A HIERARCHY OF RANDOMNESS FOR GRAPHS

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Abstract

In this paper we formulate four families of problems with which we aim at distinguishing *different levels of randomness.*

The first one is completely non-random, being the ordinary Ramsey-Turán problem and in the subsequent three problems we formulate some randomized variations of it. As we will show, these four levels form a hierarchy. In a continuation of this paper we shall prove some further theorems and discuss some further, related problems.

1. Introduction

This paper is an introduction to a field on the hierarchy of randomness with some new problems and results.

1.1. The original questions

Below graphs of order n will be considered and *"almost surely"* or *"almost every"* means that the probability of some event – in a class of n-vertex graphs – tends to 1 as $n \to \infty$.

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To start in the middle, we formulate very briefly (and informally) four questions. The paper is centered around them. The questions are as follows:

> Fix a graph property, i.e., a class $\mathbb{P}$ of graphs (closed under isomorphism) and an integer $r \geq 2$.
>
> (**DD**) How many edges guarantee for a graph G_n that if we r-color its edges arbitrarily, we *always* find a monochromatic subgraph $G_n^* \subseteq G_n$, with $G_n^* \in \mathbb{P}$?
>
> (**DR**) How many edges guarantee for a graph G_n that in *almost all* r-edge-colorings, we find a monochromatic subgraph of $G_n^* \subseteq G_n$, with $G_n^* \in \mathbb{P}$?
>
> (**RD**) How many edges guarantee for a *random graph* G_n almost surely, that r-coloring its edges arbitrarily, we *always* find a monochromatic subgraph of G_n, $G_n^* \in \mathbb{P}$?
>
> (**RR**) How many edges guarantee for a random graph G_n *almost surely*, that r-coloring its edges at random, *almost all* the r-colorings contain a monochromatic $G_n^* \in \mathbb{P}$?

Notation

We shall restrict our considerations mostly to ordinary graphs without loops and multiple edges. $G_n, H_n, \ldots$ will denote graphs with n vertices, $e(G)$, $v(G)$ and $\chi(G)$ will denote the number of edges, vertices in the graph G, and the chromatic number, respectively.

A *graph property* $\mathbb{P}$ is a *set of graphs* and $G \in \mathbb{P}$ means that "G has property $\mathbb{P}$". A graph property $\mathbb{P}$ is assumed to be closed under isomorphism, i.e., invariant under the permutation of vertices. $\mathbb{P}$ is called **monotone** (upward) if adding an edge to an $H_n \in \mathbb{P}$, we get an $H_n^* \in \mathbb{P}$.

Examples.

(1) For a fixed family $\mathcal{L}$ of sample graphs, $\mathbb{P}_{\mathcal{L}}$ denotes the family of graphs containing some $L \in \mathcal{L}$.
(2) $\mathbb{P}_{\mathcal{H}}$ denotes that G has a Hamiltonian cycle.
(3) $\mathbb{P}_{\chi \geq k}$ denotes that $\chi(G) \geq k$.
(4) $\mathbb{P}_{\mathbf{D}}$ denotes the property that G has diameter $\leq D$

(5) For a given function $d = d(n) \geq 0$, $\mathbb{P}_{\mathsf{dmax}}$ denotes the property that the maximum degree is $\geq d(n)$.
(6) For a given function $d = d(n) \geq 0$, $\mathbb{P}_{\mathsf{dmin}}$ denotes the property that the minimum degree is $\geq d(n)$.
(7) For a fixed constant $\alpha \in (0, 1]$, $\mathbb{P}_d(\alpha)$ is the property that G has a subgraph of size at least αn with minimal degree $\geq d(n)$.
(8) $\mathbb{P}_{d-\text{reg}}$: G has a d-regular subgraph.
(9) $\mathbb{P}^*_{d-\text{reg}}$: G has a d-regular spanning subgraph.
(10) $\mathbb{P}_{NonPlanar}$: family of non-planar graphs.

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Below we use $a(n) \sim b(n)$ if both $\frac{a(n)}{b(n)}$ and $\frac{b(n)}{a(n)}$ are bounded, $a(n) \approx b(n)$ if $\frac{a(n)}{b(n)} \to 1$. $a(n) \gg b(n)$ means that $\frac{a(n)}{b(n)} \to \infty$. We shall use the notation $a(n) \succ b(n)$ if for some constant $c > 0$, $a(n) \geq (1 + c)b(n)$.

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Our investigation is strongly related to three basic topics in graph theory: Extremal Graph Theory, Ramsey Theory, and Random Graphs. Here we list some of the basic definitions and notations:

1. For a monotone graph property $\mathbb{P}$, $\text{ext}(n, \mathbb{P})$ is the maximum number of edges a graph $G_n \notin \mathbb{P}$ can have. We call a graph $S_n \notin \mathbb{P}$ extremal for $\mathbb{P}$, if it has $\text{ext}(n, \mathbb{P})$ edges.

2. $G_n \in \mathcal{G}_{n,p}$ means that G_n is a random graph with binomial distribution, where the edge-probability is $p = p(n)$.

3. $\mathbf{ER}(n, \mathbb{P})$ is the Erdős-Rényi **weak threshold** for property $\mathbb{P}$, in the uniform model (see Theorem 4.4).

4. The Ramsey number $R(\mathcal{L}_1, \dots, \mathcal{L}_r)$ is the maximum integer q for which K_q can be r-colored without any monochromatic $L \in \mathcal{L}_i$ of color i, (for any $i \in [1, r]$.)

For related literature see, [59], [5], [50], [34], [25], [7], [2],

2. Four levels of Deterministic and Random Ramsey problems

We could consider two types of problems: the **vertex-problems**, where we increase the number of vertices and suddenly some **"phase-transition"** occurs, and the **edge-problems**, where for a given n we consider graphs on n vertices and increase the number of edges. Ramsey theorem is the typical case of the vertex-phase-transition: if we increase the number of vertices of the graph G_n, then – after a while – either G_n or the complementary graph $\overline{G}_n$ has the regarded property. For vertex-problems (and also for hypergraph problems) our knowledge is very poor in the fields discussed here.

In this paper we are interested primarily in the *edge-phase-transitions* connected to *Ramsey properties*: for fixed n we consider graphs G_n on n vertices and gradually increase $e(G_n)$ from 0 to $\binom{n}{2}$. Concerning a fixed property $\mathbb{P}$, – which now will be some "Ramsey Property" – for some number of edges, $f(n)$, we have a radical change in the structure of the graph, and we are interested in finding this $f(n)$. This $f(n)$ will be called the **threshold** function.

All the *vertex problems* will be discussed in a continuation of this paper. The sharp difference between the edge-problems and the vertex problems is that for the edge-problems it turned out that most of the problems reduce to already known problems, while for vertex problems we have many deep and interesting questions but very few answers.

In this paper all colorings are *edge-colorings*.

Random edge-colorings

There are several ways to define random colorings. To make the picture simpler and clearer, we agree to use the "uniform edge-coloring":

Definition 2.1 (Random edge-coloring)**.** A random r-edge-coloring of a graph G_n is a coloring when the edges are colored by $1, \dots, r$ and for each edge we choose each color uniformly and independently. The subgraph of G_n defined by the edges of color i will be denoted by $G_n^{[i]}$.

2.1. Threshold functions

We shall always assume that a *monotone* property $\mathbb{P}$ is fixed: for *non-monotone* properties most questions *we regard here* do not make sense. In some *other* non-monotone cases the phenomena completely change. Two typical *non-monotone* properties showing many difficulties are

(i) $e(G)$ is even

and

(ii) G contains an $L \neq K_p$ as an *induced* subgraph.

The four general problems we discuss here can be formulated as follows:

Beside fixing a monotone property $\mathbb{P}$ we also fix a color-number r. To avoid trivialities or degenerate cases, we shall always assume that

$$(*) \qquad \text{in any considered } r\text{-coloring of } K_n \text{ some } K_n^{[i]} \in \mathbb{P}.$$

Definition 2.2 (Deterministic-Deterministic). $f^r_{\mathbf{DD}}(n, \mathbb{P})$ is the minimum Γ for which for every r-coloring of every G_n of $\Gamma := f^r_{\mathbf{DD}}(n, \mathbb{P})$ edges, at least one of the "color graphs" $G_n^{[i]}$ has property $\mathbb{P}$.[1]

Observe that there is no randomness in this definition and $f^r_{\mathbf{DD}}(n, \mathbb{P})$ is uniquely defined for any fixed n. (The family of considered graphs is non-empty, by $(*)$.)[2]

Clearly, $f^1_{\mathbf{DD}}(n, \mathbb{P}) = \operatorname{ext}(n, \mathbb{P})$.

The next definition is related to the *usual uniform threshold function* (the binomial version is analogous). Here we use the uniform model, and we do not ask about random graphs but about *random colorings of deterministic graphs.*

Definition 2.3 (Deterministic-Random). We call $f^r_{\mathbf{DR}}(n, \mathbb{P})$ a weak **DR**-threshold function if

(a) for

$$\frac{f(n)}{f^r_{\mathbf{DR}}(n, \mathbb{P})} \to \infty, \quad \text{as} \quad n \to \infty,$$

[1]Here we often think of small subgraphs: $G_n^{[i]} \in \mathbb{P}$ may mean that $G_n^{[i]}$ contains a triangle, or any other (small) subgraph. On the other hand, it may also mean that $G_n^{[i]}$ is hamiltonian: sometimes we thing about spanning subgraphs.

[2]We cannot forget $(*)$: for the property that "G is hamiltonian", for two or more colours this does not hold and therefore our functions are not defined. See e.g. Section 5.3.

for every graph G_n with $f(n)$ edges in almost every r-coloring at least one of the color-graphs $G_n^{[i]}$ has property $\mathbb{P}$;

(b) on the other hand, if

$$\frac{f(n)}{f^r_{\mathbf{DR}}(n,\mathbb{P})} \to 0,$$

then for every G_n of $f(n)$ edges for almost every r-coloring of G_n we have $G_n^{[i]} \notin \mathbb{P}$, for $i = 1, \dots, r$.[3]

This threshold function is often a sharp threshold function, (see below) moreover it is often uniquely determined, or determined up to a very small additive error term.

The weak threshold functions are determined only up to a multiplicative constant: if $f(n)$ is a threshold function, then $c(n) \cdot f(n)$ is as well, for any bounded function $c(n) > 0$ for which $1/c(n)$ is also bounded.

Definition 2.4 (Random-Deterministic)**.** $f^r_{\mathbf{RD}}(n,\mathbb{P})$ is a weak **RD**-threshold function assuming that

(a) if

$$\frac{f(n)}{f^r_{\mathbf{RD}}(n,\mathbb{P})} \to \infty, \quad \text{as} \quad n \to \infty,$$

then in almost every G_n of $f(n)$ edges, in every r-coloring at least one of the color graphs $G_n^{[i]}$ $(i = 1, \dots, r)$ has property $\mathbb{P}$; while

(b) if

$$\frac{f(n)}{f^r_{\mathbf{RD}}(n,\mathbb{P})} \to 0, \quad \text{as} \quad n \to \infty,$$

then almost every G_n of $f(n)$ edges has an r-coloring where the color graphs $G_n^{[i]} \notin \mathbb{P}$ for $i = 1, \dots, r$.

This area became a widely investigated research field. Among the first related results we should mention the paper of Łuczak. Ruciński and Voigt [39] on monochromatic triangles and Füredi's paper [32] on graphs in which every 2-coloring contains a monochromatic C_4. Many papers of Rödl, Ruciński [45], [46], and others should also be mentioned here. For some more details see Section 4.4.

[3]In the uniform model, when we speak of a graph of f edges, we assume that f is integer-valued.

Definition 2.5 (Random-Random)**.** We call $f^r_{\mathbf{RR}}(n,\mathbb{P})$ a weak **RR**-threshold function if

(a)

$$\frac{f(n)}{f^r_{\mathbf{RR}}(n,\mathbb{P})} \to \infty,$$

implies that for almost all graphs G_n with $f(n)$ edges, for almost all r-colorings of (at least) one of the color-graphs $G_n^{[i]}$ is in $\mathbb{P}$;

(b) on the other hand,

$$\frac{f(n)}{f^r_{\mathbf{RR}}(n,\mathbb{P})} \to 0,$$

implies that for almost all G_n of $f(n)$ edges, for almost all r-colorings, we have $G_n^{[i]} \notin \mathbb{P}$, for $i = 1,\ldots,r$.

Sharp thresholds

The **sharp threshold functions** were defined already by Erdős and Rényi [23]. We shall define $g^r_{\mathbf{RD}}$, $g^r_{\mathbf{DR}}$, $g^r_{\mathbf{RR}}$ similarly to the threshold functions $f^r_{\mathbf{RD}}$, $f^r_{\mathbf{DR}}$, $f^r_{\mathbf{RR}}$ above:

Definition 2.6 (Sharp threshold, uniform)**.** Let $U,V \in \{D,R\}$. We call $g^r_{\mathbf{UV}}(n,\mathbb{P})$ a sharp threshold function for "UV" if there exist two functions $g^-_{\mathbf{UV}}$ and $g^+_{\mathbf{UV}}$ such that

$$g^-_{\mathbf{UV}}(n,\mathbb{P}) = (1-o(1))g^r_{\mathbf{UV}}(n,\mathbb{P})$$

and

$$g^+_{\mathbf{UV}}(n,\mathbb{P}) = (1+o(1))g^r_{\mathbf{UV}}(n,\mathbb{P})$$

and $e(G_n) = g^+_{\mathbf{UV}}$ implies "YES" while $e(G_n) = g^-_{\mathbf{UV}}$ implies "NO" in the corresponding question in §1.1 for r colors and $\mathbb{P}$.

We discuss here questions related to the existence of "sharp threshold" only in particular cases. There are many very interesting results on the existence of sharp thresholds, see e.g., many results of Friedgut and others. Here we refer the reader only to some papers related to our approach, like Friedgut and Krivelevich, [30], Friedgut, Rödl, Ruciński and Tetali [31], and also, for **graph properties** (more precisely, in a more general setting) to Friedgut and Kalai [29], and Friedgut [27].

For a very recent survey, see Friedgut, [28].

All the *definitions* for ordinary graphs can be extended to hypergraphs and digraphs.

Basic Questions

Having these definitions, we are interested in the following problems:

1. When do these threshold functions exist? The weak threshold exists in all the four cases. This will shortly be discussed in Section 2.2.
2. Which are the basic relations (inequalities) among our threshold functions when the property $\mathbb{P}$ and r are fixed? (Mostly we fix r, but occasionally $r \to \infty$ slowly. See, e.g., the next section, Claim 3.1, Theorems 4.1, 4.2, etc.)
3. How are the threshold functions related to other, more well known graph theoretical functions? (E.g., connections to Ramsey or Turán numbers, see Theorems 4.1, 4.11,...)
4. What are the order-relations between these functions?
5. Which graph-theoretical properties of $\mathbb{P}$ influence the threshold functions, and how? See, e.g., Theorem 4.11.

Altogether we are interested here in at least 10 functions: the $g^r_{\mathbf{DD}}(n,\mathbb{P})$, the binomial and uniform versions of the other three thresholds, the extremal function $\mathrm{ext}(n,\mathbb{P})$, (which coincides with $g^1_{\mathbf{DD}}(n,\mathbb{P})$) the binomial and uniform versions of the Erdős-Rényi threshold (which coincides with $f^1_{\mathbf{RR}}(n,\mathbb{P})$ and also with $f^1_{\mathbf{RD}}(n,\mathbb{P})$,...)

The structure of this paper is as follows. In the next section we discuss the existence of threshold functions, then we state some basic inequalities relating the above threshold functions to each other. The results formulated there will be proved in the subsequent sections. In Section 4 we shall turn to the *Local Properties* $\mathbb{P}$ and show that $g_{\mathbf{DD}}$ is the Turán-Ramsey function, $f_{\mathbf{DR}}$ is almost the Turán-extremal function, $f_{\mathbf{RD}}$ is described by Rödl and Ruciński, and $f_{\mathbf{RR}}$ is essentially the same as the Erdős-Rényi threshold. The non-trivial separation results will follow from these characterizations.

2.2. Existence of weak threshold functions

The function $g_{\mathbf{DD}}(n,\mathbb{P})$ is deterministic: there nothing is needed to be proved.

It is easy to prove the existence of $f_{\mathbf{DR}}(n, \mathbb{P})$:

(a) if $e(G_n) \le \text{ext}(n, \mathbb{P})$, then $G_n \notin \mathbb{P}$ may occur and this G_n, when r-colored, has neither a monochromatic $G_n^{[i]} \in \mathbb{P}$.

(b) If $e(G_n) > r \cdot \text{ext}(n, \mathbb{P})$ then at least one $G_n^{[i]}$ will have at least $\text{ext}(n, \mathbb{P})$ edges and therefore will be in $\mathbb{P}$.

A general result of Bollobás and Thomason [12] (see the appendix, p29) implies the existence of threshold functions for monotone graph properties as well. So it also implies the existence of the threshold functions $f^r_{\mathbf{RD}}(n, \mathbb{P})$ and $f^r_{\mathbf{RR}}(n, \mathbb{P})$. Indeed,

(c) for any $\mathbb{P}$ we may define $\mathbb{P}^*_r$ as the set of those graphs G_n for which in any r-coloring G_n there is a monochromatic $G_n^{[i]} \in \mathbb{P}$. $\mathbb{P}^*_r$ is a monotone graph property and therefore it has a threshold function f^*_r. This is just what we needed.

(d) The existence of the threshold function for $f_{\mathbf{RR}}$ is also very simple: If we fix a threshold function $\mathbf{ER}(n, \mathbb{P})$ for $\mathbb{P}$ and take a graph G_n with $o(\mathbf{ER}(n, \mathbb{P}))$ edges, then even without coloring it, almost surely $G_n \notin \mathbb{P}$, and of course, coloring G_n in r colors, we get subgraphs that will be neither in $\mathbb{P}$. The other side of our assertion is trivial in the binomial model. There we can refer to the fact that if G_n is a random graph with binomial edge distribution, with edge probability $p(n)$ and we randomly and independently r-color its edges, then each $G_n^{[i]}$ is a random graph with binomial distribution and edge probability $p(n)/r$. Now the standard technique used to prove the equivalence of the two models for monotone properties also shows that if $\omega(n) \to \infty$ and we take a random graph G_n with $\ge \omega(n)\mathbf{ER}(n, \mathbb{P})$ edges, and randomly r-color its edges, then each $G_n^{[i]} \in \mathbb{P}$, almost surely.

We know much less about the existence of sharp thresholds.

3. Basic Inequalities

Now we know that the weak thresholds exist in all the four cases. Below, having inequalities for weak thresholds, (since these functions are determined only up to constants) we mean that one can normalize the function so that the corresponding inequalities hold.[4] To emphasize that we speak of sharp

[4] Inequalities where one can normalize the sides may often be problematic. Actually, one standard way to normalize is to take that very number of edges for which the corresponding probability is as close from below to $\frac{1}{2}$ as possible, see e.g., Bollobás and Thomason,

threshold (or in case of $g_{\mathbf{DD}}$ about a uniquely defined number), we shall often write g instead of f.

Claim 3.1. *For every monotone graph property $\mathbb{P}$, for every n if the corresponding functions are defined and $r \geq 2$*

$$\mathrm{ext}(n,\mathbb{P}) \leq f^r_{\mathbf{DR}}(n,\mathbb{P}) \leq g^r_{\mathbf{DD}}(n,\mathbb{P}) \leq r \cdot \mathrm{ext}(n,\mathbb{P}) \tag{1}$$

and

$$\mathbf{ER}(n,\mathbb{P}) \leq f^r_{\mathbf{RR}}(n,\mathbb{P}) \leq f^r_{\mathbf{RD}}(n,\mathbb{P}) \leq r \cdot \mathrm{ext}(n,\mathbb{P}). \tag{2}$$

If $\mathrm{ext}(n,\mathbb{P}) \prec \binom{n}{2}$,[5] *then*

$$\mathrm{ext}(n,\mathbb{P}) \prec g^r_{\mathbf{DD}}(n,\mathbb{P}). \tag{3}$$

The proof of the claim is trivial from the definitions.

It also immediately follows from (1) and (2) that $f_{\mathbf{RD}} = o(f_{\mathbf{RD}})$ is impossible. More precisely, there is a constant $c = c(r,\mathbb{P})$ for which

$$f_{\mathbf{DR}} \geq c \cdot f_{\mathbf{RD}}.$$

Perhaps the most interesting question we could not settle is whether there is a real hierarchy, i.e, a given order-relation among the above functions. *Here* this boils down to the following:

What can be said about the relation between $f^r_{\mathbf{DR}}(n,\mathbb{P})$ and $f^r_{\mathbf{RD}}(n,\mathbb{P})$, resp. $g_{\mathbf{DR}}(n,\mathbb{P})$ and $g_{\mathbf{RD}}(n,\mathbb{P})$?

In the last part of this paper we shall see that in some degenerate cases $g^r_{\mathbf{DR}}(n,\mathbb{P}) < g^r_{\mathbf{RD}}(n,\mathbb{P})$, say if $r=2$ and the property $\mathbb{P}$ is that "the graph is connected". However, we do not know if

Problem 3.2. *Is $g^r_{\mathbf{DR}}(n,\mathbb{P}) \prec g^r_{\mathbf{RD}}(n,\mathbb{P})$ possible*

(a) for $g^r_{\mathbf{DR}}(n,\mathbb{P}) \geq cn^2$?.

(b) for $g^r_{\mathbf{DR}}(n,\mathbb{P}) = o(n^2)$?

Problem 3.3. *What do we know about the orders of magnitude of these functions for general $\mathbb{P}$?*

We can answer this question only for some special classes of properties.

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[12]. Then our inequalities hold. In many cases one can also overcome the problem of normalization by forgetting about the functions and speak about the corresponding families of graphs.

[5] i.e., $\mathrm{ext}(n,\mathbb{P}) < \binom{n}{2} - cn^2$ for some constant $c > 0$,

4. Local Properties

To simplify our notation, if $\mathbb{P}_{\mathcal{L}}$ is the graph property that

$$L \subseteq G \quad \text{for some} \quad L \in \mathcal{L},$$

then $f_{\mathbf{UV}}(n, \mathbb{P}_{\mathcal{L}})$ will be abbreviated to $f_{\mathbf{UV}}(n, \mathcal{L})$, and if $\mathcal{L} = \{L\}$, then we write $f_{\mathbf{UV}}(n, L)$. If $\mathcal{L}$ is **finite**, then we speak of "local" properties. All other properties will be called "global".

First we restrict ourselves to "local" properties.

We should remark here that the *extremal graph problems* behave completely differently if $\mathcal{L}$ contains bipartite graphs from the cases when $\mathcal{L}$ contains no bipartite L's. This difference will be inherited by our problems related to $f_{\mathbf{DR}}$, as well.

To formulate our results, put

$$t = t(\mathcal{L}) = \min_{L \in \mathcal{L}} \chi(L). \tag{4}$$

$$d_1 := d_1(\mathcal{L}) := \min_{L \in \mathcal{L}} \max_{F \subseteq L} \frac{e(F)}{v(F)}, \tag{5}$$

and

$$d := d(\mathcal{L}) := \min_{L \in \mathcal{L}} \max_{F \subseteq L} \frac{e(F) - 1}{v(F) - 2}. \tag{6}$$

Let $q = q(r, \mathcal{L})$ be the minimum integer for which there exists an $m = m(r, \mathcal{L})$ such that if $K_q(m, \ldots, m)$ is r-colored then it must contain a monochromatic $L \in \mathcal{L}$. By [40], if $\mathcal{L}$ contains a bipartite graph, then $q = 2$. On the other hand, if $t > 2$ then $q > t$.

For general $\mathcal{L}$ we have

Theorem 4.1 (Non-degenerate Case)**.** *Let $\mathcal{L}$ be a finite family of graphs and $r \geq 2$. Then,*

(a) for $t = \min_{L \in \mathcal{L}} \chi(L) \geq 3$ we have the following relations:

$$\begin{aligned}
r \cdot \mathrm{ext}(n, \mathcal{L}) &\succ g^r_{\mathbf{DD}}(n, \mathcal{L}), & (7)\\
g^r_{\mathbf{DD}}(n, \mathcal{L}) &\approx \mathrm{ext}(n, K_q) = \left(1 - \frac{1}{q-1}\right)\binom{n}{2} + O(1), & (8)\\
g^r_{\mathbf{DR}}(n, \mathcal{L}) &\approx \mathrm{ext}(n, \mathcal{L}) = \left(1 - \frac{1}{t-1}\right)\binom{n}{2} + o(n^2), & (9)\\
f^r_{\mathbf{RD}}(n, \mathcal{L}) &\sim n^{2-(1/d)}, & (10)\\
f^r_{\mathbf{RR}}(n, \mathcal{L}) &\sim \mathbf{ER}(n, \mathcal{L}) \sim n^{2-(1/d_1)} & (11)
\end{aligned}$$

(b) For $t := \min_{L\in\mathcal{L}} \chi(L) \geq 3$ the above relations imply that

$$f^r_{\mathbf{RR}}(n,\mathcal{L}) \ll f^r_{\mathbf{RD}}(n,\mathcal{L}) \ll g^r_{\mathbf{DR}}(n,\mathcal{L}) \prec g^r_{\mathbf{DD}}(n,\mathcal{L}). \tag{12}$$

Theorem 4.2 (Degenerate case)**.** *Let $\mathcal{L}$ be a finite family of graphs and $r \geq 2$. For the sake of simplicity, in (15) below we exclude the forests from $\mathcal{L}$. Then,*

(c) for $t = \min_{L\in\mathcal{L}} \chi(L) = 2$ we have the following relations:

$$\begin{aligned} g^r_{\mathbf{DD}}(n,\mathcal{L}) &\approx r \cdot \mathrm{ext}(n,\mathcal{L}) = o(n^2), & (13)\\ g^r_{\mathbf{DR}}(n,\mathcal{L}) &\approx \mathrm{ext}(n,\mathcal{L}), & (14)\\ f^r_{\mathbf{RD}}(n,\mathcal{L}) &\sim n^{2-(1/d)}, & (15)\\ f^r_{\mathbf{RR}}(n,\mathcal{L}) &\sim \mathbf{ER}(n,\mathcal{L}) \sim n^{2-(1/d_1)} & (16) \end{aligned}$$

(d) Further, the above relations imply that

$$f^r_{\mathbf{RR}}(n,\mathcal{L}) \ll f^r_{\mathbf{RD}}(n,\mathcal{L}) < g^r_{\mathbf{DR}}(n,\mathcal{L}) \prec g^r_{\mathbf{DD}}(n,\mathcal{L}). \tag{17}$$

To prove this, we shall need the following two theorems

Theorem 4.3 (Erdős-Stone-Simonovits [24], [17], [49])**.** *Given a (finite or infinite) family $\mathcal{L}$ of forbidden graphs, with*

$$t = t(\mathcal{L}) = \min_{L\in\mathcal{L}} \chi(L). \tag{18}$$

Then

$$\mathrm{ext}(n,\mathcal{L}) = \mathrm{ext}(n,K_t) + o(n^2) = \left(1 - \frac{1}{t-1}\right)\binom{n}{2} + o(n^2). \tag{19}$$

[6]

Theorem 4.4 (Erdős-Rényi, [23])**.** *Let $\mathcal{L}$ be a finite family of graphs and*

$$d_1(\mathcal{L}) = \min_{L\in\mathcal{L}} \max_{F\subseteq L} \tfrac{e(F)}{v(F)}. \tag{20}$$

(i) The binomial threshold function is $\frac{1}{n^{1/d_1(\mathcal{L})}}$: for $p = \omega(n)\cdot\frac{1}{n^{1/d_1(\mathcal{L})}}$, if $G_n \in \mathcal{G}_{n,p}$, then

$$\lim_{n\to\infty} \mathsf{Prob}(L \subseteq G_n \text{ for some } L \in \mathcal{L}) = \begin{cases} 1 & \text{if } \omega(n) \to \infty; \\ 0 & \text{if } \omega(n) \to 0; \end{cases}$$

[6] Mostly this is used for one graph L but we need it for finite graph families $\mathcal{L}$.

(ii) In the uniform model the threshold function is

$$\mathbf{ER}(n,\mathcal{L}) := n^{2-1/d_1(\mathcal{L})}. \tag{21}$$

For infinite families we have to be careful: we cannot simply take the infimum of the exponents. If, e.g., $\mathcal{L}$ is the family of all graphs with minimum degree 4, then the Erdős-Rényi threshold will be at least $n^{3/2}$ for each $L \in \mathcal{L}$, while any random or non-random graph with $\Gamma := 4n$ will contain at least one of them: the threshold for $\mathcal{L}$ will be linear.

4.1. The Deterministic-Deterministic case

Though this section is related to the local properties, yet in many cases the proofs work for more general classes of $\mathbb{P}$.

To describe $g^r_{\mathbf{DD}}$ we distinguish two cases:

(a) $\mathrm{ext}(n,\mathbb{P}) = o(n^2)$,

(b) Properties where $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ with $t \geq 3$, where $\mathcal{L}$ may be infinite as well.

One could think that if $\mathrm{ext}(n,\mathbb{P}) \geq cn^2$, then

$$g^r_{\mathbf{DD}}(n,\mathbb{P}) \prec r \cdot \mathrm{ext}(n,\mathbb{P}). \tag{22}$$

A counterexample to this is given in Claim 5.6.

Problem 4.5. *Are there natural conditions ensuring (22)?*

For the important particular case $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ (see Example (1)) in Section 1.1) Theorems 4.6 and 4.1 completely answer this question.

We start with (a).

Theorem 4.6. *If* $\mathrm{ext}(n,\mathbb{P}) = o(n^2)$*, then, for every fixed* r*,*

$$g^r_{\mathbf{DD}}(n,\mathbb{P}) \approx r \cdot \mathrm{ext}(n,\mathbb{P}). \tag{23}$$

Obviously, on the one hand, (23) cannot hold if $\mathrm{ext}(n,\mathbb{P}) > c\binom{n}{2}$ and $r > 1/c$. On the other hand, there are also many examples where for any fixed r, we have the even stronger

$$g^r_{\mathbf{DD}}(n,\mathbb{P}) = r \cdot \mathrm{ext}(n,\mathbb{P}) \quad \text{for infinitely many} \quad n. \tag{24}$$

(Observe that here we have "=", not only "≈".)

First we prove Theorem 4.6. We shall need the following assertion.[7]

Lemma 4.7. *If G_n is an arbitrary graph with $e(G_n) = o(n^2)$ and π is a random permutation of the vertices, then, almost surely,*

$$|E(G_n) \cap E(\pi(G_n))| = o(e(G_n)), \tag{25}$$

where $\pi(G_n)$ is the image of G_n under the vertex permutation. [8]

We leave the proof of the lemma to the reader.

Proof of Theorem 4.6 (Outline). Fix an extremal graph S_n (for $\mathbb{P}$). By the assumption, $e(S_n) = o(n^2)$. By Lemma 4.7 we can put on n vertices r copies of S_n (permuting their vertices in an appropriate way) so that any two of them intersect in at most $o(e(S_n))$ edges. Deleting the edges in the $\binom{r}{2}$ intersections, we get an r-colored G_n with $(r - o(1))\,\mathrm{ext}(n, \mathcal{L})$ edges, where the color-graphs $G_n^{[i]} \notin \mathbb{P}$. This proves $g_{\mathbf{DD}}(n, \mathbb{P}) \geq (r - o(1))\,\mathrm{ext}(n, \mathbb{P})$. The upper bound (contained in (1)) is trivial. ∎

This implies (13) in Theorem 4.2.

The case when $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$.

The answer to the **DD**-problem in the case $\mathbb{P} = \mathbb{P}_{\mathcal{L}}$ $(r \geq 2)$ is a special case of the Ramsey-Turán problem to determine $\mathbf{RT}^*(n, \mathcal{L}, \dots, \mathcal{L}, \mid m)$, where this function is defined below and its asymptotic value is given by Theorem 4.8. For a more detailed description of the situation, see [51].

Below we reduce the **DD**-problem for this case (apart from some error-term) to Theorem 4.8.

Ramsey-Turán problems. *Given r families of sample graphs $\mathcal{L}_1, \dots, \mathcal{L}_r$ and an integer m. Find the maximum number of edges a graph G_n on n vertices can have under the condition that it can be r-colored so that the i^{th} color contains no $L \in \mathcal{L}_i$ for $i = 1, \dots, r$ and the independence number $\alpha(G_n) \leq m$. Denote by $\mathbf{RT}^*(n, \mathcal{L}_1, \dots, \mathcal{L}_r \mid m)$ this maximum. Put $\mathbf{RT}(n, \mathcal{L}_1, \dots, \mathcal{L}_r) := \mathbf{RT}^*(n, \mathcal{L}_1, \dots, \mathcal{L}_r, n)$.*

[7]Based on a paper of Chung and Graham it seems to us that much earlier Joel Spencer arrived to exactly this result, see [55].

[8]If $G_n \cap \pi(G_n)$ is the graph on the common vertex set having the common edges, then we could also write that $e(G_n \cap \pi(G_n)) = o(e(G_n))$.

Generally we do not know if such graphs exist at all. The special case $\mathbf{RT}(n, \mathcal{L}_1 \dots, \mathcal{L}_r)$ means that we have no restriction on the independence number.[9]

Obviously $g^r_{\mathbf{DD}}(n, \mathcal{L}) = \mathbf{RT}(n, \overbrace{\mathcal{L}, \dots, \mathcal{L}}^{r})$: they are identical, just the notation is different.

This Ramsey-Turán problem (and therefore the problem of determining $g^r_{\mathbf{DD}}(n, \mathbb{P})$) can be solved as follows.

Theorem 4.8 (T. Sós [52], Burr-Erdős-Lovász [14])**.** *Let $q := q(\mathcal{L}_1, \dots, \mathcal{L}_r)$ be the smallest integer such that if m is sufficiently large and we r-edge-color the complete q-partite graph $K_q(m, \dots, m)$, then there will be an $i \le r$ for which we shall have a monochromatic $L \in \mathcal{L}_i$ in the i^{th} color. Then*[10]

$$\mathbf{RT}(n, \mathcal{L}_1, \dots, \mathcal{L}_r) = \operatorname{ext}(n, K_q) + o(n^2). \tag{26}$$

Thus $g^r_{\mathbf{DD}}(n, \mathcal{L}) = \operatorname{ext}(n, K_q) + o(n^2)$ which proves (8) of Theorem 4.1 Theorem 4.8 is an almost immediate consequence of the Erdős-Stone theorem [26]. For the details see [19] or [51]. Also, it can easily be obtained from the *equivalence principle* described in the next section.

One could ask: what if we have an arbitrary $\mathbb{P}$? (Not only a $\mathbb{P}_{\mathcal{L}}$!) Can we reduce the problem of $g_{\mathbf{DD}}(n, \mathbb{P})$ to some kind of a Ramsey-Turán problem? Define $\mathbf{RT}(n, \mathbb{P}_1, \dots, \mathbb{P}_r)$ as the maximum number of edges a graph G_n can have under the condition that it can be r-colored so that the color-graphs $G_n^{[i]} \notin \mathbb{P}_i$ $(i = 1, \dots, r)$. Again, $g_{\mathbf{DD}}(n, \mathbb{P}) = \mathbf{RT}(\mathbb{P}, \dots, \mathbb{P})$.

However, often these quantities do not exist: Condition $(*)$ on p5 is just to exclude the trivial exceptions in such cases.

4.2. Detour: An equivalence principle

Each Ramsey-Turán problem of the simpler type (i.e., when we do not have any upper bound on the *independence number* $\alpha(G_n)$) is equivalent to an ordinary Turán problem:

> Given the color-number r and the (finite or infinite) families $\mathcal{L}_1, \dots, \mathcal{L}_r$ of forbidden graphs, define $\mathcal{M}$ as the family of vertex-

[9]Typically we are interested in estimating $\mathbf{RT}^*(n, L_1, \dots, L_r, \mid o(n))$ see [51], but here we may restrict our attention to $\mathbf{RT}(n, L_1, \dots, L_r)$. (See also Section 4.2.)

[10]We formulated their result slightly more generally.

minimal[11] graphs M such that in any r-coloring of M, there is a monochromatic $L \in \mathcal{L}_i$, of the i^{th} color, for some $i \in [1, n]$. Then

$$\mathbf{RT}(n, \mathcal{L}_1, \ldots, \mathcal{L}_r) = \operatorname{ext}(n, \mathcal{M}). \tag{27}$$

This trivial observation implies that many of the results (error terms, structural stability of the extremal graphs...) that we know for ordinary extremal graph problems, automatically generalize to this simpler type of Ramsey-Turán problems.

Remark 4.9. Unfortunately, no such theorem exists for the general Ramsey-Turán problems, where we consider a sequence (G_n) of graphs and beside the coloring condition also assume that $\alpha(G_n) = o(n)$. To see this we quote the surprising Szemerédi-Bollobás-Erdős theorem, [58], [10] according to which

$$\mathbf{RT}^*(n, K_4 \mid o(n)) = \frac{1}{8}n^2 + o(n^2).$$

Since

$$\operatorname{ext}(n, \mathcal{L}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2)$$

for any $\mathcal{L}$, for some *integer* p, therefore the positive extremal densities are at least $\frac{1}{4}$, while the density in the above mentioned Ramsey-Turán problem (with the "extra condition" $\alpha(G_n) = o(n)$) is $\frac{1}{8}$. So it cannot be equivalent to any ordinary extremal graph problem.

See also Erdős, Hajnal, T. Sós and Szemerédi [20] and Bollobás [9] for related topics, or [51] for a survey on Ramsey-Turán type problems, Sudakov [57] for some newer results in the field.

4.3. The Deterministic-Random case

Almost Local Properties, DR

For *Local Properties*, for **DR**, we always have a sharp threshold. The existence of the sharp threshold was stated in Theorem 4.1, by stating $f^r_{\mathbf{DR}}(n, \mathcal{L}) \approx \operatorname{ext}(n, \mathcal{L})$, and will be generalized in the next theorem. We shall go slightly beyond properties $\mathbb{P}_{\mathcal{L}}$.

[11]This minimality can be omitted, but then we may get infinite families $\mathcal{M}$ in cases where otherwise we would get a finite $\mathcal{M}$.

Definition 4.10. Fix r. Let us call $\mathbb{P}$ **log-concentrated** if (a) any *edge-minimal* $G_\nu \in \mathbb{P}$ has at most $\frac{1}{2}\log_r \operatorname{ext}(\nu, \mathbb{P})$ edges, and (b) $\operatorname{ext}(\nu, \mathbb{P}) \to \infty$.

Here (b) is to exclude a few trivial cases. (a) says that any $G_\nu \in \mathbb{P}$ contains a small $G_h \in \mathbb{P}$. $\frac{1}{2}$ could be replaced by any $\gamma < 1$.

Theorem 4.11. *If $\mathbb{P}$ is log-concentrated then the sharp threshold $g^r_{\mathbf{DR}}(n, \mathbb{P})$ exists and*

$$g^r_{\mathbf{DR}}(n, \mathbb{P}) \approx \operatorname{ext}(n, \mathbb{P}).$$

Proof. We know that $f^r_{\mathbf{DR}}(n, \mathbb{P}) \geq \operatorname{ext}(n, \mathbb{P})$. We need only that

$$f^r_{\mathbf{DR}}(n, \mathbb{P}) < (1 + \varepsilon) \operatorname{ext}(n, \mathbb{P}). \tag{28}$$

Fix an $\varepsilon > 0$ and take a G_n with $E := (1 + \varepsilon) \operatorname{ext}(n, \mathbb{P})$ edges. It contains a subgraph $H_1 \in \mathbb{P}$ with at most

$$\phi = \phi(n) = \frac{1}{2} \log_r \operatorname{ext}(n, \mathbb{P})$$

edges. Delete its edges from G_n and take in the remaining G' an H_2 of at most $\phi(n)$ edges. Iterate this in $\mu := \lceil \varepsilon \operatorname{ext}(n, \mathbb{P})/\phi(n) \rceil$ steps. This way we get μ edge-disjoint $\mathbb{P}$-graphs. The probability that for a random r-coloring none of them will be monochromatic is

$$\leq (1 - r^{-\phi})^\mu \approx \exp\left(-\frac{\mu}{r^\phi}\right) \approx \exp\left(-\frac{\varepsilon E}{\phi\sqrt{E}}\right) \to 0 \quad \text{if} \quad n \to \infty.$$

This proves (28). ∎

If $\mathcal{L}$ is finite and $\mathbb{P} := \mathbb{P}_{\mathcal{L}}$, then $\mathbb{P}$ is trivially log-concentrated. This proves (9) of Theorem 4.1 and 14 of Theorem 4.2. In this case we have an even sharper estimate:

Theorem 4.12. *Let $\mathcal{L}$ be finite, r be fixed, and let $\omega(n) \to \infty$ be arbitrary. If*

$$e(G_n) > \operatorname{ext}(n, \mathcal{L}) + \omega(n),$$

then with probability tending to 1, a random r-coloring of G_n contains a monochromatic copy of some $L \in \mathcal{L}$.

Loosely we could say that for any $\omega(n) \to \infty$

$$0 \le g^r_{\mathbf{DR}}(n, \mathcal{L}) - \operatorname{ext}(n, \mathcal{L}) \le \omega(n) \quad \text{if} \quad n > n_0(\mathcal{L}, \omega).$$

However, this is not quite correct, since g was defined only up to an $(1+o(1))$ factor. The proof is easy and roughly the same as the proof of Theorem 4.11.

The finiteness of $\mathcal{L}$ *cannot* be dropped, see Claim 5.1.

There are cases when this is sharp: we need $\omega(n) \to \infty$. In some other cases $\omega(n)$ can be dropped, see Theorem 4.15.

Remark 4.13. The phenomenon described in the above theorem is actually the following: if $\mathcal{L}$ is finite, then determining $f^r_{\mathbf{DR}}(n, \mathcal{L})$ or $g^r_{\mathbf{DR}}(n, \mathcal{L})$ is the same as determining, when will G_n have $\psi(n)$ edge-disjoint copies of subgraphs from $\mathcal{L}$ with $\psi(n) \to \infty$.

Remark 4.14. Theorem 4.12 can easily be extended to digraphs, multigraphs or hypergraphs. Observe the very weak dependence on r.

4.3.1. Weak dependence on the number of colors

One could ask,

> When do the threshold functions depend on the number of colors and when are they (almost) independent?

Speaking of the **DD** case, we restrict ourselves to the simplest case of $g_{\mathbf{DD}}(n, L, \dots, L)$ and assume that $\chi(L) \ge 3$. This function "strongly" depends on the number of colors, since the corresponding Ramsey numbers strictly increase when we increase the number of colors and $g_{\mathbf{DD}}$ is around the corresponding extremal function $\operatorname{ext}(n, K_{R(\dots)})$, which increases as R increases. (For the bipartite case this dependence is even stronger, by Theorem 4.6.)

Contrary to this, the dependence on r is "negligible" in the **DR** case.

4.3.2. Eliminating the error term?

One could ask if $\omega(n)$ is really needed in Theorem 4.12. There are cases where it is needed, in some others $g_{\mathbf{DR}}(n, \mathcal{L}) = \operatorname{ext}(n, \mathcal{L})$ for $n > n_0(L)$. One of the the simplest cases when we need $\omega(n) \to \infty$ is if $\mathbb{P} = \{G_n : P_3 \subseteq G_n\}$, and more generally, $\mathbb{P} = \{G_n : P_k \subseteq G_n\}$.

Below we shall show that for $\mathcal{L} := \{K_p\}$ the additive error term $\omega(n)$ can be discarded. More generally, let us call an edge $e \in E(L)$ "critical" if $\chi(L - e) < \chi(L)$. One general phenomenon in extremal graph theory is that for sample graphs L with *critical edges* the things are simpler: almost everything is the same as for the complete graphs, at least if n is sufficiently large. Among others, for $t = \chi(L)$,

$$\operatorname{ext}(n, L) = \operatorname{ext}(n, K_t) \quad \text{if} \quad n > n_0(L).$$

We shall prove the following, general result.

Theorem 4.15 (Critical edge)**.** *Let L be a fixed t-chromatic graph ($t \geq 3$) with an edge e for which $\chi(L-e) < t$. Then, for any fixed r and $n > n_0(L, r)$, we have*

$$g^r_{\mathbf{DR}}(n, L) = \operatorname{ext}(n, K_t) = \operatorname{ext}(n, L).$$

Proof of Theorem 4.15. It is enough to ensure ℓ copies of L having a common edge e and otherwise being edge-disjoint. If $\ell \to \infty$, then these copies will ensure (almost surely) a monochromatic L.

Let us consider $T(tm, t, 1)$: the graph obtained from $K_{t-1}(m, \dots, m)$ by adding an edge to it. A theorem of Simonovits [49] (generalizing some results of Erdős) asserts that

$$\operatorname{ext}(n, T(tm, t, 1)) = \operatorname{ext}(n, K_t) \quad \text{if} \quad n > n_0(L).$$

So for $e(G_n) > \operatorname{ext}(n, K_t)$ we shall have a $T(tm, t, 1)$ for $m > \ell \cdot v(L)$, which contains ℓ copies of L having one common edge e_0 but otherwise being edge-disjoint. Any r-coloring of this $T(tm, t, 1)$ contains a monochromatic L (namely, of the color of e_0) with probability tending to 1, as $n \to \infty$ (and therefore $m \to \infty$). ∎

We close this part with the following

Problem 4.16. *Does there exist the sharp threshold function $f^r_{\mathbf{DR}}(n, \mathbb{P})$ for every monotone $\mathbb{P}$?*

4.4. Random-Deterministic case

Here we are interested in the problem: when, at which edge level will a random graph (almost surely) imply some Ramsey property, say have – for all r-edge-colorings – a monochromatic subgraph L?

We have to emphasize that there are very many related, deep results in this field. It would go far beyond the scope of this paper even to attempt to describe them. Also, there are very many related open problems. One of the places to look for such results is the book of Janson, Łuczak and Ruciński [35]. Here we mention a few related papers: [39], [46], [45].

We formulate here just one important result:

Theorem 4.17 (Rödl-Ruciński [45]). *Fix a color-number $r > 2$. Assume that L is* **not** *a star-forest, or if $r = 2$, then L is not the union of a star-forest and paths P_3. Define*[12]

$$d := d(L) := \max_{M \subseteq L} \frac{e(M) - 1}{v(M) - 2}. \tag{29}$$

Then there exist two constants, $c > 0$ and $C > 0$ such that if $p > C/\sqrt[d]{n}$, then for almost all $G_{n,p}$ every r-coloring of $G_{n,p}$ contains a monochromatic L. If, on the other hand, $p \le c/\sqrt[d]{n}$ then for almost all $G_{n,p}$ there exists an r-coloring[13] *of $G_{n,p}$ not containing monochromatic L's.*

This means that if $\mathcal{L}$ contains no star-forests or path, then $f^r_{\mathbf{RD}}(n, \mathcal{L}) \sim n^{2-\frac{1}{d}}$, which gives (10) of Theorem 4.1 and (15) of Theorem 4.2.

It is worth noticing that here we have something between the weak and sharp thresholds: multiplying f by a large but fixed constant we get probability 1, by a small constant, we get probability 0: (not ε and $1 - \varepsilon$).

We formulated this result for the binomial model since the original version was also formulated for that one. Here we shall prove a much weaker, almost trivial assertion.

Claim 4.18. *For every $\mathcal{L}$ with $\min_{L \in \mathcal{L}} \chi(L) \ge 3$ there is a $c_{\mathcal{L}}$ such that*

$$f^r_{\mathbf{RD}}(n, \mathcal{L}) < n^{2-c_{\mathcal{L}}}.$$

This proves $\operatorname{ext}(n, \mathcal{L}) \gg f^r_{\mathbf{RD}}(n, \mathcal{L})$ implicitly stated in Theorem 4.1.[14]

[12] as in (6)

[13] Observe that the "threshold" does not really depend on r. The larger is r, the stronger the upper and the weaker the lower bounds are. Therefore the strongest form of the lower bound is that there is a $c > 0$ for which, if $p < c/\sqrt[d]{n}$ then even in the two-colorings the probability of monochromatic L's tends to 0.

[14] Actually, here the difference between a single L and a family $\mathcal{L}$ disappears. Fix any $L \in \mathcal{L}$. If we know Claim 4.18 for an $L \in \mathcal{L}$ then we know it for the whole $\mathcal{L}$. To get the inequality $\operatorname{ext}(n, \mathcal{L}) \gg f^r_{\mathbf{RD}}(n, \mathcal{L})$, we may pick an $L \in \mathcal{L}$ of minimum chromatic number. Then by Theorem 4.3, we know that if $\chi(L) \ge 3$, then $\operatorname{ext}(n, \mathcal{L}) \approx \operatorname{ext}(n, L)$. This implies that if we know this inequality for one L, then we know it for graph families as well.

Proof of Claim 4.18. By Ramsey theorem, we know that there exists an integer $R = R(r)$ such that if we edge-color K_R in r colors, it always contains a monochromatic copy of this L. By the Erdős–Rényi Theorem (see Theorem 4.4) if G_n is a random graph (either with uniform or with binomial distribution), and if

$$\frac{e(G_n)}{n^{2-\frac{2}{R-1}}} \to \infty,$$

then the probability $\mathsf{Prob}(K_R \subseteq G_n) \to 1$. This proves the claim. ∎

The meaning of the Rödl-Ruciński theorem is that if $M \subseteq L$ is the "densest" subgraph of L,[15] then the threshold $f^r_{\mathbf{RD}}(n, L)$ is the same as the "moment function" in Proposition 6.1 (in the Appendix): that edge-number Γ where the expected number of copies of M in G_n, $\mathbb{N}(M \subseteq G) = c \cdot \Gamma$.[16] [17]

Problem 4.19. *Can one prove a more general theorem on the order of magnitude of $f^r_{\mathbf{RD}}(n, \mathbb{P})$, for general $\mathbb{P}$ [18]?*

4.5. The Random-Random case

The problem of $f^r_{\mathbf{RR}}$ reduces to the famous Erdős-Rényi threshold result [19] (both for the binomial and the uniform models).

Since a random r coloring of a random graph $G_n \in \mathcal{G}_{n,p}$ is a collection of r graphs $G^{[i]}_n \in \mathcal{G}_{n,p/r}$, one easily sees the following

Theorem 4.20. *For any fixed $r > 0$, using d_1 defined in (5),*

$$f^r_{\mathbf{RR}}(n, \mathcal{L}) \sim n^{2-\frac{1}{d_1(\mathcal{L})}} \quad \text{as} \quad n \to \infty.$$

Since $d_1(\mathcal{L}) < d(\mathcal{L})$, this implies that $f^r_{\mathbf{RD}}(n, \mathcal{L}) \gg f^r_{\mathbf{RR}}(n, \mathcal{L})$, proving the corresponding statement of Theorems 4.1 and 4.2.

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[15] Here the "densest" is the one where $\frac{e(M)-1}{v(M)-2}$ attains its maximum for $M \subseteq L$.

[16] While p or Γ are small, $\mathbb{N}(M \subseteq G) \leq \Gamma$ but as Γ increases, $\mathbb{N}(M \subseteq G)$ becomes much larger, and as soon as we have many $M \subseteq G_n$, each of them can easily be extended into many $L \subseteq G_n$.

[17] We would like to thank A. Ruciński for turning our attention to [31] and for his valuable remarks on this topics.

[18] Observe that this is a special case of the earlier Problem 3.3

[19] originally formulated only for balanced graphs, where "balanced" means that if $F \subseteq L$, then $\frac{e(F)}{v(F)} \leq \frac{e(L)}{v(L)}$. The general case can be found, e.g., in Bollobás [6].

Claim 4.21. *If $\mathbb{P}$ is monotone and there is a sharp threshold $\mathbf{ER}(n,\mathbb{P})$, then $g^r_{\mathbf{RR}}(n,\mathbb{P})$ also exists (the threshold is sharp) and*

$$g^r_{\mathbf{RR}}(n,\mathbb{P}) \approx r \cdot \mathbf{ER}(n,\mathbb{P}).$$

4.6. Remarks on Bipartite Graphs

Our problems for some cases are more difficult for bipartite graphs because we do not know enough about the corresponding extremal problems. The bipartite extremal problems are difficult (a) partly because we do not have good enough upper bounds, (b) partly because in most cases where we have promising upper bounds, the lower bounds are missing and seem to be hopeless.

Direct constructions

As to the function $g^r_{\mathbf{DD}}(n,\mathcal{L})$, we used random methods to prove the related results, e.g., Lemma 4.7. The surprising part is that in some cases Theorem 4.6 immediately follows from some old results connected to the so called *polarized partition relations*, see e.g. Chvátal [15], Berge and Simonovits [3], Sterboul [56].

There are some cases where explicit constructions also work. This is formulated in the Claim and Remark below.

Claim 4.22. *It is possible to color the edges of a K_n by $\approx \sqrt{n}$ colors so that each color class has approximately the same number of edges, $\approx \frac{1}{2}n\sqrt{n}$, and no monochromatic C_4 occurs (except in one color class).*

Sketch of the proof. It is enough to consider the case $n = p^2$ where p is a prime. The vertices of K_n are the pairs (a,b) taken mod p and the color of the edge joining (a,b) to (a',b') is $i = aa' + bb' \pmod p$. Then one can check that $e(G_n^{[i]}) \approx \frac{1}{2}n\sqrt{n}$ and $C_4 \not\subseteq G_n^{[i]}$, for each $i \neq 0$. ∎

Remark 4.23. This idea extends to several sporadic cases, e.g., combining a new result of Füredi [33] with a generalization of the Brown construction [13] we can prove by an explicit construction that for every $r \le (\frac{1}{2} - \varepsilon)\sqrt[3]{n}$, the upper bound in Theorem 4.6 is (asymptotically) sharp for $L = K(3,3)$ as well.

For $\chi(L) \geq 3$ we know that $f^r_{\mathbf{DR}}(n, \mathcal{L}) \geq \left\lfloor \frac{n^2}{4} \right\rfloor$, while $f^r_{\mathbf{RD}}(n, \mathcal{L}) = O(n^{2-c})$. This separates the order of magnitude of $f^r_{\mathbf{DR}}(n, \mathcal{L})$ and $f^2_{\mathbf{RD}}(n, \mathcal{L})$. For $\chi(L) = 2$ we do not know the order of magnitude of $\mathrm{ext}(n, \mathcal{L})$, we have only that $f^r_{\mathbf{RD}}(n, \mathcal{L}) = o(f^r_{\mathbf{DR}}(n, \mathcal{L}))$ cannot happen.

However, we have

Conjecture 4.24. *For the bipartite case, if L is not a forest, then*

$$f^r_{\mathbf{DR}}(n, L) \gg f^r_{\mathbf{RD}}(n, L). \tag{30}$$

In several "sporadic" cases, we know (30). E.g., we know it for the following cases.

1. For $L = C_{2k}$, we can separate $\mathrm{ext}(n, L)$ and the moment function $\Gamma(n, L, c)$, using results of Margulis [42], Lubotzky-Phillips-Sarnak [38] or of Lazebnik-Ustimenko-Woldar [37] [20].

2. In all the cases when $\mathcal{L}$ is finite and the extremal number is obtained by some algebraic construction: for $K_{a,b}$ if $a = 2, 3$, for $K_{a,b}$ if $b > (a-1)!$,

3. For the cube graph Q_8, where the conjectured lower bound is still missing but we know at least, that

   $$\mathrm{ext}(n, Q_8) \geq \mathrm{ext}(n, C_4) \approx cn^{3/2} \gg f_{\mathbf{RD}}(n, Q_8) \sim n^{2-\frac{6}{11}},$$

   proving (30). (At the end we used Theorem 4.17.)

We know (30) for many further particular cases as well.

We do not know (30) in those cases, when the known best lower bound for $\mathrm{ext}(n, \mathcal{L})$ is obtained by random methods, (mostly from the "first moment method" of Erdős, see [16], etc. or Simonovits [48]). Just the contrary, since in these cases, by Theorem 4.17 the "first moment method" yields exactly that very bound (i.e., the threshold functions in Proposition 6.1) that is given by Theorem 4.17 for $f_{\mathbf{RD}}$ and since we conjecture that the bound of the first moment method is far from the truth, we are confident that (30) holds.

However, we have no way to approach the general case.

[20] In several places the first author misstated the corresponding result, due to a misprint, writing C_{2k} instead of C_k.

5. Global Properties

Below we turn our attention to "global" properties. Many of the thresholds below will be "sharp" thresholds.

5.1. Separating $\mathrm{ext}(n,\mathbb{P})$ and $f^r_{\mathbf{DR}}(n,\mathbb{P})$

(a) Let us consider the case when $\mathbb{P}$ means that G contains a cycle. Clearly, $\mathrm{ext}(n,\mathcal{C}) = n-1$, where $\mathcal{C}$ is the set of all cycles. Thus, $g^r_{\mathbf{DD}}(n,\mathcal{C}) = r(n-1)$.

(b) Until now we had examples showing that $\mathrm{ext}(n,\mathbb{P}) \approx g_{\mathbf{DR}}(n,\mathbb{P})$. Below we give two examples where they are relatively far.

Theorem 5.1 (All cycles)**.** *Let $\mathcal{C}$ be the family of all cycles. Then, for $r \geq 2$,*

$$f^r_{\mathbf{DR}}(n,\mathcal{C}) \geq \left(\frac{6}{5} - o(1)\right) \mathrm{ext}(n,\mathcal{C}) \geq 1.2(n-1) - o(n). \tag{31}$$

The more colors we use the easier the proof is. So we shall give two proofs: in the second one we shall assume that $r \geq 3$, but the first one works for $r = 2$ as well.

Actually, we can prove the following sharper theorem. Denote by $\mathbf{g}(G)$ the girth of G.

Theorem 5.2 (High girth)**.** *Fix an integer $d \geq 3$ and $c > 0$. Let G_m be an arbitrary graph with maximum degree at most d and $\mathbf{g}(G_m) > c\log_{d-1} m$. Let*

$$\ell > \frac{(1/c)+1}{\log_{d-1} r}. \tag{32}$$

If we replace each edge of G_m with a path $P_{\ell+1}$ (i.e. with a path of ℓ edges), then we get a graph G_n with $n = m + (\ell-1)e(G_m)$ vertices such that if we r-color its edges, then almost surely, as $m \to \infty$ the resulting graph will have no monochromatic cycles.

Proof of Theorem 5.1 from Theorem 5.2. We choose $r = 2$, and $d = 3$ and we consider graphs G_m in which almost all vertices are of degree 3, with girth $\geq \log_3 m = \log_2 m/\log_2 3$. $(c > 0.63)$ By Theorem 5.2, $\ell = 2$ works. So we get a G_n with $n = 5/2m$ and $e(G_n) \approx 3m = \frac{6m}{5}$, proving Theorem 5.1.

To get the appropriate graphs, we use the construction of Biggs and Hoare, [4], as shown by Wiess [60] yields 3-regular graphs G_m with $\mathbf{g}(G_m) \geq$

$\frac{4}{3}\log_2 m$, proving Theorem 5.1. If we do not wish to use algebraic constructions, the Erdős-Rényi theory can be used: for $d = 3$, using the edge-deletion method for random graphs we get a G_m with $e(G_m) \approx \frac{3}{2}m$, and $\mathbf{g}(G_n) \geq \log_3 m$. The above argument, used with $\ell = 2$, also proves Theorem 5.1.

(The second proof is postponed after the proof of Theorem 5.2.)

Proof of Theorem 5.2. Consider a G_m satisfying the conditions of the theorem. Construct G_n from it, as described above. Then, r-coloring the edges of G_n, each hanging path remains monochromatic with probability $\leq 1/r^{(\ell-1)}$. What is more important, each fixed path $P_{\mathbf{g}}$ of G_m corresponds to a path $P_{\ell\mathbf{g}} \subseteq G_n$ and will be monochromatic with probability $r^{-\ell\mathbf{g}+1}$. (Here we also took into account that we have r possible colors for the monochromatic path.)

Let $\mathbf{g} = \mathbf{g}(G_m)$. Since G_m contains at most $\frac{1}{2}md(d-1)^{\mathbf{g}}$ paths $P_{\mathbf{g}+1}$, the probability that in the colored G_n we have a monochromatic path of length $\mathbf{g} \cdot \ell$ can be estimated by

$$\frac{md}{2} \cdot \frac{(d-1)^{\mathbf{g}-1}}{r^{\mathbf{g}\ell-1}} < mr\left(\frac{d-1}{r^\ell}\right)^{\mathbf{g}} = o(1). \tag{33}$$

Indeed, we know that $(d-1)^{\mathbf{g}} > m^c$. Put $\gamma = \ell \cdot \log_{d-1} r$. By $r^\ell = (d-1)^\gamma$, we have

$$mr\left(\frac{d-1}{r^\ell}\right)^{\mathbf{g}} = mr\left(\frac{1}{(d-1)^{\gamma-1}}\right)^{\mathbf{g}} = \frac{mr}{((d-1)^{c\log_{d-1} m})^{\gamma-1}} = \frac{mr}{(m^c)^{\gamma-1}} = o(1)$$

if $c(\gamma-1) > 1$, i.e., $\gamma > 1+\frac{1}{c}$. Since $\gamma = \ell \cdot \log_{d-1} r$, we get that if (32) holds, then (33) also holds. This completes our proof. ∎

For the next proof we shall need

Theorem 5.3 (Erdős–Rényi, Theorem 4c of [23]). *if*

$$\lim_{n\to\infty} \frac{e(G_n)}{n} = c < \frac{1}{2},$$

then for any function $\omega_n \to \infty$ the (induced) tree-components of G_n cover almost surely at least $n - \omega_n$ vertices.

This theorem can be extended to the binomial model as well.

2nd proof of Theorem 5.1, using random graphs. One could produce graphs satisfying the conditions of Theorem 5.2, using the Erdős-Rényi uniform graph model or using the binomial model. One technical problem to fight would be that vertices of degrees larger than 3 could occur. So we shall use the binomial model and some of their results proved originally for the uniform model.

Consider a binomially distributed random graph G_n with edge probability $p = \frac{\lambda}{n}$, for any fixed $\lambda < 3$. The expected number of cycles of length ℓ is $\frac{1}{\ell}(pn)^\ell$. So the expected number of cycles of length smaller than h is

$$\sum_{\ell<h} \frac{1}{\ell}(pn)^\ell < \frac{2}{h}(pn)^h.$$

So we may color $o(n)$ edges GREY and the remaining edges BLACK and then the girth of the BLACK graph will be at least $h = \log_\lambda n$. Now we denote the BLACK part of G_n by G_n^B and color the edges of G_n in RED-BLUE-GREEN uniformly, (thus also 3-coloring the edges of G_n^B uniformly). The 6 graphs defined by the three colors will be binomially distributed random graphs with edge-probability $\frac{\lambda}{3n}$. If they were from the uniform distribution, we could directly apply Theorem 5.3 with $\omega_n = \log\log n$. The 3-coloring ruins all the cycles longer than $\log\log n$ and the deletion of GREY edges ruins all the cycles shorter than $\log_3 n$: G_n^B shows that

$$f^3_{\mathbf{DR}}(n, \mathcal{C}) \geq \left(\frac{3}{2} - o(n)\right) n.$$

∎

For $\mathcal{C}$ we know that $\operatorname{ext}(n, \mathcal{C}) = n - 1$ and $g^r_{\mathbf{DD}}(n, \mathcal{C}) = r \cdot (n-1)$ for any fixed r. We do not know the value of $f_{\mathbf{DR}}$.

Problem 5.4. *Is it true that $f^2_{\mathbf{DR}}(n, \mathcal{C}) \approx \frac{3}{2}n$? If not, can one prove at least, that*

$$f^2_{\mathbf{DR}}(n, \mathcal{C}) < (2 - c)n? \tag{34}$$

Why could one believe this? Because whenever we have a graph G_n having vertices of degree 2, we may replace the corresponding induced path by one edge and the probability that in the new graph (in a random 2-coloring) all the cycles will be $\geq$ 2-colored increases.

Problem 5.5. *Take any of the constructions on regular graphs with bounded degree and high girth, say, the 4-regular Margulis graph described in [41], or some Ramanujan graphs with bounded degrees, see e.g., [38] or [42]. Can one prove that a random edge-coloring with 2 colors (or with r colors for some larger but fixed r) almost surely will have no monochromatic cycles?*

Next we consider the max-degree problem, but only for the case when $d_{\max}(n)/\log n \to \infty$. For $g_{\mathbf{DD}}$ the problem is trivial.

Claim 5.6 (The max-degree case)**.** *Fix an integer r and a function, $d = d(n)$ for which $d(n)/\log n \to \infty$. Let $G_n \in \mathbb{P}_{\mathsf{dmax}}$ mean that the maximum degree in G_n is at least $d(n)$. Then*

$$\lim_{n\to\infty} \frac{f^r_{\mathbf{DR}}(n, \mathbb{P}_{\mathsf{dmax}})}{\operatorname{ext}(n, \mathbb{P}_{\mathsf{dmax}})} = \lim_{n\to\infty} \frac{g^r_{\mathbf{DD}}(n, \mathbb{P}_{\mathsf{dmax}})}{\operatorname{ext}(n, \mathbb{P}_{\mathsf{dmax}})} = r.$$

Proof. Clearly, $\operatorname{ext}(n, \mathbb{P}_{\mathsf{dmax}}) = \frac{1}{2}nd(n) + O(1)$ and, by (1), it is enough to prove $f^r_{\mathbf{DR}}(n, \mathbb{P}_{\mathsf{dmax}}) \geq (r - o(1)) \cdot \operatorname{ext}(n, \mathbb{P}_{\mathsf{dmax}})$. Take an arbitrary Δ-regular graph G_n for $\Delta = \lceil (1-\varepsilon)rd(n)\rceil$. [21] There exists a $c = c(\varepsilon) > 0$ for which, if we color the edges of a G_n in r colors, randomly, uniformly, then the probability that for a fixed vertex x of original degree $\Delta \gg \log n$ in the i^{th} color has degree $\delta^i(x) \geq (1+\varepsilon)\frac{\Delta}{r}$ is smaller than $2^{-c\Delta} = o(\frac{1}{n})$. So the maximum degree in each color will be, almost surely, smaller than $d(n)$. Therefore

$$f_{\mathbf{DR}}(n, \mathbb{P}_{\mathsf{dmax}}) \geq \frac{1}{2}\Delta n \geq \frac{1}{2}(1-\varepsilon)rd(n)n \geq (1-\varepsilon)r \cdot \operatorname{ext}(n, \mathbb{P}_{\mathsf{dmax}}) - O(1). \blacksquare$$

5.2. Connectedness

Let $\mathbb{P}_{\mathbf{Conn}}$ be the graph property that G is connected.

Claim 5.7.
$g^2_{\mathbf{DD}}(n, \mathbb{P}_{\mathbf{Conn}}) = \binom{n}{2} = g^2_{\mathbf{RD}}(n, \mathbb{P}_{\mathbf{Conn}})$.
$g^2_{\mathbf{DR}}(n, \mathbb{P}_{\mathbf{Conn}}) = \binom{n}{2} - (n-2)$.

$$g^r_{\mathbf{RR}}(n, \mathbb{P}_{\mathbf{Conn}}) = \frac{1}{2} r\, n \log n + O(n \log\log n)$$

[21] If n is odd, we allow one vertex of degree $\Delta - 1$.

We mention that the sharp threshold is a corollary of the corresponding Erdős-Rényi theorem.

Remark 5.8. Observe that until now we saw only properties $\mathbb{P}$ where we had $g^2_{\mathbf{DR}}(n, \mathbb{P}) > g^2_{\mathbf{RD}}(n, \mathbb{P})$ but now, for $\mathbb{P} = \mathbb{P}_{\mathbf{Conn}}$ we have the opposite inequality.

One can go somewhat further in analysing this situation: if (G_n) is a sequence of connected graphs and G_n has a vertex x for which the edge-connectivity $\eta(G_n - x) \to \infty$, then almost all 2-colorings of G_n contain a connected, monochromatic spanning subgraph.

5.3. Hamiltonicity and 1-factor

Let $\mathbb{P}$ be any one of the following properties:

(a) G_n has a Hamilton cycle, or that

(b) G_n has a Hamilton path, or that

(c) G_n contains a 1-factor. In the last case we shall restrict ourselves to even values of n.

Trivially, for these properties Condition $(*)$ on p5 does not hold.

However, the corresponding functions $g^r_{\mathbf{DR}}(n, \mathbb{P})$ exist:

Claim 5.9. *If $\omega(n) \to \infty$ and $e(G_n) = \binom{n-1}{2} + \omega(n)$, then almost all random r-colorings of G_n contain a monochromatic Hamilton cycle. Hence $g^2_{\mathbf{DR}}(n, \mathbb{P}_{\mathcal{H}}) = \binom{n-1}{2}$.*[22]

Claim 5.10. *If $e(G_n) = \binom{n-1}{2} + 1$, then almost all random r-colorings of G_n contain a monochromatic Hamilton path and (for n even) a 1-factor.*

Claim 5.11. $g^r_{\mathbf{RR}}(n, \mathbb{P}_{\mathcal{H}}) = \frac{1}{2} r\, n \log n$. *and the same holds for the 1-factor.*

Claim 5.12 (Stopping Rule)**.** *Fix a function $\omega(n) \to \infty$ and use the stopping rule model, stopping when $\deg_{\min}(G_n) \geq \omega(n)$. Then almost all the r-colorings of G_n contain monochromatic Hamiltonian cycles, for fixed number r of colors.*

To prove the Claim, one can reduce it to the uncolored case: to results proved by Ajtai, Komlós, and Szemerédi [36, 1], and Bollobás [8]. (The first breakthrough on the Hamiltonicity of random graphs is due to Pósa, [43].)

One would get very similar results for the Hamiltonian path.

[22] Here one has to be slightly careful: as we have defined the sharp threshold, any function $\frac{n^2}{2} + o(n^2)$ would do.

Problem 5.13. *Can we prove some reasonable inequality, comparing $f^r_{\mathbf{RD}}(n,\mathbb{P})$ and $f^r_{\mathbf{RR}}(n,\mathbb{P})$ for general (monotone) $\mathbb{P}$?*

6. Appendix

We often need/use the **"moment" function**: given an L and a constant $c \in (0,1)$, $p = p_M = p_M(n,L,c)$ will be that very probability for which the **expected number** of copies of $L \subseteq G_n \in \mathcal{G}_{n,p}$ is $cp\binom{n}{2}$. One can easily see that,

Proposition 6.1. *For every graph L,*

$$p(n,L,c) = \frac{c'}{n^{\frac{v(L)-2}{e(L)-1}}}. \tag{35}$$

and the corresponding uniform edge-number is

$$\Gamma(n,L,c) = c'' \cdot n^{2-\frac{v(L)-2}{e(L)-1}}, \tag{36}$$

where $c' > 0$, $c" > 0$ depend only on L and c.

— · —

The existence of a threshold function for some random event $\mathcal{A}(\lambda)$, – depending on some parameter λ – means (at least in our cases) that for any $\varepsilon > 0$, if for some λ_0 and λ_1 the probability

$$\mathsf{Prob}(\mathcal{A}(\lambda_0)) \geq \varepsilon \quad \text{and} \quad \mathsf{Prob}(\mathcal{A}(\lambda_1)) > 1-\varepsilon.$$

then

$$\frac{\lambda_1}{\lambda_0} = O_\varepsilon(1).$$

Bollobás and Thomason proved a general existence theorem [12] on the existence of threshold functions, for monotone properties.

Theorem 6.2 (Bollobás and Thomason)**.** *Every non-trivial monotone increasing* **property of subsets of sets** *has a threshold function.*

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