Simonovits proved that

$$\mathbf{ex}(n, C_4, C_5) = \left(\frac{n}{2}\right)^{3/2} + o(n^{3/2}).$$

Their proof uses sharp supersaturated graph theorems on walks length k in graphs. It follows from some well known inequalities non-negative matrices [41] (see comments below) that

Theorem 15. *If $E = e(G_n)$ and $d = 2E/n$ is the average degree, then G_n contains at least $(1/2)nd^k$ walks of length k (where we regard the walks $(x_0, \ldots, x_k)$ and $(x_k, \ldots, x_0)$ as identical). Further, if $d \to \infty$, then G_n contains at least $(1/2)nd^k + o(nd^k)$ paths of length k.*

[5] Simonovits, 2010: Some of our conjectures listed here were later disproved, see e.g. [42]

Theorem 15 is sharp: a regular graph contains exactly $(1/2)nd^k$ walks of length k and $(1/2)nd^k - o(nd^k)$ paths of length k if $d \to \infty$. This theorem was conjectured by Simonovits, proved by Godsil for walks of even length and by Faudree and McKay for $k = 3^s$. Finally Godsil found some matrix inequalities [41] immediately implying it for all walks. Below, we sketch how one can use it to prove the upper bound in

Theorem 16.

$$\left(\frac{n}{2}\right)^{3/2} - o(n^{3/2}) \le \mathbf{ex}(n, C_4, C_5) \le \left(\frac{n}{2}\right)^{3/2} + O(n).$$ [6]

To prove this theorem we take a graph of G_n with E edges, cou- how many walks of length three occur in it, and conclude that there exists an edge (x, y) contained in more than n paths of the form (x, y, z, t) or (y, x, z, t). Hence, we find two paths of the form (x, y, z, t) and (x, y, z', t) or two paths of the form (x, y, z, t) and (y, x, z', t). In the first case we find a C_4, in the second one a C_5. (In trying to gi- precise proofs for such theorems, the main difficulty is always to get rid of the "coincidences": above, for example, we have to ensure that z z', otherwise our C_4 or C_5 is degenerate.)

12. Final Remarks

Supersaturated extremal theorems can be used in many other cases, too. Thus, for example, we may use supersaturated theorems on some L to prove supersaturated theorems on other ones. This happened in the proof of the recursion theorem on supersaturated graphs (Theorem 14). A similar situation occurred in [22], where we gave a recursion theorem yielding supersaturated hypergraph theorems. The recursion uses results on r–uniform hypergraphs to prove supersaturated theorems on $(r+1)$-uniform hypergraphs. Right now, one of the most intriguing supersaturated graph conjectures, which does not seem completely hopeless (though we cannot prove it), is Conjecture 8.

[6] Simonovits, 2010: Actually, $O(n)$ should here be replaced by $o(n^{3/2})$. This was pointed out for me by Benny Sudakov and Jacques Verstraete. In our original proof with Erdős we treated distinguishing the walks and paths slightly too loosely. The proof can easily be corrected.

One more remark: applying supersaturated graph theorems to prove extremal graph theorems or other supersaturated graph results seems to be a fairly powerful method. The reason for this may be that, in these applications we may use rough averaging methods and convexity arguments, which are not greatly affected by the irregularities in the structure of our graph.

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