DIMACS Series in Discrete Mathematics
and Theoretical Computer Science

HOW TO SOLVE A TURÁN TYPE EXTREMAL GRAPH PROBLEM? (LINEAR DECOMPOSITION)

Miklós Simonovits

Dedicated to the memory of PAUL ERDŐS

ABSTRACT. The main purpose of this paper is to show that for many forbidden graphs L, "selected at random", some old, almost forgotten asymptotic or quasi-asymptotic results provide easy, almost immediate (and often "exact") solutions of the corresponding Turán type extremal problems. In some sense, the Petersen graph will be the "pretext" to formulate our theorems.

We shall get the Turán extremal number of the Petersen graph as an easy consequence of our results:

• for $n > n_0$ the following graph $H_{n,2,3}$ is the (only) extremal graph for the Petersen graph $\mathbb{P}_{10}$: one fixes a Turán graph $T_{(n-2),2}$ on $n-2$ vertices and joins two further vertices x and y to each other and to all the vertices of $T_{(n-2),2}$.

• We shall prove good general asymptotics for wide classes of excluded graphs (Theorem 2.2). Our results also provide algorithmic solutions of the corresponding extremal graph problems.

• We shall also prove an analog of the Andrásfai-Erdős-Sós theorem for some generalized Petersen graphs (Theorem 4.2).

The "motivating" theorems will lead to some further, more general theorems as well.

1. INTRODUCTION

1.1. Turán type extremal problems. In a Turán type extremal problem a family $\mathcal{L}$ of – so called – sample graphs is fixed and we consider only graphs G_n on n vertices not containing subgraphs $L \in \mathcal{L}$, where a "subgraph" means a not necessarily "induced" one.

PROBLEM 1.1 (Turán type). Given a family $\mathcal{L}$ of forbidden graphs, what is the maximum number of edges a graph G_n of order n can have without containing any $L \in \mathcal{L}$?

1991 *Mathematics Subject Classification.* Primary 05C35.

Research partially supported by HU-NSF grants OTKA T026069 and T16389, by AKP and also by DIMACS, New Jersey.

The maximum number of edges such a graph can have will be denoted by $\text{ext}(n, \mathcal{L})$. Let $T_{n,p}$ be the Turán graph on n vertices and p classes, i.e. n vertices be partitioned into p classes $A_1, \dots, A_p$ as uniformly as possible, and two vertices be joined iff they belong to different classes. In 1940 Turán [**28**], (see also [**29, 30**]) proved his famous theorem on the extremal number of K_{p+1}, according to which $T_{n,p}$ is the (only) extremal graph:

THEOREM 1.2 (Turán Theorem). *(a) $T_{n,p}$ contains no K_{p+1} and*
(b) if $e(G_n) \ge e(T_{n,p})$ and $G_n \ne T_{n,p}$ then $G_n \supseteq K_{p+1}$.

It is known from the Erdős–Stone-Simonovits theorem [**16, 12**] that $\text{ext}(n, \mathcal{L})$ is asymptotically determined by

$$p(\mathcal{L}) = \min_{L \in \mathcal{L}} \chi(L) - 1.$$

Namely,

$$\text{ext}(n, \mathcal{L}) = \text{ext}(n, K_{p+1}) + o(n^2).$$

Clearly, $\text{ext}(n, \mathcal{L}) = o(n^2)$ iff $p(\mathcal{L}) = 1$, which case will be called **degenerate**. (For corresponding stronger structural results see [**8, 22**].) The remainder terms in the above theorem depend primarily on the **Decomposition Class** $\mathcal{M}$ of $\mathcal{L}$. To define the decomposition class we introduce some notation.

Given two graphs Z and W, their **product** $Z \otimes W$ (often called by others their join) is the graph obtained by taking their vertex disjoint copies and joining each vertex of Z to each vertex of W. Given the graphs $U_1, \dots, U_p$, with pairwise disjoint vertex-sets, their product $\prod U_i$ is the graph obtained by joining each vertex of U_i to each vertex of U_j for each $i \ne j$.

DEFINITION 1.3. Given a family $\mathcal{L}$ of forbidden graphs with $p = p(\mathcal{L}) = \min_{L \in \mathcal{L}} \chi(L) - 1$, we shall call its **Decomposition Class** the family $\mathcal{M}$ of graphs M for which there exists an $L_0 = L_0(M) \in \mathcal{L}$ and a t such that

$$L_0 \subseteq M \otimes K_{p-1}(t, \dots, t).$$

In other words, a graph M is in the decomposition class if putting[1] it into a class A_i of a large $T_{n,p}$, the resulting graph *will* contain a forbidden $L \in \mathcal{L}$. Clearly, $p(\mathcal{M}) = 1$, the extremal graph problem of the decomposition class is always degenerate.

Whether we can solve a Turán type extremal problem or not depends primarily on the corresponding decomposition class $\mathcal{M}$. In some sense all the Turán type extremal problems reduce to the solution of the extremal graph problem of $\mathcal{M}$:

(a) If there is a tree or a forest in $\mathcal{M}$, then we can solve the problem in all known cases where we can solve the extremal graph problem of $\mathcal{M}$. This is what can be called linear decomposition: this is equivalent with saying that

$$\text{ext}(n, \mathcal{L}) - \text{ext}(n, K_{p+1}) = O(n).$$

(b) I conjecture that

CONJECTURE 1.4. *If $\mathcal{M}$ contains no trees, (neither forests) then some (or all?) extremal graphs for $\mathcal{L}$ for $n \ge n_0(\mathcal{L})$ are obtained by taking p graphs $U_1, \dots, U_p$ of order $n/p + o(n)$ and joining each vertex of each U_i to each vertex of each U_j $(i \ne j)$. Then we say that the extremal graphs are of* **product forms**.

[1] "putting" means selecting $v(M)$ vertices in this class and joining them so that the resulting subgraph be isomorphic to M.

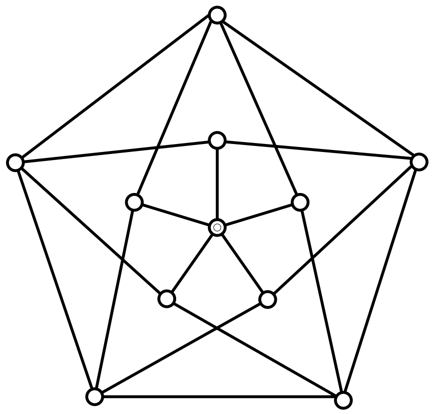

FIGURE 1:
Grötzsch Graph

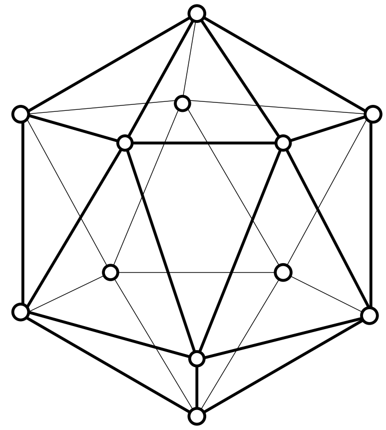

FIGURE 2:
Icosahedron graph

(We know that generally only $o(n^2)$ edges are missing between the classes and here the extra requirement is that **all** the edges between different classes must belong to the graph.)

One can easily see that – if Conjecture 1.4 holds for $\mathcal{L}$, then – U_i are also some extremal graphs for some degenerate families $\mathcal{M}_{i_n} \supseteq \mathcal{M}$. The details can be found in [**25**]. (See also our work with J. Griggs and G. Rubin Thomas, [**18**].)

Watch out! The main statement in (a) is not a theorem: it is an observation. Further, I think that Conjecture 1.4 is one of the most important, and most intriguing questions in this theory.

1.2. Some examples. Having proved his theorem, Turán also asked for the determination of $\text{ext}(n, L)$ for various other sample graphs L, among others, for the determination of $\text{ext}(n, P_k)$, or of $\text{ext}(n, L)$, if L is the graph of a platonic polyhedron: cube Q_8, octahedron O_6, icosahedron I_{12}, dodecahedron D_{20}. (The tetrahedron, K_4 was covered by his theorem.)

> *His aim was not so much to determine the extremal numbers for these particular sample graphs but to discover some new phenomena by solving the above special cases.*

(i) The question for P_k was answered by Erdős and Gallai [**11**].

(ii) Surprisingly enough, the cube Q_8 seems to be the most intractable. Erdős and I gave an upper bound $\text{ext}(n, Q_8) = O(n^{8/5})$, [**13**], which we conjectured to be sharp but we cannot prove this. As a matter of fact, there are no reasonable lower bounds known for this case. We cannot even prove that

$$\frac{\text{ext}(n, Q_8)}{n^{3/2}} \to \infty.$$

(iii) Erdős and I reduced the problem of the octahedron to the problem of C_4, [**14**]. We have proved that if S_n is extremal for the octahedron graph O_6, and n is sufficiently large, then $S_n = G_1 \otimes G_2$ where G_1 is extremal for C_4, G_2 is extremal for P_3 and $v(G_1) - v(G_2) = o(n)$.

(iv) The problems of the icosahedron and dodecahedron were solved by the author in [**23**] and [**24**]. To solve these problems, the author developed a special theory in [**24**] for the extremal graph problems where the decomposition class contains a path:

$$L \subseteq P_v \otimes K_{p-1}(v, \dots, v) \text{ for some } L \in \mathcal{L}, \text{ for } v = v(L). \tag{1.1}$$

The solution of the dodecahedron and icosahedron problems helped to understand a lot about some kind of "non-degenerate" extremal graph problems. One could ask whether attacking some other special graphs, say the Grötzsch graph, or the Petersen graph, would that have lead to a completely different theory?

The surprising answer is that – as far as it can be judged – the theorem solving the problem of the Petersen graph would have led to *exactly* the same theory as the problem of the Dodecahedron graph. The problem of the Grötzsch graph was solved and is a subcase of some other results. Let us call an edge e of G color-critical if $\chi(G-e) < \chi(G)$. An old result of mine (for a special case proved first by Erdős) [**22, 23**]) asserts that

THEOREM 1.5 (Critical edge). *If $\chi(L) = p+1$ and L contains a color-critical edge, then $T_{n,p}$ is the (only) extremal for L, for $n > n_1$.*

This immediately implies Turán's theorem for large values of n and also solves the problem of the Grötzsch graph, since it is 4-chromatic and all its edges are critical. The icosahedron problem turned out to be much deeper and led to a more general theory. (The reason seems to be that there are almost extremal graphs for I_{12} which are fairly different from the extremal graph. In other words, the extremal graphs for I_{12} are less stable than for D_{20}.)

In this paper – among others – we shall determine the Turán and Zarankiewicz extremal numbers of the Petersen graph.[2] Further, we shall also prove an analog of the Andrásfai-Erdős-Sós theorem for Petersen graphs. These "motivating" theorems will lead to some more general theorems as well.

> *One aim of this paper is to illustrate on some new examples, how some old theorems or some old methods can be used to answer many new cases. New extremal problems can often immediately be solved by some old, almost forgotten general theorems.*

For the general theory of Extremal graph problems see e.g. [**3, 26**].

1.3. NOTATION. Here we also include some of the notations used above. We shall restrict our considerations to simple graphs, that is, to graphs without loops and multiple edges. Given a graph G, $e(G)$, $v(G)$ and $\chi(G)$ will denote the number of edges, vertices, and the chromatic number of G. We shall also use subscripts to indicate the number of vertices: $G_n, H_n, \dots, S_n \dots$ will always denote graphs of order n. Given a subset X of vertices in the vertex set $V(G)$ of the graph G, $e(X)$ will denote the number of edges both endvertices of which belong to X, and if Y is another set of vertices in $V(G)$, and $X \cap Y = \emptyset$, then $e(X,Y)$ will be the number of edges one endvertex of which is in X the other in Y. A graph of chromatic number $\leq p$ will be called p–colorable.

[2]The Zarankiewicz problem will be treated rather superficially.

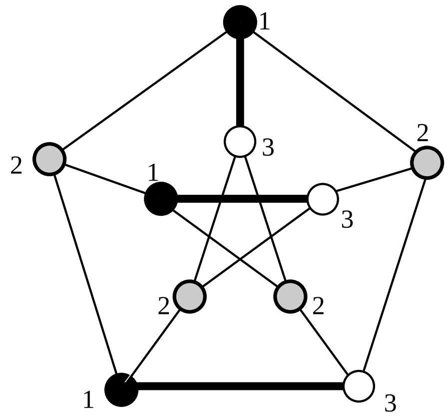

FIGURE 3:
Petersen graph

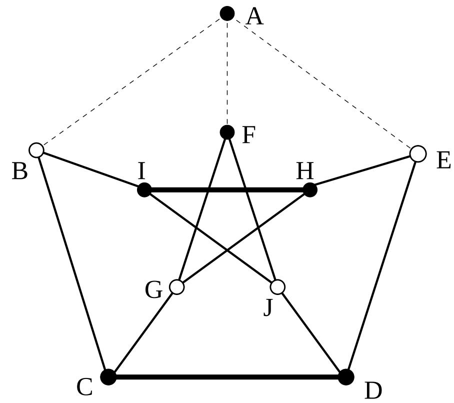

FIGURE 4:
Petersen Truncated

Special graphs. P_k denotes the path on k vertices. We denote the Petersen graph by $\mathbb{P}_{10}$ (while P_{10} is a path of 10 vertices!). $K_p(n_1, \ldots, n_p)$ is the complete p-partite graph with n_i vertices in its ith class. $T_{n,p}$ is the Turán graph on n vertices and p classes, i.e. n vertices are partitioned into p classes $A_1, \ldots, A_p$ as uniformly as possible, (the sizes of the classes differ by at most 1) and two vertices are joined iff they belong to different classes. $T_{n,p,k}$ is obtained from a $T_{n,p}$ by putting k new independent edges into one of its maximum size classes. (Of course, $k \le n/(2p)$ is assumed.) One of the most important special graphs of this paper is $H_{n,p,t} = K_{t-1} \otimes T_{n-t+1,p}$: the graph where $n-t+1$ vertices are partitioned into p classes $A_1, \ldots, A_p$ as uniformly as possible and any two of these vertices are joined iff they belong to different classes. The remaining $t-1$ vertices have full degree $n-1$, i.e. they are joined to each other and to all the vertices in $\cup A_i$. Finally, $C_p[n]$ will denote the following "cyclical" graph: pn vertices are partitioned into p classes $A_1, \ldots, A_p$, of size n each, and all the vertices of A_i are joined to all the vertices of A_{i-1} and A_{i+1}, where the indices are counted mod p. (In some places we shall misuse the notation, writing $C_p[\frac{1}{p}m]$ even if m is not divisible by p and meaning that m vertices are distributed in p classes as uniformly as possible. Which are the smaller or larger classes we do not care.)

— * * * —

Since the publication of [**23, 24**], many results have been proved which would follow (at least for $n > n_0(\mathcal{L})$) almost immediately from the theorems of [**23, 24**]. Here I shall prove – among others – four theorems in connection with the Petersen graph, (Theorems 1.6, 1.11, Lemma 3.2, Theorem 4.2) the first two of which are immediate corollaries of the theory built up in [**23, 24**].

THEOREM 1.6 (Petersen Extremal). *For $n > n_0$ $H_{n,2,3}$ is the (only) extremal graph for the Petersen graph $\mathbb{P}_{10}$.*

This theorem immediately follows from Theorem 2.2 of [**23**]:

THEOREM 1.7 ($H_{n,p,t}$-theorem). *(i) Let $L_1, \dots, L_\lambda$ be given graphs with $\min \chi(L_i) = p+1$. Assume that omitting any $t-1$ vertices of any L_i we obtain a graph of chromatic number $\geq p+1$, but L_1 can be colored in $p+1$ colors so that the subgraph of L_1 spanned by the first two colors is the union of t independent edges and (perhaps) of some isolated vertices. Then, for $n > n_0(L_1, \dots, L_\lambda)$, $H_{n,p,t}$ is the (only) extremal graph.*

(ii) Further, there exists a constant $C > 0$ such that if G_n contains no $L_i \in \mathcal{L}$ and

$$e(G_n) > e(H_{n,p,t}) - \frac{n}{p} + C,$$

then one can delete $t-1$ vertices of G_n so that the remaining G_{n-t+1} is p–colorable.

This theorem is strongly connected with Theorem 1.5. It is natural to ask if the uniqueness holds here as well or not:

OPEN PROBLEM 1.8. *Is there a family $\mathcal{L}$ of forbidden graphs for which for $n > n_0$ $H(n,p,t)$ is extremal but it is not the unique extremal graph?*

REMARK 1.9. The condition on L_1 is equivalent to that $L_1 \subseteq T_{m,p,t}$ for some m. One could also formulate this by saying that the decomposition class contains the graph consisting of t independent edges.

The meaning of (ii) is that the extremal structure is stable in some sense. To understand this stability better, we introduce the notion of chromatic properties, first only in its simplest form.

DEFINITION 1.10 ($\mathcal{B}_{p,t}$–property). We shall say that a graph has property $\mathcal{B}_{p,t}$ if one cannot delete $t-1$ vertices from it to get a p–colorable graph.

We shall not distinguish a property of graphs from the set of graphs having this property. If a graph $G \notin \mathcal{B}_{p,t}$ and $H \subseteq G$, then $H \notin \mathcal{B}_{p,t}$ either. To have such a property means that the graph is "big" in some sense, to not have means, that it is "small".

The assumptions of Theorem 1.7 include that each forbidden $L_i \in \mathcal{B}_{p,t}$ but $H_{n,p,t} \notin \mathcal{B}_{p,t}$. So trivially $H_{n,p,t}$ contains no forbidden L_i's. The meaning of Theorem 1.7 is that this property dominates the above extremal problem in the sense that the extremal graph is that very G_n which

(∗) does not have the property $\mathcal{B}_{p,t}$, and
(∗∗) has the maximum number of edges in $\overline{\mathcal{B}_{p,t}}$.

Theorem 1.7 immediately implies many known results, e.g. a result of Erdős and Gallai, [**11**] on independent edges, generalized by Erdős to independent triangles, [**5**] and by J. W. Moon [**21**] to independent K_{p+1}'s. Moon's theorem asserts that if L is the disjoint union of t vertex-disjoint K_{p+1}, then $H_{n,p,t}$ is the extremal graph for $n > n_0(p,t)$. Theorem 1.7 can also be applied to the dodecahedron graph: for D_{20}, $H_{n,2,6}$ is the extremal graph for $n > n_0$. (The reader can see 6 edges on Figure 6 the deletion of which yields a bipartite graph. On the other hand, it is not too difficult to check that the deletion of any 5 points leaves at least one odd cycle unchanged.) Applying Theorem 1.7 to $\mathbb{P}_{10}$ we get

THEOREM 1.11 (Petersen, Stability). *There exists a constant $C > 0$ such that if*

$$e(G_n) > e(H_{n,2,3}) - \frac{1}{2}n + C,$$

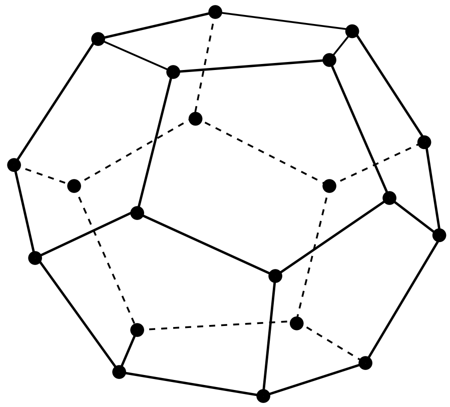

FIGURE 5:
Dodecahedron

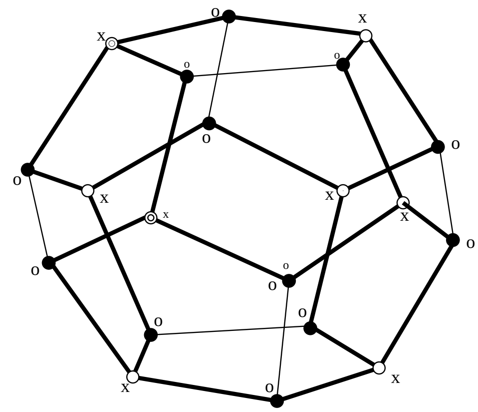

FIGURE 6:
Dodecahedron, 6 edges deleted, 2-colored

and $\mathbb{P}_{10} \not\subseteq G_n$, *then one can delete 2 vertices of* G_n *so that the resulting graph* G_{n-2} *be bipartite.*

Proof of Theorems 1.6 and 1.11 from Theorem 1.7. We show that $\mathbb{P}_{10}$ satisfies the conditions of Theorem 1.7 with $p = 2$ colors and $t = 3$ independent edges. Observe that

(a) the Petersen graph $\mathbb{P}_{10}$ is 3–chromatic (see Figure 1),

(b) one can color it in RED and BLUE so that the BLUE vertices are independent and the RED vertices span 3 independent edges: $\mathbb{P}_{10} \subseteq T_{n,2,3}$. (This can be seen on Figure 1 if we call colors 1 and 3 BLUE and 2 RED.

Further,

(c) deleting any 2 vertices of $\mathbb{P}_{10}$ we still have a 3-chromatic graph.

To prove property (c), observe that if we delete 2 vertices of $\mathbb{P}_{10}$, then (by symmetry) we may always assume that one of them is the A not seen on Figure 2. The remaining pentagons, $[BCDJI]$, $[CDEHG]$ and $[FGHIJ]$ cannot be represented by one vertex. ∎

Theorem 1.7/(ii) is sharp for $T_{n,p,t}$, but this does not prove e.g. that it is also sharp for the Petersen graph. In principle it could happen that, applying Theorem 1.7/(ii) the $-\frac{1}{2}n$ could be replaced by something much more negative. Construction 1.12 below shows that Theorem 1.11 is also sharp: Theorem 1.7 is always sharp when we apply it to a graph L not containing K_3.

CONSTRUCTION 1.12. *Let* $Z_{n,p,t}$ *be the graph obtained as follows. Take a Turán graph* $T_{n,p}$ *and fix* $t+1$ *vertices* $x_1, \dots, x_{t-2}$, y_1, y_2, y_3 *in its first class. Add the edges* y_1y_2, y_2y_3, *and* y_3y_1 *to the graph and also join each* x_i, $i = 1, \dots, t-1$ *to all the remaining vertices but to* y_1, y_2, y_3.

To prove the sharpness of Theorem 1.11 observe that

$$e(Z_{n,p,t}) = e(H_{n,p,t}) - \frac{n}{p} + O(1).$$

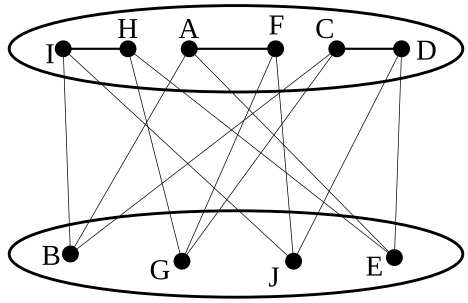

FIGURE 7:
Petersen embedded

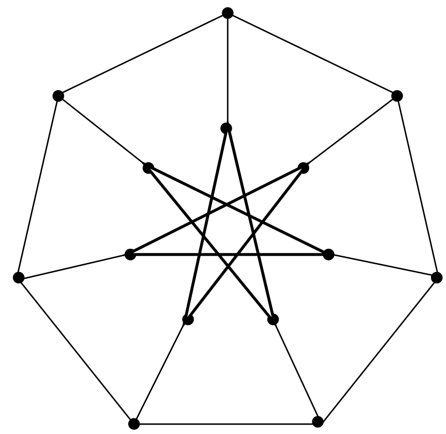

FIGURE 8:
Generalized Petersen Graph

Further, $Z_{n,p,t}$ has the following properties:

(a) if $K_3 \not\subseteq L \in \mathcal{B}_{p,t}$, then $L \not\subseteq Z_{n,p,t}$.

(b) $Z_{n,p,t} \in \mathcal{B}_{p,t}$.

Below we shall prove both (a) and (b) and they will obviously imply that Theorem 1.11 is sharp. To prove (a) observe that if $Z_{n,p,t}$ contained an L, then deleting e.g., y_1y_2 we would obtain a $Z_n^* \notin \mathcal{B}_{p,t}$ also containing L, but $L \in \mathcal{B}_{p,t}$, a contradiction. To prove (b) observe that to make $Z_{n,p,t}$ into a p-colorable graph by the deletion of some vertices one has to ruin *all* the $K_{p+1} \subseteq Z_{n,p,t}$. To achieve this, one has to delete at least 2 vertices of the triangle $y_1y_2y_3$ and all the vertices $x_1, \dots, x_{t-2}$, or at least cn other vertices. Indeed, each edge y_iy_j is contained in c_1n^{p-1} K_{p+1}'s and a vertex z forms at most $\binom{n}{p-2}$ K_{p+1} with e.g. y_1y_2. Therefore one has to delete either at least one of y_1, y_2, or at least c_2n other vertices. So we really have to delete at least two of $\{y_1, y_2, y_3\}$. Similarly, since each x_i is contained in c_3n^p copies of K_{p+1}, we have to delete all the x_i's as well, unless we are willing to delete c_4n further vertices. This completes the proof of the sharpness.

Properties of type $\mathcal{B}_{p,t}$ play fundamental role in some extremal graph problems (see e.g. [**23**]), above all, in cases where $H_{n,p,t}$ is the extremal graph. We can generalize the extremal graph problems, asking

PROBLEM 1.13. What is the maximum of $e(G_n)$ if G_n contains no $L \in \mathcal{L}$ and has property $\mathcal{B}_{p,t}$?

Assuming property $\mathcal{B}_{p,t}$ mostly decreases the extremal number: the extra condition *mostly* rules out $H_{n,p,t}$ and *mostly* in these cases there are no other extremal graphs for $\mathcal{L}$. However, this never changes the extremal number much, since we can always change the edges of a $H_{n,p,t}$ by rearranging and deleting $O(n)$ edges and increasing the chromatic number arbitrary high. The situation completely changes if we ask

PROBLEM 1.14. What is the maximum of the **minimum degree** a graph G_n can have if it contains no $L \in \mathcal{L}$ and has property $\mathcal{B}_{p,t}$?

We shall describe this situation in Section 4.

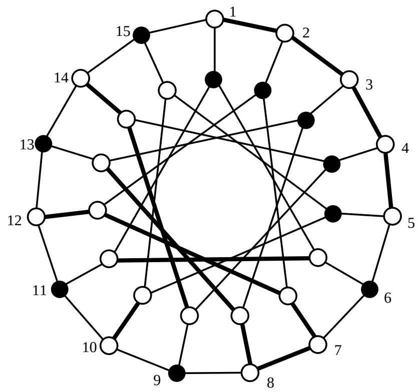

FIGURE 9:
Generalized Petersen,
$k = 15, \ell = 5$

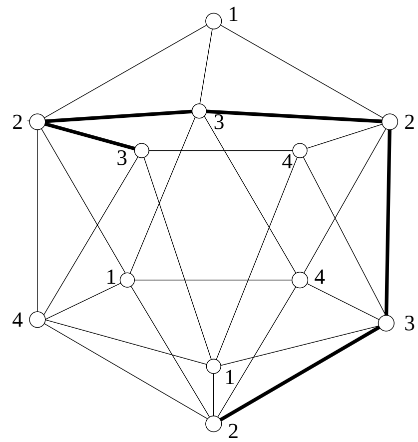

FIGURE 10:
Icosahedron and P_6 in it

Above property $\mathcal{B}_{3,3}$ played an important role in our arguments, below we will go one step further to define the so called **Chromatic properties** or **Chromatic conditions**.

DEFINITION 1.15 (Symmetrical subgraphs). Let T_1 and T_2 be connected subgraphs of G. They are called symmetrical in G if either $T_1 = T_2$ or
(i) $V(T_1) \cap V(T_2) = \emptyset$, and
(ii) there are no edges joining T_1 to T_2, and
(iii) there exists an isomorphism $\omega : T_1 \to T_2$ such that for every $x \in T_1$, $u \in G - T_1 - T_2$, x is joined to u if and only if $\omega(x)$ is joined to u.
$T_1, \dots, T_k$ are symmetric if for every $1 \le i < j \le k$, T_i and T_j are symmetric.

The connectedness is assumed to rule out some trivial cases. We shall use this definition to state that under some conditions there are sequences of symmetrical extremal graphs: assuming the connectedness in this definition we get stronger statements.

DEFINITION 1.16 (Chromatic condition). A property $\mathcal{A}$ of graphs will be called chromatic condition if
(i) $G \in \mathcal{A}$ and $H \supset G$ implies $H \in \mathcal{A}$.
(ii) If $\rho = \rho(\mathcal{A})$ is a sufficiently large integer, then the following holds: if $T_1, \dots T_\rho$ are symmetric subgraphs of an $\mathcal{A}$-graph G, then $G - T_\rho$ is also an $\mathcal{A}$-graph.
(iii) there are graphs of property $\mathcal{A}$ and of arbitrary high girth.

Here (iii) is assumed only to rule out the uninteresting cases. Condition (ii) may seem to be artificial, however, this is the main point in the definition. It means that certain p-colorings of a part of a graph can automatically be extended to the whole graph, using the symmetry. E.g. if $\mathcal{A}$ is the family of at least q-chromatic graphs, then $\rho = 2$ can be taken.

EXAMPLE 1.17.

- The property $\mathcal{B}_{p,t}$ that one cannot delete t vertices of G to get a p–colorable graph is one of the typical chromatic conditions. (Here $\rho = t + 2$ will do.)
- The property $\mathcal{B}_{p,t,h,\Delta}$ that one cannot delete $t \geq h$ vertices of G, at least h of which are of degree at most Δ, to get a p–colorable graph, is a more complicated one, but still a chromatic property.

Given a chromatic condition $\mathcal{A}$, we may generalize our previous problem:

PROBLEM 1.18. What is the maximum number of edges a graph $G_n \in \mathcal{A}$ can have if it contains no $L \in \mathcal{L}$.?

Here mostly Definition 1.15(iii) ensures the existence of such graphs.

DEFINITION 1.19. Given a family $\mathcal{L}$ of graphs and a chromatic property $\mathcal{A}$, we call G_n $(\mathcal{L}, \mathcal{A})$–extremal if $G_n \in \mathcal{A}$, it contains no $L \in \mathcal{L}$ and has maximum number of edges under these conditions.

2. EXTREMAL GRAPH PROBLEMS WITH LINEAR DECOMPOSITION

In [**23**] I have given a fairly general theorem from which one can "almost algorithmically" solve many involved extremal graph problems. (The expression "almost algorithmically" will be explained in Remarks 2.3-2.4.)

As it is mentioned above, there I formulated a theorem (Theorem 2.7) which covered the case of the Dodecahedron graph and would have covered the case of the Petersen graph as well, assumed I would have considered it. To illustrate the applicability of Theorem 2.7, below we shall consider two different generalizations of the Petersen graph. The first one is the Kneser graph, the second one the cyclic generalization.

2.1. The Kneser graph. The Petersen graph is a special case of the Kneser graph.

DEFINITION 2.1. Given a set A of a elements and an integer $b < a/2$, define the Kneser graph $Z(a, b)$ as a graph on $\binom{a}{b}$ vertices, where the vertices are the b–tuples of A and two of these b-tuples are joined iff they are disjoint.

It is known and easy to see that $\mathbb{P}_{10} = Z(5, 2)$.

Obviously, the Kneser graphs $Z(a, b)$ can be colored in $p = a - 2b + 2$ colors as follows. For $a - 2b$ fixed elements $x_1, \dots, x_{a-2b}$ of A we fix $a - 2b$ colors, say $\psi_1, \dots, \psi_{a-2b}$. Whenever a b-tuple B contains x_j (a fixed element), color the corresponding vertex B of $Z(a, b)$ by ψ_j. If there are more than one such x_j's, use any of these colors. Now the vertices of $Z(a, b)$ not colored as yet correspond to the b-tuples of the $2b$–element set $M = A - \{x_1, \dots, x_{a-2b}\}$. Therefore they span a 1-factor of $Z(a, b)$ which can be colored with two further colors. This shows that

- $\chi(Z(a, b)) \leq a - 2b + 2$;
- One can color $Z(a, b)$ in $a - b + 1$ colors so that the first color-class spans a 1-factor, the others form independent sets of vertices.

It was a longstanding conjecture of Kneser, a deep theorem of Lovász [**20**] (proved slightly later in a simpler way by Bárány [**2**]) that the above upper bound is sharp:

$$\chi(Z(a, b)) = a - 2b + 2.$$

The next theorem is "kind of a generalization" of Theorem 1.7 and makes the solution of the extremal graph problem of any Kneser graph "almost" algorithmic.

THEOREM 2.2. *Let $\mathcal{L}$ be finite, $p(\mathcal{L}) = p$ but for some $L \in \mathcal{L}$*

$$L \subseteq T_{2pv,p,v} \text{ for } v := v(L), \tag{2.1}$$

then there exists an $n_0(\mathcal{L})$ such that, for $n > n_0$, there is an extremal graph U_n for $\mathcal{L}$, for which

(i) one can delete v vertices of U_n to get a $T_{n-v,p}$.

(ii) Furthermore, if $v := \max_{L\in\mathcal{L}} v(L)$, then one can also delete $m < v^2 \cdot 2^v$ vertices of U_n to get a $T_{n-m,p}$ where all the vertices of $x \in U_n - T_{n-m,p}$ are either joined to all the vertices of $T_{n-m,p}$ or to $p-1$ classes of $T_{n-m,p}$ completely but not at all to the remaining one. (This exceptional class may depend on x.)

REMARK 2.3. This is indeed an **algorithmic solution** of the extremal graph problem where $Z(a, b)$ is the excluded graph, or more generally, of any extremal problem of any finite $\mathcal{L}$ satisfying the conditions of Theorem 2.2. Indeed, in Theorem 2.2 the sequences (U_n) are completely given by the graphs $S_m := U_n - T_{n-m,p}$ and by the patterns according to which the vertices of S_m are joined to $T_{n-m,p}$. So we have to check only finitely many graph sequences (U_n), for finitely many S_m and finitely many ways of joining S_m to $T_{n-m,p}$ and decide if a graph $L_i \in \mathcal{L}$ is contained in U_n for large n, (i.e., the sequence is "good") or not (it is "bad"), Then we have to choose a "good" graph sequence (U_n) with the maximum number of edges: given two graph sequences, one can easily decide, which has more edges, though in principle this may depend on divisibility properties of n as well. Here we should be **careful** with the **infinite** families $\mathcal{L}$. If $\mathcal{L}$ is infinite, then we have to assume that there exists an **oracle** which tells us, if a sequence of graphs above is good or not ...

REMARK 2.4. The earlier results were weaker in the sense that they have not provided explicit (or implicit) upper bounds on the number of vertices to be deleted in Theorem 2.2(ii) and therefore having found the extremal graph one could not be sure if that is the extremal graph. (Those years we did not care so much about the algorithmic aspects.)

One could hope that the Kneser graphs satisfy the conditions of Theorem 1.7 and then one would get a much nicer looking result: $H_{n,p,s}$ would be the extremal graphs for $p = a - 2b + 1$ and $s := \frac{1}{2}\binom{2b}{b}$. However, generally this is not the case. As Z. Füredi pointed me out, sometimes one can delete much fewer vertices. The colorings of the Kneser graph can be much more complicated than the above one. A paper of P. Frankl and Z. Füredi [**17**] (disproving a conjecture of Erdős) gives a lot of information on this. Among others it contains the following construction. Let us consider e.g $Z(2b + 1, b)$. This is 3-chromatic. The above calculated $s = \frac{1}{2}\binom{2b}{b}$. If we split the base set into

$$X_1 \cup X_2 \text{ with } |X_1| = b + 1 \text{ and } |X_2| = b,$$

and color by RED those b-tuples which intersect X_1 in more than $\frac{1}{2}|X_1|$ elements, and color by BLUE those b-tuples which intersect X_2 in more than $\frac{1}{2}|X_2|$ elements, then the number of badly colored b-tuples is $\approx \frac{c}{\sqrt{b}}\binom{2b}{b} = o(s)$.

OPEN PROBLEM 2.5 (Kneser Coloring). *Which is the minimum $p(a, b)$ such that one can delete p vertices of the Kneser graph to get an $a - 2b + 1$-chromatic graph? And for edges?*

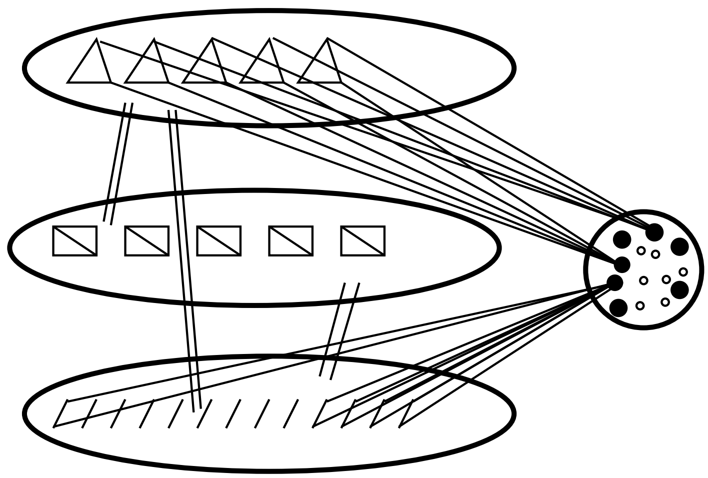

FIGURE 11:
Sequences of symmetrical graphs

Now we turn to the formulation of the general (old) theorem from which all *known* (?) Turán type extremal graph theorems with linear decomposition follow.

DEFINITION 2.6. (Family of symmetrical graphs) $\mathbb{D}(n,p,r)$ is the class of graphs G_n satisfying the following symmetry condition:

(i) It is possible to omit $\le r$ vertices of G_n so that the remaining graph G^* is a product (of graphs of almost equal order):

$$G^* = \prod_{\ell \le p} G^{m_\ell} \qquad \text{where} \qquad \left| m_\ell - \frac{n}{p} \right| \le r.$$

(ii) For every $\ell \le p$, there exist connected graphs $H_{p,j} \subseteq G^{m_\ell}$ and isomorphisms

$$\omega_{\ell,j} \ : \ H_{\ell,1} \to H_{\ell,j}$$

such that $H_{\ell,j}$ $(j = 1, \dots, k_\ell)$ are symmetric subgraphs of G_n, $v(H_{\ell,j}) \le r$ and

$$G^{m_\ell} = \sum_{j \le k_\ell} H_{\ell,j},$$

where the sum $\sum H_{\ell,j}$ is the vertex-disjoint union.

The graphs $H_{\ell,j}$ will be called the “blocks”, the vertices in $G_n - G^*$ will be called “exceptional”.

THEOREM 2.7 (Symmetrical extremal graphs, [**23**]). *Assume that a finite family $\mathcal{L}$ of forbidden graphs with $p = p(\mathcal{L})$, and a chromatic condition $\mathcal{A}$ are given. If for some $L \in \mathcal{L}$ and $v := v(L)$,*

$$L \subseteq P^v \otimes K_{p-1}(v, v, \dots, v), \tag{2.2}$$

then there exists a constant $r = r(L)$ such that, for every n, $\mathbb{D}(n,p,r)$ contains an extremal graph for $(\mathcal{L}, \mathcal{A})$. Furthermore, if for every $R > r$ there exists an n_R such that for $n > n_R$ $\mathbb{D}(n,p,R)$ contains only one extremal graph, then for sufficiently large n this is the only extremal graph. (Of course, the families $\mathbb{D}(n,p,R)$ form nested sequences, so this common graph sequence is the same.)

Now we turn to the proof of Theorem 2.2. The proof will be cut into two parts, the first being formulated in the following theorem.

THEOREM 2.8. *Assume that a finite family $\mathcal{L}$ of forbidden graphs with $p = p(\mathcal{L})$, and a chromatic condition $\mathcal{A}$ are given. If some $L \in \mathcal{L}$ is contained in a $T_{2pv,p,v}$, then – applying Theorem 2.7 –*

(i) all the blocks of $\mathbb{D}(n,p,r)$ will consist of isolated vertices: the product graph G^* will be a Turán graph $T_{n^*,p}$.

(ii) If there is no chromatic condition involved and s is the maximum integer for which $K_{p+1}(s,m,m,\dots,m)$ contains no forbidden $L \in \mathcal{L}$ (even if m is very large), then

$$\text{ext}(n,\mathcal{L}) = \text{ext}(n,K_{p+1}) + \frac{s}{p}n + O(1). \tag{2.3}$$

(iii) Further, – still assuming the absence of the chromatic condition [3] *– each vertex x of the "exceptional set" $U_n - G^*$ is joined either to all the vertices of this Turán graph or to all the vertices of $p-1$ classes and to none of the remaining class (but the remaining class may depend on x).*

Proof. (a) Let I^s denote the set of s independent vertices and $H^*_{n,p,s+1} = I^s \otimes T_{n-s,p}$. By the definition of s, $H^*_{n,p,s+1}$ contains no forbidden subgraphs:

$$e(H^*_{n,p,s+1}) \le \text{ext}(n,\mathcal{L}).$$

This proves the lower bound in (2.3).

(b) We apply Theorem 2.7. Let U_n be extremal for $\mathcal{L}$ as described in Theorem 1.7. If a block contained an edge, then v blocks of that class would provide v independent edges, so U_n would contain a $T_{2pv,p,v}$, which was excluded. So the blocks are isolated vertices, the graph $U_n - S_m$ is a Turán graph. The second part of (i) follows from the definition of the Symmetrical Graph Sequences. Observe now that if there are ζ exceptional vertices joined to all the classes of this Turán graph and η vertices joined to fewer than $p-1$ classes, then

$$e(U_n) \le e(T_{n.p}) + \frac{\zeta - \eta}{p}n + O(1). \tag{2.4}$$

The number of "exceptional vertices" joined to all the classes of $T_{n^*,p}$ is at most s, by the definition of s. This proves the upper bound in (2.3) and also that the number of these vertices is *exactly* s. The only thing to be checked is that the "exceptional vertices" cannot be joined to fewer than $p-1$ classes. If we had an exceptional vertex y joined to at most $p-2$ classes, then

$$e(U_n) < \text{ext}(n,K_{p+1}) + \frac{s-1}{p}n + O(1) < e(H^*_{n,p,s+1})$$

would follow, a contradiction. ∎

The argument of (b) is not necessarily true if we extend the statement to extremal problems with chromatic conditions!

Proof of Theorem 2.2. Let U_n be an extremal graph for $\mathcal{L}$ described in Theorem 2.8. The classes of the Turán graph $T_{n^*,p}$ will be denoted by $A_1,\dots,A_p$. We classify the vertices of $U_n - T_{n^*,p}$ as follows. W is formed by those s vertices which are joined to $T_{n^*,p}$ completely. $D_i \subseteq U_n - T_{n^*,p} - W$ is the class of vertices joined to $T_{n^*,p} - A_i$ completely, but not to A_i, $i = 1,\dots,p$. By the assumption

[3] Or we could say that the chromatic condition contains all graphs.

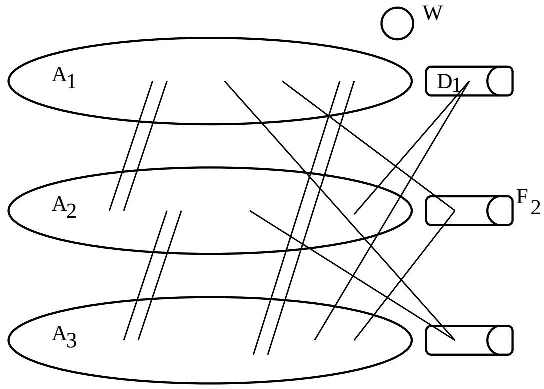

FIGURE 12:
The vertex-partition

on $\mathcal{L}$, $T_{2pv,p,t} \not\subset U_n$ and so there cannot be t independent edges between A_i and $W \cup D_i$. This and $t \le [v/(2p)]$ imply (i): the edges in $A_i \cup D_i$ can be represented by $2(t - |W|)$ vertices. So it is enough to delete $2p(t - |W|) + |W| \le v$ vertices to get a Turán graph.

To prove (ii) first we define the "horizontal"[4] and the "missing" edges. The edge xy is "horizontal" if x, y belong to the same $A_i \cup D_i$. A *non-edge* is a "missing edge" if x and y belong to different $A_i \cup D_i$'s. Let us fix in each D_i a maximal set of independent edges. The endvertices of these edges form the sets S_i. Put $F_i = D_i - S_i$. The vertices in F_i can be classified according to their connection to $(\cup S_i) \cup W$. There are at most 2^v classes. We show that each $x \in F_i$ is connected to all but at most v vertices of $V(U_n) - A_i - D_i$. (In other words, the "missing"-degree of x is at most v.) Indeed, if there were at least $v+1$ "missing edges" incident to x, then adding these edges to U_n and deleting all the edges joining x to $(\cup S_i) \cup W$ we would get a U_n^* not containing forbidden subgraphs but having more than $e(U_n)$ edges, a contradiction.

Now, if there is a class $Q \subseteq F_i$ with $|Q| \ge v^2$ (in the above classification), then all the vertices $x \in Q$ are joined to all the vertices of $\cup_{i \ne j} F_j$. Otherwise there is a "missing edge" xy and we add xy to the graph U_n, getting a forbidden subgraph $L \subseteq U_n + xy$ and then we could replace x by an $x' \in Q$ joined to all the vertices in $N(x)$ and to y too: otherwise the "missing edge"-degree of y would be too large. Thus $L \subseteq U_n - x$, a contradiction again. This means that the vertices of Q are symmetric to each other.

Now, if we "increase" the size of Q by h (where h can be negative and positive as well), at the cost of changing the size of A_i in the other direction, then we get a family of graphs $U_{n,h}$ and $e(U_{n,h})$ will be a linear function of h. Since $U_{n,0}$ is extremal, and neither $U_{n,1}$, nor $U_{n,-1}$ contains forbidden subgraphs, $e(U_{n,h})$ must be constant. So we can replace U_n by $U_{n,-|Q|}$. This means that we can eliminate Q getting an other extremal graph. We can eliminate all the classes of size at least v^2. This completes our proof. ∎

[4]The "horizontal" edges will be used only later.

2.2. Cyclic generalization of the Petersen graphs.

DEFINITION 2.9. A generalized Petersen graph $\mathbb{P}_{k,\ell}$ is a 3-regular graph defined as follows: $x_1, \ldots, x_k$ are the "outer" vertices, $y_1, \ldots, y_\ell$ are the "inner" vertices, x_i is joined to y_i and $x_{i\pm 1}$, y_i is joined to x_i and $x_{i\pm\ell}$, (where the indices are counted mod k).

Observe that generally the "inner" and "outer" vertices behave differently, they play the same role only if $(k, \ell) = 1$: otherwise inside we have many disjoint cycles.

THEOREM 2.10. *A generalized cyclic Petersen graph $\mathbb{P}_{k,\ell}$ can be colored in RED and BLUE so that the RED vertices are independent and the BLUE vertices can be covered by a path.*

This implies that our theorems can be applied to $\mathbb{P}_{k,\ell}$ as well.

Proof of Theorem 2.10. There are many explicit colorings but here we shall be concise. We shall return to the detailed discussion of this topic in [**27**]. The cases $\ell = 1, 2$ can be treated separately and we skip them. ($\ell = 1$ is trivial.)

(a) Let us start with an observation. If we color the vertices of a 3-regular graph in RED and BLUE so that the RED vertices are independent and the BLUE ones form a forest, then we are home: any BLUE vertex having 3 BLUE neighbors can (one by one) be changed into RED. So we get the desired coloring: the RED vertices remain independent and the BLUE ones will have only BLUE degrees at most 2. Further, no BLUE cycle can arise. So the graph spanned by the BLUE vertices will be the union of paths.

(b) This gives an easy solution for ℓ even: color $y_1, \ldots, y_\ell$ by RED, $x_1, \ldots, x_\ell$ by BLUE, and then color the remaining y_i's by BLUE, the arc $x_{\ell+1}, x_{\ell+2}, \ldots, x_k$ in RED-BLUE-RED- ... -RED. (This arc has now odd length.) We do not verify this simpler case, rather the next, slightly more involved one.

(c) If ℓ is odd, we may slightly change the above coloring: the outside path $x_{\ell+1}, x_{\ell+2}, \ldots, x_k$ can be colored by a

RED-BLUE-BLUE-RED-BLUE-RED-BLUE ... -RED-BLUE-RED.

One can easily see that this works. (a) the RED points are independent. (b) One has to show that there is no cycle with all blue vertices. $y_1, \ldots, y_\ell$ are RED and the BLUE path $x_1, \ldots, x_\ell$ is separated by RED vertices from the remaining parts: so we may forget it. The only thing to be checked is that no BLUE cycle occurs (where a BLUE cycle means that all its vertices are BLUE). Indeed, if we delete the inside RED vertices, then the cycles inside are ruined. Call an edge BLUE if it joins BLUE vertices. There are no BLUE edges outside,except $(x_{\ell+2}, x_{\ell+3}$, see Figure 5(a). and iw there were a BLUE cycle in the graph and we would walk around it, then arriving at any outside vertex but these two we are stuck. On the other hand, if we use the edge $(x_{\ell+2}, x_{\ell+3}$, then we otherwise must stay completely inside. Distinguishing the two cases $(k, \ell) = 1$ or not, we may easily cope with this case. ∎

CONJECTURE 2.11. *If L is a 3-regular graph of girth at least 5, then it can be two-colored in RED and BLUE so that all the RED vertices are independent and the BLUE vertices form a subgraph which can be covered by a path.*

(Maybe, this conjecture is not too difficult.)

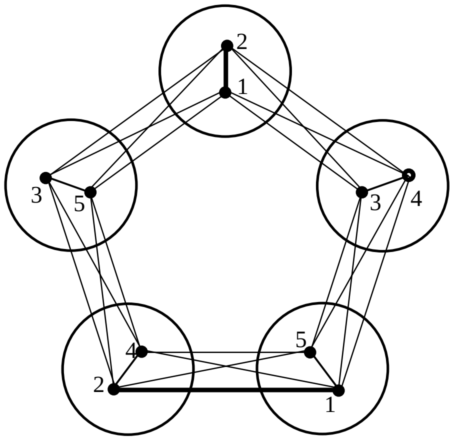

FIGURE 13:
The Łuczak graph

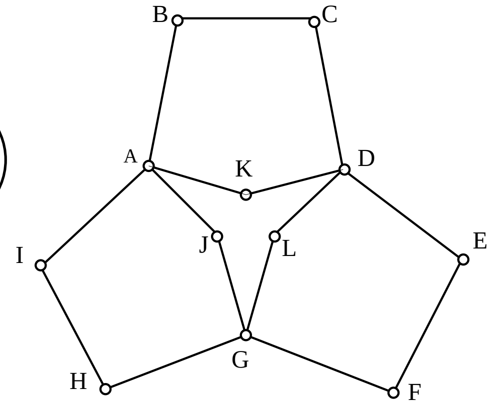

FIGURE 14:
Nešetřil Graph

2.3. The case of the Łuczak graph. To show the power of a method one can select certain cases "at random" and show that the method works in those cases. Below I shall discuss the case of two such graphs. The first one, L_{10} was used (as a counterexample for some graph entropy question) by Tomasz Łuczak, and is given in Figure 13. So I shall call it the Łuczak graph. The other graph, N_{12}, was used in a lecture by Jarik Nešetřil and below will be called Nešetřil graph. (These two graphs and the corresponding results can be generalized and we shall return to this in another paper [**27**]

PROBLEM 2.12. Determine $\text{ext}(n, L_{10})$. What are the extremal graphs?

This is really easy: One can see that L_{10} is 5-chromatic and that removing any vertex of L_{10} it remains 5-chromatic, but one can color it in 5 colors so that the first two colors span 2 independent edges (see Figure 13). Hence we can apply Theorem 1.7:

THEOREM 2.13. *For L_{10}, $H_{n,4,2}$ is the (only) extremal graph, for $n > n_0(L_{10})$.*

2.4. The Nešetřil graph. The Nešetřil graph consists of 3 pentagons $C_5(a_i, b_i, c_i, d_i, e_i)$ where the pentagons are glued together at the b_i's and e_i's cyclically: $b_1 = e_2$, $b_2 = e_3$, $b_3 = e_1$. Denote this graph by N_{12}. Relabel them as seen in Figure 14.

THEOREM 2.14. *There exists an $n_0(N_{12})$ such that for $n > n_0$, $H_{n,2,2}$ is the (only) extremal graph.*

REMARK 2.15. Theorem 2.14 does not follow from Theorem 1.7. Below we shall use the labeling of Figure 14.

(a) One can delete 3 independent edges, e.g. BC, EF, and IH to get a bipartite graph from N_{12} and the omission of 2 edges is obviously not enough.

(b) We could apply Theorem 1.7 if we could show that the omission of any 2 vertices leaves us with a 3-chromatic graph. The extremal graph would be $H_{n,3,2}$. However, this is not the case: the omission of A and L results a tree.

Proof of Theorem 2.14.

(a) $N_{12} \not\subset H_{n,2,2}$, since $N_{12} \in \mathcal{B}_{2,1}$ but $H_{n,2,2} \notin \mathcal{B}_{2,1}$.

(b) Apply Theorems 2.7 and 2.8 to an extremal graph U_n: there is a $T_{n^*,2} \subseteq U_n$ with $n^* = n - O(1)$. All the blocks are 1-vertex blocks and there can be at most one "exceptional" vertex $u \in S_n - T_{n^*,2}$ joined to $p = 2$ classes, otherwise, by Remark 2.15(b), $N_{12} \subseteq U_n$. (As a matter of fact, a $H_{\nu,2,3} \subseteq U_n$ for some $\nu \approx n/2$. So all the fixed graphs $L \notin \mathcal{B}_{2,2}$ can be built up in U_n, for $n > n_0(L)$.) If there is no exceptional vertex joined to both classes of $T_{n^*,2}$, then

$$e(U_n) \le e(T_{n,2}) + O(1) < e(H_{n,2,2}).$$

So there is exactly one vertex u joined to both classes. All the other "exceptional" vertices are joined to a class of $T_{n^*,2}$ completely, and to the other not at all. If there is a vertex $v \in S_n - T_{n^*,2}$ joined to (say) the second class of $T_{n^*,2}$ completely and to at least one vertex of the first class, then we can again find an $N_{12} \subseteq U_n$ (because N_{12} can be turned into a tree by deleting a vertex of degree 4 and an edge from the "opposite" pentagon. Now we may partition $V(U_n) - u$ into two classes according to whether they are joined to the first class of $T_{n^*,2}$ or the second one and the above observation yields that the vertices in these classes are independent. So $U_n \in \mathcal{B}_{2,1}$ and therefore either $e(U_n) < e(H_{n,2,2})$ or $U_n = H_{n,2,2}$. ∎

3. ZARANKIEWICZ TYPE PROBLEMS

Now we would like to answer the following question.

How large minimum degree ensures for a graph G_n a Petersen graph $\mathbb{P}_{10} \subseteq G_n$?

These type of questions are called Zarankiewicz type extremal problems.

PROBLEM 3.1 (Zarankiewicz type). Given a family $\mathcal{L}$ of forbidden graphs, what is the maximum of the minimum degree $\delta(G_n)$ a graph G_n of order n can have without containing any $L \in \mathcal{L}$?

Let us denote the maximum in such a problem by $\operatorname{dex}(n, \mathcal{L})$, assumed that $\mathcal{L}$ is a family of excluded subgraphs. It is obvious that

$$\operatorname{dex}(n, \mathcal{L}) \le \frac{2}{n} \cdot \operatorname{ext}(n, \mathcal{L}), \tag{3.1}$$

since any graph having

$$\delta(G_n) > \frac{2}{n} \cdot \operatorname{ext}(n, \mathcal{L})$$

has more than $\operatorname{ext}(n, \mathcal{L})$ edges and therefore must contain some of $L \in \mathcal{L}$.

On the other hand, in many cases some extremal graphs for the Turán type problem are almost regular. Then the difference between the Turán and Zarankiewicz type problems is negligible. If e.g. there exists an extremal graph sequence (Z_n) for the Turán problem in which the maximum degree and the minimum degree differ only by (at most) one, then

$$\operatorname{dex}(n, \mathcal{L}) = \left\lceil \frac{2}{n} \cdot \operatorname{ext}(n, \mathcal{L}) \right\rceil \tag{3.2}$$

In cases discussed by us the graph sequence $(H_{n,p,t})$ provided the extremal sequence, which (for $t > 1$) is far from being almost regular. Yet its minimum degree and average degree are near to each other. So

LEMMA 3.2. *If for $\mathcal{L}$ the extremal graph sequence is $(H_{n,p,t})$, then the solution of the Zarankiewicz problem is between*

$$\left(1-\frac{1}{p}\right)n \text{ and } \left(1-\frac{1}{p}\right)n+\frac{t-1}{p}.$$

This clarifies the situation in a weak asymptotic sense but the detailed discussion is more involved and postponed.

4. ANDRÁSFAI-ERDŐS-SÓS TYPE PROBLEMS

A completely new phenomenon occurs when we combine the Zarankiewicz type problems with chromatic condition type problems. Perhaps Andrásfai was the first to ask such questions:

PROBLEM 4.1 (Andrásfai). Determine the maximum of the minimum degree in a graph G_n under the condition

$$K_{p+1} \not\subseteq G_n \quad \text{and } \chi(G_n) > k.$$

For $k = p+1$ this was solved by B. Andrásfai, P. Erdős and Vera T. Sós, [**1**]. In the simplest case, when we assume that G_n is triangle-free and non-bipartite, the pentagon like graph $H_n := C_5[n/5]$ shows that the minimum degree can be $\frac{2}{5}n - O(1)$. For the general case of K_{p+1} and $k = p+1$ the extremal graph is $C_5[m/5] \otimes T_{n-m,p-2}$, where m is chosen so that the resulting graph be approximately regular. (This is obtained when $m = \frac{5n}{3p-1} + O(1)$.) P. Erdős and the author, [**15**] extended this result for the case where an arbitrary $p+1$-chromatic graph L with some color-critical edges is excluded. Of course, this does not cover the case of $\mathbb{P}_{10}$. To describe the case for the Petersen graph, we shall prove

THEOREM 4.2. *For every v (and $t \le v/2$) there exists a $K = K(v)$ such that if*

$$\delta(G_n) > \frac{2}{5}n + K$$

and $T_{v,2,t} \not\subseteq G_n$, then one can delete K vertices of G_n to get a bipartite graph.

REMARK 4.3.

(a) Theorem 4.2 is sharp, as shown by $C_5[\frac{1}{5}n]$. Clearly, $\delta(C_5[\frac{1}{5}n]) \ge \frac{2}{5}n - 2$ and $T_{v,2,t} \not\subseteq C_5[\frac{1}{5}n]$. Further, replacing $T_{v,2,t}$ by any graph $L \subseteq T_{v,2,t}$ we get the same sharpness if $K_3 \subseteq L$, since $C_5[\frac{1}{5}n]$ contains no K_3.

(b) Similarly, the theorem is sharp e.g. for $\mathbb{P}_{10}$: if one tries to embed $\mathbb{P}_{10}$ into $C_5[\frac{1}{5}n]$, then all the pentagons of $\mathbb{P}_{10}$ must "run around" in $C_5[\frac{1}{5}n]$. Therefore we may assume that (using the notations of Figure 2) $A \in A_1$, $B \in A_2$, $C \in A_3$, $D \in A_4$, $E \in A_5$. By symmetry we may assume that $F \in A_2$. Now we cannot put G anywhere: it must be in an A_i neighboring both A_3 and A_2, which is impossible.

(c) The theorem is *not sharp* if $\chi(L) = 3$ and $L \subseteq C_5[\mu]$ for some μ.

There are many deep, unsolved problems in this field but now we skip them. To prove Theorem 4.2 we shall use two results from extremal graph theory.

THEOREM 4.4 (Theorem 9 of [**4**]). *Let t be a natural number and let $c > 0$. Put $\ell(c) = \lceil \frac{c^{-1}-1}{2} \rceil$. Then there exists an $n_0 = n_0(t,c)$ such that if $n > n_0$ and $G_n \not\supseteq C_m[t]$ for $m = 3, 5, \dots, 2\ell(c)+1$, then G_n can be made bipartite by the omission of at most cn^2 edges.*

REMARK 4.5. For improved versions of the above theorem see a very recent result of Komlós, [**19**]. His theorem provides a much smaller $\ell(c)$, approximately the best one.

The other result we need is the following lemma.

LEMMA 4.6 (Lemma 4 of [**22**], p315). *Let M be a given positive integer and $c > 0$ be an arbitrary constant. Then there exist an M' and a $c' > 0$ such that if a set A of n elements contains M' subsets A_i each of which has at least cn elements, then there are M subsets*

$$A_{i_1}, \dots, A_{i_M}$$

among them whose intersection has more than $c'n$ elements.

(Here M' is typically much bigger than M, c' much smaller than c. The lemma is contained in a lemma of Erdős, [**7**].)

Proof of Theorem 4.2. First we fix three constants:

$$\gamma := \frac{1}{100}, \qquad \eta := \gamma^2, \qquad \varepsilon := \gamma^4, \qquad \mu = \frac{100v^2}{\varepsilon}.$$

We shall consider a G_n not containing $T_{v,2,t}$. Our proof will have two main parts:

(A) when one can delete $< \varepsilon n^2$ edges from G_n to turn it into a bipartite graph and

(B) when one cannot do this.

In the first case we shall show that we can delete $O(1)$ vertices to make G_n bipartite. In the second one we shall apply Theorem 4.4 to get a $C_{2\ell+1}[M]$ for some not too large ℓ and very large M and then we shall deduce in (C) that $G_n \supseteq C_5[\mu]$. From this we shall deduce that the minimum degree is $< \frac{2}{5}n + K$.

(A) So first we assume that G_n can be turned into a bipartite graph by deleting $< \varepsilon n^2$ edges. Let us choose the vertex-partitioning $A \cup B = V(G_n)$ for which $e(A) + e(B)$ is the minimum possible. We shall prove that A contains $O(1)$ vertices so that deleting them we ruin all the edges in A. The same will hold for B. This will prove that we can omit $O(1)$ vertices to get a bipartite graph, as stated in the theorem.

(A1) By Lemma 4.6, there exists a constant h such that if $x_1y_1, \dots, x_hy_h$ are h independent edges in A and $U_i = N(x_i) \cap N(y_i) \cap B$, $|U_i| > \varepsilon n$ for $i = 1, \dots, h$, then we may choose v of these edges so that $|\cap U_i| > v$. Therefore $T_{2v,2,v} \subseteq G_n$ a contradiction.

(A2) Below we shall call *horizontal degree* the number of edges to the same class: For a given $x \in A$ we shall call $\alpha(x) := |N(x) \cap A|$ the horizontal degree and $\beta(x) := |N(x) \cap B|$. To show that

$$\frac{2}{5}n - 2\varepsilon n \le |A|, |B| \le \frac{3}{5}n + 2\varepsilon n, \tag{4.1}$$

first we assume that $|A| \ge |B|$. By the assumption that A and B contain only $\le \varepsilon n^2$ edges, we get that the average of $\alpha(x)$ in A is at most $2\varepsilon n$. Therefore A contains an x for which $\beta(x) = d(x) - \alpha(x) \ge \frac{2}{5}n - 2\varepsilon n$:

$$|A| \ge |B| \ge \beta(x) \ge \frac{2}{5}n - 2\varepsilon n,$$

proving (4.1).

(A3) From now on $|A| \ge |B|$ will not be assumed. If $A' \subseteq A$ is the set of vertices x having horizontal degree $\alpha(x) < \eta n$, then A' contains at most h independent edges: each edge xy has endvertices joined to at least $\frac{2}{5}n - \eta n$ vertices from B. Therefore x and y have at least

$$\beta(x) + \beta(y) - |B| \geq \frac{4}{5}n - \frac{3}{5}n - 2\varepsilon n - 2\eta n > \frac{1}{10}n$$

common neighbors in B. So we may apply Lemma 4.6, as described in (A1) to see that the number of independent edges in A' is at most h, therefore they can be represented by at most $2h$ vertices.

(A4) Similarly, $|A - A'| < h$. Indeed, if $A'' := A - A'$ contains T points, [5] then we can select T independent edges $x_i y_i$ with

$$x_i \in A'', \qquad y_i \in A', \qquad \alpha(y_i) < 3\eta n.$$

Now, by the minimality condition on the partition,

$$\beta(x) \geq \alpha(x) \text{ and } \beta(x) \geq \frac{1}{2}d(x) \geq \frac{1}{5}n.$$

(A4/(i)) If $\beta(x_i) > \frac{1}{5}n + 10\eta n$, then

$$|N(x_i) \cap N(y_i) \cap B| > \frac{1}{5}n + 10\eta n + \frac{2}{5}n - \alpha(y_i) - |B| > (10\eta - 2\varepsilon)n.$$

This is enough to apply (A1).

(A4/(ii)) If $\beta(x_i) \leq \frac{1}{5}n + 10\eta n$, then both

$$\left|\alpha(x_i) - \frac{n}{5}\right| < 10\eta n, \text{ and } \left|\beta(x_i) - \frac{n}{5}\right| < 10\eta n.$$

If $|B| < \frac{3}{5}n - 6\eta n$, then the number of triangles on $x_i y_i$ is at least

$$|N(x_i) \cap N(y_i) \cap B| \geq \beta(x_i) + \beta(y_i) - |B| \geq \frac{1}{5}n + \frac{2}{5}n - 3\eta n - |B| > 3\eta n,$$

and we may apply (A1). If, on the other hand, $|B| \geq \frac{3}{5}n - 6\eta n$, then $|A| \leq \frac{2}{5}n + 6\eta n$. We estimate from below the number of K_3's of form $x_i yz$, where $z \in B \cap N(x_i)$ and $y \in N(x_i) \cap A$. Observe that each $z \in B \cap N(x_i)$ sends at least

$$\beta(z) + \alpha(x_i) - |A| \geq \frac{2}{5}n - \alpha(z) + \frac{1}{5}n - 10\eta n - \frac{2}{5}n - 6\eta n > \frac{n}{5} - 16\eta n - \alpha(z)$$

edges to $N(x_i) \cap A$. Further, $\sum \alpha(z) \leq e(B)$. So the number of triangles $x_i yz$ is at least

$$\sum_{z \in B \cap N(x_i)} |N(z) \cap N(x_i) \cap A| > \alpha(x_i) \cdot \left(\tfrac{n}{5} - 16\eta n\right) - e(B)$$

$$> \left(\tfrac{n}{5} - 10\eta n\right)\left(\tfrac{n}{5} - 16\eta n\right) > \tfrac{n^2}{30}.$$

This means that (for fixed x_i) $x_i y$ is contained in at least $\frac{n^2}{30}/\alpha(x_i) > \frac{n}{20}$ triangles on the average (taken over $y \in A \cap N(x_i)$.) Therefore we may recursively choose the edges $x_i y_i$ contained in at least $\frac{1}{30}n$ triangles $i := 1, \ldots, T$, $T = o(n)$. Lemma 4.6 now provides us with a $T_{2v,2,v} \subseteq G_n$, unless $T \leq h$. So in this case we can again represent the above edges by at most h vertices $x_1, \ldots, x_T$.

(B) Next we assume that we cannot delete εn^2 edges of G_n to get a bipartite graph. By Theorem 4.4, for some $\ell \leq \frac{1}{\varepsilon}$ and for $M = \mu \cdot 10^{1/\varepsilon}$, G_n contains an $C_{2\ell+1}[M]$, assumed that n is sufficiently large.

---

[5] Here we assume e.g. that $T < \log n$ to deduce that $T < h$. If $|A - A'|$ is larger, then choose a subset of $[\log n]$ vertices.

Let us count the number of edges between $G_n - C_{2\ell+1}[M]$ and $C_{2\ell+1}[M]$. By the minimum degree condition,

$$e(G_n - C_{2\ell+1}[M], C_{2\ell+1}[M]) > \frac{2}{5}n \cdot (2\ell+1)M - O(M^2).$$

Each vertex $x \in G_n - C_{2\ell+1}[M]$ can be classified according to $N(x) \cap C_5[M]$. There are $O(2^{(2\ell+1)M}) = O_M(1)$ classes.

Let us say that an $x \in G_n - C_{2\ell+1}[M]$ is typically connected to a class A_i of $C_{2\ell+1}[M]$ if it is joined to at least $\frac{M}{10}$ vertices in this class and there are v other vertices of $G_n - C_{2\ell+1}[M]$ joined to the same vertices of $C_{2\ell+1}[M]$. A short calculation shows that if $2\ell+1 \geq 7$, then most vertices x are typically joined to at least 3 classes and these 3 classes must be pairwise non-neighbors. This implies that we have a $k < \ell$ for which $C_{2k+1}[M^*] \subseteq G_n$, with $M^* = [M/10]$. Iterating this step at most $[\frac{1}{\varepsilon}]$ times we get that a $C_5[\mu] \subseteq G_n$, for $\mu = 100v^2/\varepsilon$.

(C) So we assume that $G_n \supseteq C_5[\mu]$. Fix one such $C_5[\mu]$ and calculate the number of edges between $C_5[\mu]$ and $G_n - C_5[\mu]$:

$$e(C_5[\mu], G_n - C_5[\mu]) \geq 5\mu\delta(G_n) - 5\mu^2 > 2\mu n - 5\mu^2. \tag{4.2}$$

(C1) We change now the meaning of "typical connection": $x \notin C_5[\mu]$ is *typically* joined to a class A_i of $C_5[\mu]$ if $|N(x) \cap A_i| \geq v$ and there are v other vertices outside $C_5[\mu]$ also joined to each $y \in N(x) \cap C_5[\mu]$.

(*) If x is joined typically to A_i and A_{i+1}, then $T_{2v,2,v} \subseteq K_3(v,v,v) \subseteq G_n$ which is excluded.

We partition $V(G_n - C_5[\mu])$ into 7 classes:

(a) W is the set of those vertices which are joined typically to A_i and A_{i+1} (for some $i = 1, \ldots, 5$). By (*), $|W| < v \cdot 32^{\mu} = O(1)$.

(b) Z is the set of vertices "poorly" joined to $C_5[\mu]$: those vertices which are joined to at most $2\mu - 10v/\varepsilon$ vertices of $C_5[\mu]$. We will show that $|Z| < \frac{1}{2}\varepsilon n$.

(c) U_i $(i = 1, \ldots, 5)$ is the set of vertices not in Z, typically joined to A_{i-1} and A_{i+1}.

A vertex in U_i is joined to $C_5[\mu]$ by less then $2\mu + 3v$ edges. This, the bound on $|W|$, the definition of Z and (4.2) imply that $|Z| < \frac{3v}{10v/\varepsilon}n < \frac{1}{2}\varepsilon n$. So deleting all the edges represented by Z we deleted fewer than $\frac{1}{2}\varepsilon n^2$ edges.

Using the method of (A1) we see that U_i contains at most $O(1)$ independent edges, otherwise $T_{2v,2,v} \subseteq G_n$ and therefore $L \subseteq G_n$. So $e(U_i) = O(n)$, moreover, all the edges of U_i can be represented by $O(1)$ vertices.

Let us call a graph pentagon-like if it is contained in some $C_5[N]$. We have just proved that we can delete $T_1 := O(1)$ vertices of G_n to get a pentagon-like graph. But it is trivial that the minimum degree of a pentagon-like graph G_k is at most $\frac{2}{5}k$. So $\delta(G_n) \leq \frac{2}{5}n + T_1$. ∎

References

[1] B. Andrásfai, P. Erdős and V. T. Sós: On the connection between chromatic number maximal clique and minimal degree of a graph, Discrete Mathematics **8** (1974), 205-218, MR**49** #4831.

[2] I. Bárány: A short proof of Kneser's conjecture, Journal of Combinatorial Theory A**25,** (1978), 325-326.

[3] B. Bollobás: Extremal Graph Theory, Academic Press, London, (1978).

[4] B. Bollobás. P. Erdős, M. Simonovits and E. Szemerédi: Extremal graphs without large forbidden subgraphs, Advances in Graph Theory (ed. B. Bollobás), Annals of Discrete Mathematics 3 (1978), 29-41, North-Holland. MR**80a**# 05119.

[5] P. Erdős; Über ein Extremalproblem in der Graphentheorie, Archiv d. Math. **13**(1962), 222–227.

[6] P. Erdős: On the structure of linear graphs, Israel J. Math. **1** (1963), 156-160 MR**28** # 4533.

[7] P. Erdős: On extremal problems of graphs and generalized graphs, Israel J. Math. **2**(3) (1965), 183-190; MR**32**# 1134.

[8] P. Erdős: Some recent results on extremal problems in graph theory (Results), Theory of Graphs (International symposium, Rome, 1966), Gordon and Breach, New York and Dunod, Paris, 1967, pp. 117-123, MR 37, #2634.

[9] P. Erdős: On some new inequalities concerning extremal properties or graphs, Theory of graphs, Proc. Coll. Tihany (1966) (Eds. P. Erdős and G. Katona) Acad. Press, N.Y., (1968) pp77-81; MR**38**# 1026.

[10] P. Erdős: The Art of Counting, Selected Writings (in Combinatorics and Graph Theory, ed. J. Spencer), The MIT Press, Cambridge, Mass., (1973) MR**58**#27144

[11] P. Erdős and T. Gallai: On maximal paths and circuits of graphs, Acta Math. Acad. Sci. Hungar. **10** (1959), 337-356; MR**22**#5591.

[12] P. Erdős and M. Simonovits: A limit theorem in graph theory, Studia Sci. Math. Hungar. 1 (1966) 51-57. (Reprinted in [**10**], 1973, pp 194-200.) 51-57; MR**34**#5702.

[13] P. Erdős and M. Simonovits: Some extremal problems in graph theory, Combinatorial Theory and its applications, I. (ed. P. Erdős et al.), North-Holland, Amsterdam, (1970), pp. 377-390; MR**46**#84. Reprinted in the Art of Counting, MIT PRESS, 1973, pp 246-259).

[14] P. Erdős and M. Simonovits: An extremal graph problem, Acta Math. Acad. Sci. Hung. 22(3-4)(1971), 275-282. MR**45**#1790.

[15] P. Erdős and M. Simonovits: On a valence problem in graph theory, Discrete Mathematics 5 (1973), 323-334. MR**49**# 7175.

[16] P. Erdős and A. H. Stone: On the structure of linear graphs, Bull. Amer. Math. Soc. **52** (1946), 1089-1091. MR**8**# 333b.

[17] P. Frankl and Z. Füredi: Extremal problems concerning Kneser graphs. Journal of Combinatorial Theory. Series B **40**(3) (1986), 270–284. MR**87i**#05118

[18] J. Griggs, M. Simonovits, G. R. Thomas: Extremal graphs with bounded densities of small subgraphs, accepted in Journal of Graph Theory, 1997

[19] J. Komlós: Covering odd cycles, Combinatorica **17**(4) (1997), pp393–400.

[20] L. Lovász: Kneser's conjecture, chromatic number, and homotopy, Journal of Combinatorial Theory A**25**(3), (1978) 319-324.

[21] J. W. Moon: On independent complete subgraphs in a graph, Canad. J. Math. 20, (1968), 95-102; MR**36**#2529; also in: International Congress of Math. Moscow, (1966), vol 13.

[22] M. Simonovits: A method for solving extremal problems in graph theory, Theory of graphs, Proc. Coll. Tihany (1966) (Eds. P. Erdős and G. Katona) Acad. Press, N.Y., (1968) 279-319. MR**38**#2056.

[23] M. Simonovits: Extremal graph problems with symmetrical extremal graphs, additional chromatic conditions, Discrete Mathematics **7,** (1974), 349-376. MR**49**#2459.

[24] M. Simonovits: The extremal graph problem of the icosahedron, Journal of Combinatorial Theory 17B (1974) 69-79; MR**51**# 7945.

[25] M. Simonovits: Extremal graph problems and graph products, Studies in Pure Math. (dedicated to the memory of P. Turán) Akadémiai Kiadó+Birkhäuser Verlag (1982)

[26] M. Simonovits: Extremal Graph Theory, Selected Topics in Graph Theory, (ed. by Beineke and Wilson) Academic Press, London, New York, San Francisco, (1983) 161-200.

[27] M. Simonovits: On the extremal graph problem of the generalized Petersen graphs, (manuscript)

[28] P. Turán: On an extremal problem in graph theory, Matematikai Lapok, **48** (1941), 436-452 (in Hungarian), (see also [**29**], [**30**]). MR**8**# 284j

[29] P. Turán: On the theory of graphs, Colloq. Math. **3** (1954), 19-30, (see also [**30**]). MR**15**# 976b.

[30] Collected papers of Paul Turán: Akadémiai Kiadó, Budapest, 1989. Vol 1-3, (with comments of Simonovits on Turán's graph theorem).

MATHEMATICAL INSTITUTE OF THE HUNGARIAN ACADEMY OF SCIENCES BUDAPEST, REÁLTANODA U 13-15, H–1053

E-mail address: `miki@math-inst.hu`