A METHOD FOR SOLVING EXTREMAL PROBLEMS IN GRAPH THEORY, STABILITY PROBLEMS

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In this paper G^n denotes a graph having n vertices, without loops and multiple edges.

I. Introduction

In 1941 the following problem was proved by P. TURÁN [1]: Determine the maximum number of edges which a graph G^n can have if it does not contain complete p-graphs. The complete p-graph will be denoted by K_p. Denote by $T^{n,d}$ the following graph: n vertices are divided into d classes each of which contains almost the same number of vertices: they contain $\left[\frac{n}{d}\right]$ or $\left[\frac{n}{d}\right]+1$ vertices. Join two vertices by an edge if and only if they belong to different classes. The graph obtained thus is denoted by $T^{n,d}$, and it is of great importance in our problems.

The answer to the problem of TURÁN is (as he proved in [1]), that $T^{n,p-1}$ does not contain K_p and has more edges than any G^n not containing K_p.

Many similar problems have been solved since that. A possible generalization of this question is the following

General problem [2]. Let $F_1, \ldots, F_l$ be given graphs. Determine the maximum number of edges a graph G^n can have if it does not contain an F_i. Determine the extremal graphs for $F_1, \ldots, F_l$, i.e., the graphs having the maximum number of edges.

(A) I have conjectured that in the general case the extremal graphs are very similar to the extremal graphs of K_p: they are very similar to $T^{n,d}$, where d depends only on $F_1, \ldots, F_l$. ERDŐS proved [2] that if $d+1$ is the minimal chromatic number of $F_1, \ldots, F_l$ and K^n is the extremal graph for $F_1, \ldots, F_l$ then

$$e(K^n) = e(T^{n,d}) + O(n^{2-c}),$$

where $e(G^n)$ denotes the number of edges of G^n and c is a positive constant depending on the F_i-s. This result states that the extremal graph has asymptotically as many edges as $T^{n,d}$.

Later it was proved by ERDŐS and myself independently, that the extremal graphs can be obtained from a $T^{n,d}$ omitting from it and adding to it

$O(n^{2-c})$ edges. Moreover, all the graphs not containing an F_i and having almost as many edges as K^n can also be obtained from $T^{n,d}$ by small number changing few edges in them: Let $\varepsilon > 0$ be arbitrary constant. There exists a constant $\delta > 0$ such that if G^n does not contain an F_i and $e(G^n) > e(T^{n,d}) - \delta n^2$ then G^n can be obtained from $T^{dn,}$ by omitting at most $[\varepsilon n^2]$ edges from it and adding at most $[\varepsilon n^2]$ new edges to it. The last part of this paper contains a proof of this and a sharpening of the statement concerning the structure of the extremal graphs.

This sharpening states:

Let $F_1, \ldots, F_l$ be given graphs, K^n be the extremal graph for them. If each F_i is at least $d + 1$ chromatic and e.g. F_1 is $d + 1$ chromatic and it has a colouring by the colours "1", ..., "d" so that only r vertices of F_1 are coloured by "1", then:

$$e(K^n) = e(T^{n,d}) + O(n^{1-\frac{1}{r}})$$

and K^n has almost the same structure as $T^{n,d}$: its vertices can be divided into d disjoint classes $A_1, \ldots, A_d$ so that the folloving conditios are fulfilled.

(a) The classes contain almost the same number of vertices: A_i contains $\frac{n}{d} + O(n^{1-\frac{1}{2r}})$ vertices.

(b) The classes contain few edges: the number of edges joining two vertices of A_i is $O(n^{2-\frac{1}{r}})$.

(c) Each vertex of K^n has the valence $\frac{n}{d} \cdot (d-1) + O(n^{1-\frac{1}{r}})$.

(d) All but $O(n^{2-\frac{1}{r}})$ edges of form (x, y), where $x \in A_i$, $y \in A_j$, $i \neq j$, are contained in K^n.[1]

The first part of this paper deals with some special problems. All our problems are strongly connected with the results of P. ERDŐS.

(B) Consider a $T^{m,d}$ and let be $s \leq \frac{m}{2d}$. Add s edges to $T^{m,d}$ so that the endpoints of our s edges are $2s$ different vertices of the same class of $T^{m,d}$ (i.e. the edges are independent and are in the same class). The graph obtained is denoted by $T(n, d, s)$.

PROBLEM 1. *Determine the maximum number of edges of the graphs having n vertices and not containing $T(n, d, s)$ as a subgraph.*

This problem was posed by ERDŐS and solved only in the special cases $d = 2$, $s = 1, 2$ (unpublished). In this paper it will be solved for any $d \geq 2$, $s \geq 1$ (for $d = 1$ it is very easy to solve it if n is large enough).

[1] Remark. The paper of ERDŐS, which is published also in this volume, states essentially the same results, which I have mentioned in this sharpening. Knowing the general structural theorem we wanted to get a sharpening of it and thus we have proved independently the same theorem by very similar methods. Alas, we have noticed it too late; because of this we publish them here, independently.

Consider a K_{s-1} and a $T^{n-s+1,d}$ (without common vertices) and join each vertex of $T^{n-s+1,d}$ to each vertex of K_{s-1}. Denote the obtained graph by $H(n, d, s)$. Then, if n is large enough, $H(n, d, s)$ is the (only) extremal graph for the problem of $T(n, d, s)$.

This result is the generalization of a result of MOON [3] and also of a result of ERDŐS and GALLAI [4], [5]: $H(n, p-1, s)$ does not contain s vertex-independent K_p. If n is large enough and G^n does not contain s independent K_p, then $e(G^n) \leq e(H(n, p-1, s))$ and the equality holds if and only if $G^n = H(n, p-1, s)$.

(C) Investigating a four-dimensional geometrical problem ERDŐS has found the following extremal problem:

Denote by $Q(r, d)$ the graph obtained from $T^{rd,d}$ by joining each vertex of it to x, where x is a vertex not contained in $T^{rd,d}$. (Clearly $Q(r, d) = H(rd+1, d, 1)$.)

PROBLEM 2. *Consider the graphs of n vertices, not containing $Q(r, d)$. Determine the maximum number of edges of these graphs. Determine the extremal graphs.*

ERDŐS solved this problem for $Q(3, 2)$. I have a method for solving such problems and when ERDŐS asked me, whether my method worked in the case of $Q(3, 2)$, I solved this problem for every $r \geq 2$, $d \geq 2$ using this method.

(D) We have seen a special case, when $T^{n,d}$ was the extremal graph. There are also many other cases when $T^{n,d}$ is the extremal graph.

PROBLEM 3. *Characterize the graph-sets $F_1, \ldots, F_l$ such that if n is large enough, $T^{n,d}$ is the extremal graph for $F_1, \ldots, F_l$.*

We solve this problem by completely proving the following result.

Let $F_1, \ldots, F_l$ be given graphs. $T^{n,d}$ is extremal graph for $F_1, \ldots, F_l$ for sufficiently large values of n if and only if each F_i. has chromatic number $\geq d+1$ but there is an F_{i_0} and an edge e in it so that $F_{i_0} - \{e\}$ is d-chromatic. Further, if each F_i has less than m vertices and there is a $k \geq md$ such that $T^{k,d}$ is extremal graph for $F_1, \ldots, F_l$, then for sufficiently large values of n $T^{n,d}$ will be the only extremal graph.

(E) Some of our problems will be called stability-problems. First we formulate which problems are called stability-problems, then we try to explain, why they are called so and lastly we give a list of the concrete stability-problems, investigated in this paper.

Let $F_1, \ldots, F_l$ be given graphs and K^n be an extremal graph of the problem of $F_1, \ldots, F_l$. Let A be a property defined for graphs. It will be said that the extremal graphs are stabil for the property A if:

(a) None of the graphs G^n having the property A contains any F_i;

(b) The extremal graphs $\{K^n\}$ have the property A;

(c) There is a function $f(n)$ tending to infinity such that if a graph G^n does not contain any F_i and has at least $e(K^n) - f(n)$ edges, then it has also the property A.

This definition can be motivated by the following heuristic argument: We notice a property A of the extremal graphs and pose the question, whether it is essential in our problem or not. Suppose, A is a property such that a graph having A cannot contain any F_i. Then K^n is a graph, having maximal number of edges among graphs having the property A. We say (heuristically) that this property has important role in our problem and K^n is extremal graph just because it has maximum number of edges among graphs having the property A, if not only the extremal graphs, but all the graphs having almost as many edges as K^n has and not containing an F_i possess the property A. This is expressed by (c).

This problem may have a positive answer and then it may be asked: what is the greatest $f(n)$ in condition (c). Thus there arises a new problem: the stability problem of $F_1, \ldots, F_l$, that is: Let S^n be a graph having maximal number of edges among graphs not containing any F_i and not having the property A, and set $f(n) = e(K^n)-e(S^n)$. What is the order of magnitude of $f(n)$? Determine S^n.

Trivially, if A is the following property: G does not contain any F_i, then we obtain a trivial and not interesting stability theorem. Because of this example we restrict the considered properties A. We are interested only in the global properties of G, i.e., in properties, which cannot be verified knowing only the small subgraphs of G. Above all, we are interested in stability theorems, where A concerns the chromatic number of G^n or similar properties.

We mentioned already a stability theorem of the general case, (see (A)) where A is the following property:

We may omit less than εn^2 edges from G^n so that the obtained graph be d-chromatic.

Some other stability theorems

(a) If G^n is $p-1$-chromatic, it does not contain K_p. It will be proved that there is a constant M such that if

$$e(G^n) > e(T^{n,p-1}) - \frac{n}{p-1} + M$$

and G^n does not contain K_p, then G^n is d-chromatic.*

(b) Let $F_1, \ldots, F_l$ be given graphs such that for sufficiently large n $T^{n,d}$ is the extremal graph. Then the statement of (a) remains valid if K_p is replaced by $F_1, \ldots, F_l$ in it.

(c) Let A_1 be the following property:

It is possible to delete $s-1$ vertices of G^n so that the remaining graph is d-chromatic. Then a graph G^n having the property A_1 does not contain $T(n, d, s)$. The extremal graph for $T(n, d, s)$ is $H(n, d, s)$ which clearly has the property A_1. A graph having at least $e\big(H(n,d,s)\big) - \dfrac{n}{d} + M$ edges

* Here the property A is that G is d-chromatic.

(where M is a suitable constant) and not containing $T(n, d, s)$ has the property A_1. The same is true for the problem of s independent $K_{d+1} = K_p$.
The results (a)–(c) are essentially the best possible. In the following part we determine also the extremal graphs of these stability theorems.

II. Definitions, notations

As we have mentioned already, we consider graphs without loops and multiple edges. If G is a graph $v(G)$, $e(G)$ and $\chi(G)$ denote the number of vertices, edges and the chromatic number of G respectively. (The chromatic number of G is the smallest integer k such that the vertices of G can be divided into k classes so that two vertices of the same class are not joined.) If x is a vertex of G, $\sigma(x)$ denotes the valence of x i.e. the number of vertices joined to x.

If G_1 is a subgraph of G or, in general, an arbitrary set of vertices and edges of G, then $G - G_1$ denotes the graph which remains after having omitted the vertices and edges of G_1 and all those edges, at least one endpoint of which belongs to G_1. If A is a set, $|A|$ denotes the number of elements of A. If G is a graph and $G_1, \ldots, G_m$ are some subgraphs of it, they are independent if no two of them have vertices in common. G^n always denotes a graph of n vertices.

In this paper a method will be presented which can be applied to solve many extremal problems. It consists of two parts, one of which is:

III. The progressive induction

In this paper there will be considered problems which are wanted to be solved only for large values of n because either the general statement does not hold for small values of n or it is very complicated to verify it. Because of this our problems will be investigated only for large values of n. On the other hand our problems are such that if we had them for certain consecutive values of n, say for $n_0, \ldots, n_0 + M_0$, then it would be easy to prove them for all $n \geq n_0$ using mathematical induction: the inductional step can be carried out easily, but the inductional base makes difficulties, since n_0 is unknown, or n_0 is so large that it is the same to prove the statement for n_0 or to prove it for all $n \geq n_0$. It seems that the mathematical induction breaks down. However, sometimes we can eliminate this difficulty, using a modified form of the mathematical induction. It will be called the *progressive induction*. It is similar to the mathematical induction and is similar to the Euclidean algorithm; it is the combination of them in a certain sense. First it will be motivated by a heuristic argument, then it will be formulated in a lemma.

Let $F_1, \ldots, F_l$ and $K^{n_0}, \ldots, K^n, \ldots$ be given graphs. Assume that we have the conjecture that $K^{n_0}, \ldots, K^n, \ldots$ are the only extremal graphs for $F_1, \ldots, F_l$ if $n > n_0$. Denote by H^n a real extremal graph for $F_1, \ldots, F_l$ $(n = n_1, \ldots)$ which is unknown yet. It is wanted to be proved that there exists an n_0 such that if $n > n_0$, then $H^n = K^n$. We conjecture not only this last statement, but the following "sharpening" of it as well.

If n_0 is the smallest integer such that if $n > n_0$, then $H^n = K^n$, then if $n \leq n_0$ though $H^n \neq K^n$ but H^n has "similar structure" and almost as many edges as K^n and this similarity "increases" as n increases. We try to express the structural difference between K^n and H^n with the help of a function (norm) $\Delta(n)$ having great or small values according to the fact that H^n and K^n differs essentially or not. Then we try to prove that $\Delta(n)$ behaves similarly as if it were strictly decreasing. And if $\Delta(n)$ has good properties, then we will obtain just the wanted result. Precisely:

Lemma of the Progressive Induction. Let $\mathfrak{A} = \bigcup_{1}^{\infty} \mathfrak{A}_n$ be a set of given elements, such that $\mathfrak{A}_n$ are disjoint finite subsets of $\mathfrak{A}$. Let B be a condition or property defined on $\mathfrak{A}$ (i.e. the elements of $\mathfrak{A}$ may satisfy or not satisfy B). It is wanted to be shown that there exists an n_0 such that if $n > n_0$ and $a \in \mathfrak{A}_n$ then a satisfies $\mathfrak{A}$.

Let $\Delta(n)$ be a function defined also on $\mathfrak{A}$ such that $\Delta(n)$ is a non-negative integer and

(a) if a satisfies B, then $\Delta(a)$ vanishes.

(b) There is an M_0 such that if $n > M_0$ and $a \in \mathfrak{A}_n$ then either a satisfies B or there exist an n' and an a' such that

$$\frac{n}{2} < n' < n, \quad a' \in \mathfrak{A}_{n'}, \quad \text{and} \quad \Delta(a) < \Delta(a') . \tag{1}$$

(This is the condition replacing the inductional step.)

Then there exists an n_0 such that if $n > n_0$, from $a \in \mathfrak{A}_n$ follows that a satisfies B.

Proof. $\mathfrak{A}_1, \ldots, \mathfrak{A}_{M_0}$ are finite classes, $\Delta(a)$ is a finite-valued function thus $S = \max \{\Delta(a) : n \leq M_0, a \in \mathfrak{A}\}$ is a finite non-negative integer. A trivial application of mathematical induction on n shows that $\Delta(a) \leq S$ for every $a \in \mathfrak{A}$. Put $M_i = 2^i M_0$ $i = 1, \ldots, s+1$.

Then B is satisfied by every $a \in \mathfrak{A}_n$ if $n \geq M_{s+1}$.

To show this notice that $\Delta(a) \leq S - i$ if $n \geq M_i$ and $a \in \mathfrak{A}_n$ $(i = 1, \ldots, s)$. This can be proved by induction. For $i = 0$ we know it already.

Assume that it holds for $i - 1$. From $n > M_i$, $a \in \mathfrak{A}_n$ it follows that either a satisfies B (and then $\Delta(a) = 0$) or there exist an n' and an a' satisfying (1). Therefore $\Delta(a) < \Delta(a')$ and from the inductional hypothesis we have

$$\Delta(a) < M - (i - 1) .$$

Since $\Delta(a)$ is an integer, $\Delta(a) \leq M - i$, which proves our statement. Write $i = S$, then: if $n > M_s$ and $a \in \mathfrak{A}_n$ then $\Delta(a) = 0$. Now we have to prove only that from $\Delta(a) = 0$ if n is sufficiently large, it follows, that a satisfies B. Let now be $n > M_{s+1}$, $a \in \mathfrak{A}_n$ Apply (b) on a. If there were an n' and an a' satisfying (1) then $\Delta(a) < \Delta(a')$ and consequently $\Delta(a) < 0$ would hold but $\Delta(a)$ cannot be negative. Thus the other alternative holds in (b): a satisfies B.

REMARK. $\frac{n}{2}$ can be replaced by any function tending to infinity (and less than n) in the condition $\frac{n}{2} < n' < n$ in (b).

IV. Some further remarks on the method used in our proofs

Generally we shall use progressive induction in our proofs or certain modification of it. A theorem of ERDŐS and STONE states that if M and d are given integers then there exists a $c > 0$ such that if $e(G^n) > e(T^{n,d}) + n^{2-c}$, then G^n contains a $T^{M(d+1),d+1}$ ([6], [3]). Generally, it will be considered an extremal graph K^n and a $T^{Md,d}$ will be selected in it by using the theorem of ERDŐS and STONE. Then we classify the vertices of K^n with the help of this $T^{Md,d}$ (here M is a great but fixed integer) and estimate $e(K^n) - e(K^{n-Md})$. This makes possible to use progressive induction giving an estimation on $\Delta(n) - \Delta(n - Md)$ if H^n is the conjectured extremal graph and $\Delta(n) = e(K^n) - e(H^n)$.

V. A characterization of the problems, for which $T^{n,d}$ is the extremal graph

PROBLEM. Characterize the sets of graphs $F_1, \ldots, F_l$ such that $T^{n,d}$ is an extremal graph for $F_1, \ldots, F_l$ if n is sufficiently large.

It will be seen that a condition, which is trivially necessary for that $T^{n,d}$ to be an extremal graph is also sufficient for this. More exactly:

THEOREM 1. (a) Let $F_1, \ldots, F_l$ be given graphs, such that $\chi(F_i) \geq d + 1$ $(i = 1, \ldots, l)$ but there are an F_{i_0} and an edge e in it such that $\chi(F_{i_0} - \{e\}) = d$. Then there exists an n_0 such that if $n > n_0$ then $T^{n,d}$ is the only extremal graph for $F_1, \ldots, F_l$.

(b) The converse statement is also true. Moreover if $k = \max v(F_i)$ and there is at least one $T^{m,d}$ with $m \geq kd$ which is extremal graph for $F_1, \ldots, F_l$ then for $n > n_0(F_1, \ldots, F_l)$ $T^{n,d}$ is the only extremal graph for $F_1, \ldots, F_l$ and $\chi(F_i) \geq d + 1$, $(1, 2, \ldots, l)$, but there are an F_{i_0} and an edge e in it such that $\chi(F_{i_0} - \{e\}) = d$.

Instead of giving direct proof of Theorem 1 (which was given in an unpublished paper of ours) we reduce Theorem 1 to a special case of it:

We recall that $T(rd, d, 1)$ is the graph obtained from $T^{rd,d}$ adding a new edge to it. Trivially $\chi(T^{rd,d}) = d + 1$ but if e is the extra edge of it then $\chi(T(rd, d, 1) - \{e\}) = \chi(T^{rd,d}) = d$. Thus Theorem 1(a) can be applied on $F = T(rd, d, 1)$.

THEOREM 1.* *There exists an $n_0(r, d)$ such that if $n > n_0$ then $T^{rd,d}$ is the only extremal graph for $T(rd, d, 1)$.*

We know that Theorem 1* follows from Theorem 1. Now it will be shown how Theorem 1 follows from Theorem 1*.

Suppose that Theorem 1* holds. Let $F_1, \ldots, F_l$ be given graphs satisfying the conditions of Theorem 1(a). From $\chi(F_{i_0} - \{e\}) = d$ we have that if $r > v(F_{i_0})$, then $F_{i_0} \subseteq T(rd, d, 1)$. There is an n_0 such that if $n > n_0$ $T^{n,d}$ is the only extremal graph for $T(rd, d, 1)$. It will be proved that if $n > n_0$, $T^{n,d}$ is the only extremal graph for $F_1, \ldots, F_l$ as well. Since $\chi(F_i) > d$, $T^{n,d}$ does not contain any F_i. On the other hand, if $e(G^n) \geq e(T^{n,d})$ and $G^n \neq T^{n,d}$, G^n contains a $T(rd, d, 1)$ (according to Theorem 1*) and thus G^n contains an F_{i_0}. Therefore $T^{n,d}$ is the only extremal graph for $F_1, \ldots, F_l$ (if $n > n_0$).

Thus Theorem 1(a) is the consequence of Theorem 1*.

Suppose now that $F_1, \ldots, F_l$ satisfy the conditions of Theorem 1(b) Since $T^{m,d}$ contains all the graphs G^k such that $\chi(G^k) \leq d$ and $T^{m,d}$ does not contain any F_i consequently $\chi(F_i) \geq d + 1$ $(i = 1, \ldots, l)$. Since $e\big(T(m, d, 1)\big) = e(T^{m,d}) + 1$ and $T^{m,d}$ is extremal graph, $T(m, d, 1)$ must contain an F_{i_0} Notice that it is possible to omit from $T(m, d, 1)$ an edge so that the remaining graph is d-chromatic. From $F_{i_0} \subseteq T(m, d, 1)$ follows that F_{i_0} has also this property. Thus we have proved a part of Theorem 1(a) showing that $\{F_1, \ldots, F_l\}$ satisfies the conditions of Theorem 1(a). Apply Theorem 1(a), thus we obtain the other part of Theorem 1(b): if n is sufficiently large, $T^{n,d}$ is the only extremal graph for $F_1, \ldots, F_l$.

Therefore, instead of Theorem 1 it is enough to prove Theorem 1*. Theorem 1* was conjectured by ERDŐS. ERDŐS proved it if $d = 2$ and then I proved Theorem 1, generalizing ERDŐS's result but only later noticing, that it contains Theorem 1*.

Later a theorem will be proved containing Theorem 1* as a very special case. However Theorem 1* will be proved here in a direct way because this is the most beautiful and the simplest, but characteristic case of my method and the other proofs are the variants of this one.

PROOF. Let K^n be an extremal graph (for $T\big(rd, d, 1)\big)$. It will be shown that, if n is sufficiently large then $K^n = T^{n,d}$.

Since $T^{n,d}$ does not contain $T(rd, d, 1)$.

$$e(T^{n,d}) \leq e(K^n) .$$

Hence $\Delta(n) = e(K^n) - e(T^{n,d})$ is a non-negative integer (not depending on the choice of K^n if there are different extremal graphs of n vertices). Select a $T^{Md,d} \subseteq K^n$ applying Theorem ERDŐS–STONE on K^n where $M = 3r$.

The theorem will be proved by progressive induction, where $\mathfrak{A}_n$ is the set of extremal graphs having n vertices, B states that $K^n = T^{n,d}$ and $\Delta(n)$ is defined already. According to the Lemma of the Progressive Induction, it is enough to show that if $K^n \neq T^{n,d}$, then $\Delta(n - Md) > \Delta(n)$ (if n is large enough). Since K^n does not contain $T(rd, d, 1)$, $T^{Md,d}$ is a spanned subgraph of K^n (i.e. two vertices of $T^{Md,d}$ are joined by an edge of K^n if and only if they are joined by an edge of $T^{Md,d}$. Denote by $\widetilde{K}$ the graph $K^n - T^{Md,d}$, by e_K the number of edges joining $\widetilde{K}$ and $T^{Md,d}$. Clearly

$$e(K^n) = e(T^{Md,d}) + e_K + e(\widetilde{K}) . \tag{2}$$

Similarly, select a $T^{Md,d}$ in $T^{n,d}$, then $T^{n,d} - T^{Md,d} = T^{n-Md,d}$ holds and if e_T denotes the number of edges of $T^{n,d}$ joining $T^{Md,d}$ with T^{n-Md}, then we have

$$(3) \qquad e(T^{n,d}) = e(T^{Md,d}) + e_T + e(T^{n-Md}).$$

From this it follows that

$$\Delta(n) = e(K)^n - e(T^{nd,d}) = (e_K - e_T) - \{e(K^{n-Md}) - e(T^{n-Md})\} -$$
$$- \{e(K^{n-Md}) - e(\widetilde{K})\} \leq e_K - e_T + \Delta(n - Md).$$

(Since $\widetilde{K}$ does not contain $T(rd, d, 1)$ and $v(\widetilde{K}) = n - Md$ thus we have $e(K^{n-Md}) - e(\widetilde{K}) \geq 0$.)

Since from $e_k < e_T$ follows $\Delta(n) < \Delta(n - Md)$, it is enough to show that either $e_k < e_T$ or $K^n = T^{n,d}$. Clearly $e_T = (n - Md)(d - 1) \cdot M$, since each vertex of $T^{n-Md,d}$ is joined to $(d - 1)M$ vertices of $T^{Md,d}$. Let us estimate e_K now. In order of this, split the vertices of $\widetilde{K}$ into the following classes:

If $B_1, \ldots, B_d$ denotes the classes of $T^{Md,d}$ then any $x \in \widetilde{K}$ is joined to a suitably $B_{i(x)}$ only by $r - 1$ edges, otherwise K^n would contain a $Q(r, d)$ and consequently a $T(rd, d, 1)$, too.

Let $x \in D$ if $x \in \widetilde{K}$ is joined to $T^{Md,d}$ by less than $(d - 1)M$ edges. Further, if $x \in \widetilde{K} - D$, there is a $B_{i(x)}$ such that x is joined to less than $r - 1$ vertices of $B_{i(x)}$ and since $x \in D$, $i(x)$ is uniquely determined, moreover, if $j \neq i(x)$, x is joined to B_j by more than $(d - 1) \cdot M - r - (d - 2) M = 2r$ edges. But trivially, x is not joined to the vertices of $B_{i(x)}$ at all, otherwise $r - 1$ vertices of $B_{i(x)}$ (at least one of which is joined to x), x and r vertices from each other B_j joined to x would determine a $T(rd, d, 1) \subseteq K^n$. Thus x is not joined to $B_{i(x)}$, but since it is joined to the $(d - 1) \cdot M$ vertices of $T^{Md,d}$, it is joined to all the other vertices of $T^{Md,d}$. In this case let $x \in C_i\big(i = i(x)\big)$.

Thus $\widetilde{K}$ is the disjoint union of $C_1, \ldots, C_d$, D. Clearly

$$e_K \leq (n - Md) \cdot (d - 1) M - |D| = e_T - |D|.$$

We know that to show that either $\Delta(n) < \Delta(n - Md)$ or $K^n = T^{n,d}$ it would be sufficient to prove that from $e_k \geq e_T$ follows $K^n = T^{n,d}$. Thus it will be enough to show that if D is empty, then $K^n = T^{n,d}$. But it is true, since if $|D| = 0$, $\widetilde{K}$ is the disjoint union of $C_1, \ldots, C_d$. Two arbitrary vertices of $B_i \cup C_i$ must not be joined, since if $x, y \in B_i \cup C_i$ were joined, x, y, and $r - 2$ other vertices of B_i and r vertices of each $B_j (j \neq i)$ (which clearly are joined to x and y) would determine a $T(rd, d, 1) \subseteq K^n$. Thus $B_i \cup C_i$ does not contain edges. Hence $\chi(K^n) = d$. Easy to see that $T^{n,d}$ can be characterized also with the following property: $\chi(T^{n,d}) = d$ and it has more edges than any other d-chromatic graph. Thus $e(K^n) \leq e(T^{n,d})$ and the equality holds only if $K^n = T^{n,d}$. But from the extremality of K^n we have $e(K^n) \geq e(T^{n,d})$ and thus $K^n = T^{n,d}$. Qu.e.d.

This proof shows that it is a rather important property of $T^{n,d}$ that it is d-chromatic, and that *it* has more edges than the other d-chromatic graphs.

The problem arises naturally, whether all the graphs not containing $T(rd, d, 1)$ and having almost $e(T^{n,d})$ edges, are d-chromatic, or not.

An unpublished paper of mine proves that if $T^{n,d}(n \geq n_0)$ are the extremal graphs for $F_1, \ldots, F_l$, then there exists a constant $c > 0$ such that all the graphs not containing any F_i, and having more than $e(T^{n,d}) - cn$ edges are d-chromatic graphs. Erdős determined the greatest possible value of c in the case of $T(2r, 2, 1)$. He proved the existence of a constant M such that if G^n does not contain $T(2r, 2, 1)$ and $e(G^n) \geq e(T^{n,2}) - \frac{n}{2} + M$, then $\chi(G^n) = 2$. It was also given a graph by Erdős showing that his result cannot be essentially improved, and generally the conjecture, that if G^n does not contain $T(rd, d, 1)$ and $e(G^n) \geq e(T^{n,d}) - \frac{n}{d} + M$, then $\chi(G^n) = d$ cannot be improved.

Consider a $T^{n,d}$ and let x and y be two vertices in the first class of it, $z_1, \ldots, z_k$ $\left(k = \left[\frac{n}{d}\right]\right)$ be the vertices of the second one. Join x to y and omit the edges (x, z_i) if $1 \leq i \leq k_0 < k$ and (y, z_i) if $k_0 + 1 \leq i \leq k$, where k_0 is a fixed integer. The obtained graph Γ^n is clearly d-chromatic and it does not contain $T(rd, d, 1)$. Since $K_{d+1} \subseteq T(rd, d, 1)$ it is enough to show that Γ^n does not contain K_{d+1}. Suppose the contrary: Omitting all the edges (x, z_i), or all the edges (y, z_i), we obtain d-chromatic graphs not containing K_{d+1}. Hence K_{d+1} contains at least one edge of form (x, z_i) and an edge (y, z_i). But since (x, z_i) is an edge of K_{d+1}, $i > K_0$ and thus $y \in K_{d+1}$ cannot be joined to z_i. Thus K_{d+1} contains two vertices which are not joined. This contradiction proves our statement. Γ^n is not determined uniquely, it has the parameter k_0.

Clearly $e(\Gamma^n) = e(T^{n,d}) - \left[\frac{n}{d}\right] + 1$ $\left(\text{if } k = \left[\frac{n}{d}\right] \text{ what can be assumed}\right)$. Thus we have a $d + 1$ chromatic graph of $e(T^{n,d}) - \left[\frac{n}{d}\right] + 1$ edges not containing $T(rd, d, 1)$ which shows, that optimal c equals at most $\frac{1}{d}$.

I proved that the conjecture of Erdős is also true in the general case: $c_{\max} = \frac{1}{d}$, and I determined the graphs attaining the maximum number of edges amongst the graphs not containing $T(rd, d, 1)$ and having chromatic numbers greater than d.

Later I generalized these results and these generalizations will be proved in the next two paragraphs.

VI. The problem of $T(rd, d, s)$

The problem of $T(rd, d, s)$ will be investigated in this and in the next paragraphs. First of all recall, that $T(rd, d, s)$ denotes the graph obtained from $T^{rd,d}$ by putting s independent new edges into a class of it.

PROBLEM. *Consider the graphs not containing $T(rd, d, s)$ and having n vertices. Determine the maximum number of edges of these graphs and determine also the graphs attaining the maximum.*

In the following part $r \geq 2, d \geq 2, 1 \leq s \leq \frac{r}{d}$ will be fixed.

DEFINITION. A graph G^n will be called a *"good"* graph if it has $s-1$ vertices such that the graph G^{n-s+1} remaining after the deleting of this $s-1$ vertices is d-chromatic. (In other words: G^n contains a d-chromatic spanned subgraph G^{n-s+1}). The other graphs will be called *"bad"* graphs. Clearly, if G^n is good, it does not contain s independent K_{d+1} (otherwise G^{n-s+1} would contain a K_{d+1} but $\chi(G^{n-s+1}) < \chi(K_{d+1})$). Since $T(rd, d, s)$ contains s independent K_{d+1}, a "good" graph does not contain $T(rd, d, s)$ either.

Denote by $H(n, d, s)$ that very graph which is "good" and has more edges than any other "good" graph. There exists such a $H(n, d, s)$ and it has the following structure.

Join each vertex of a K_{s-1} to each vertex of a $T^{n-s+1,d}$. Thus we obtain $H(n, d, s)$. (The maximality property of $H(n, d, s)$ is a trivial consequence of the fact that $T^{n-s+1,d}$ has more edges than any other d-chromatic G^{n-s+1}).

THEOREM 2. *There exists an n_0 such that if $n > n_0$, then $H(n, d, s)$ is the only extremal graph for the problem of $T(rd, d, s)$.*

Moreover, there is a stability theorem on the $T(rd, d, s)$ similar to the stability theorem of $T(rd, d, 1)$.

THEOREM 3. *There is a constant M such that if G^n does not contain $T(rd, d, s)$ and $e(G^n) > e(H(n, d, s)) - \frac{n}{d} + M$ then G^n is "good" graph i.e.: it contains a d-chromatic spanned subgraph of $n-s+1$ vertices.*

Theorem 2 can be proved in a direct way by progressive induction, but it follows also from Theorem 3.

Suppose that Theorem 3 is proved already. If L_n is the extremal graph for the problem of $T(rd, d, s)$ then $e(L^n) \geq e(H(n, d, s))$ since $H(n, d, s)$ does not contain $T(rd, d, s)$. If L^n were a bad graph, then from Theorem 3 we should have

$$e(L^n) \leq e(H(n, d, s)) - \frac{n}{d} + M < e(H(n, d, s))$$

if $n > Md$. Thus L^n is a "good" graph for large values of n.

But then, from the maximality property of $H(n, d, s)$ (among the good graphs) and from $e(L^n) \leq e(H(n, d, s))$ we have $L^n = H(n, d, s)$ what is wanted to be proved. Thus it will be proved only in Theorem 3.

REMARK. The proof of Theorem 3 is much longer and much more difficult, than the proof of Theorem 2, since it contains colouring problems. Just because of this the following lemmas will be needed in its proof.

DEFINITION. If $A_1, \ldots, A_d$ are d classes of vertices and we join each pair of vertices of different classes, the obtained graph is called a *complete d-partite graph*. A complete d-partite graph determines its classes uniquely.

LEMMA 1. *If $\chi(G^n) = d$ and $A_1, \ldots, A_d$ are the sets of vertices having the i-th colour at a fixed colouring of G^n with d colours and m_i denotes the number of vertices of the i-th class of $T^{n,d}$ $\left(\text{i. e. } m_i = \left[\frac{n}{d}\right] \text{ or } m_i = \left[\frac{n}{d}\right] + 1 \text{ and } \Sigma m_i = n\right)$ further $|A_i| = m_i + s_i$, then*

$$e(G^n) \leq e(T^{n,d}) - \Sigma \binom{|s_i|}{2}.$$

PROOF. It is enough to prove the lemma with the assumption that if $i \neq j$, $x \in A_i$, $y \in A_j$ then x and y are joined, i.e. G^n is a complete d-partite graph. For the sake of simplicity compare $e(\bar{G}^n)$ and $e(\bar{T}^{n,d})$, where $\bar{G}^n$ and $\bar{T}^{n,d}$ are the complementary graphs of G^n and $T^{n,d}$ respectively. $\bar{G}^n$ and $\bar{T}^{n,d}$ are the disjoint unions of $K_{m_i+s_i}$'s and of K_{m_i} 's, respectively. Hence

$$e(\bar{G}^n) = \Sigma \binom{m_i + s_i}{2} = \Sigma \binom{m_i}{2} + \frac{1}{2} \Sigma (2m_i - 1) s_i + \frac{1}{2} \Sigma s_i^2.$$

Since $\Sigma s_i = 0$ and $\Sigma \binom{m_i}{2} = e(\bar{T}^{n,d})$ thus

$$e(\bar{G}^n) = e(\bar{T}^{n,d}) + \frac{1}{2} \Sigma s_i^2 + \Sigma m_i s_i.$$

Here $|m_i s_i| \leq \frac{1}{2} \Sigma |s_i|$.

Indeed $\sum_{i=1}^{d} |s_i| = \sum_{m_i = \left[\frac{n}{d}\right]} |s_i| + \sum_{m_i = \left[\frac{n}{d}\right]+1} |s_i|$. Without loss of generality it may be assumed that the second sum of the right hand side is the greater one, then $\sum_{m_i = \left[\frac{n}{d}\right]} |s_i| \leq \frac{1}{2} \sum_{i=1}^{d} |s_i|$. Therefore, and since

$\Sigma s_i = 0$ and $m_i = \left[\frac{n}{d}\right]$ or $m_i = \left[\frac{n}{d}\right] + 1$,

$$|\Sigma m_i s_i| = \left| \Sigma \left(m_i - \left[\frac{n}{d}\right] - 1\right) s_i \right| = \left| \sum_{m_i = \left[\frac{n}{d}\right]} s_i \right| \leq \frac{1}{2} \sum_{i=1}^{d} |s_i|.$$

Thus

$$e(G^n) \leq e(T^{n,d}) - \frac{1}{2} (\Sigma s_i^2 - |s_i|) = e(T^{n,d}) - \Sigma \binom{|s_i|}{2}.$$

LEMMA 2. *If c is a given positive constant, there exists an n_0 such that if $n > n_0$, $\chi(G^n) = d$ and $e(G^n) \geq e(T^{n,d}) - cn$ then G^n contains a $T^{\nu d,d}$, where $\nu \geq \left[\frac{1}{6cd^3} n\right] \geq Cn$ ($C > 0$ is a constant).*

PROOF. Colour the vertices of G^n with d colours and let A_i denote the set of vertices of the i-th colour. For the sake of simplicity x and y will be said to be joined by a red edge if they are of different colours and *they are not joined* in G^n. If $|A_i| = m_i + s_i$ and the number of red edges is t, then clearly

$$e(G^n) \leq e(T^{n,d}) - \sum \binom{|s_i|}{2} - t. \tag{4}$$

$e(G^n) \geq e(T^{n,d}) - cn$ therefore $s_i = O(\sqrt{n})$ and $t \leq cn$. Consider those vertices of G^n which are joined to more than $2cd$ vertices by *red* edges. The number of these vertices is less than $\frac{n}{2d}$. Omit these vertices. The remaining graph G^* will be also d-chromatic, it will have the classes $A_1^*, \ldots, A_d^*$ and every $x \in A_i^*$ is joined at most with $2cd$ other vertices by red edges. Further $|A_i^*| > \frac{n}{3d}$.

Now the desired $T^{\nu d,d}$ can be constructed in G^* as follows. Select recursively a sequence of vertices $x_{1,1}, \ldots, x_{1,d}; x_{2,1}, \ldots, x_{2,d}; \ldots, x_{\nu 1}, \ldots, x_{\nu d}$ so that $x_{ij} \in A_j$ and x_{ij} be joined to all the $x_{k,l}$ ($k \neq i$) selected already. It is possible to select $d \cdot \nu$ vertices x_{ij} $\left(i = 1, \ldots, \left[\frac{n}{6cd^3}\right] = \nu,\ j = 1, \ldots, d\right)$ in this way, since the number of vertices of A_j^* joined by a red edge to at least one of the vertices selected out before selecting x_{ij} is less than $d \cdot \nu \cdot 2cd \leq \leq 2d^2c \cdot \frac{n}{6d^3c} = \frac{n}{3d}$ and because of this A_i^* contains at least one vertex joined to all the vertices $x_{k,l}$ ($k \neq i$) selected already.

These vertices x_{ij} form a $T^{\nu d,d}$ where $\nu = \left[\frac{n}{6cd^3}\right]$. Qu.e.d.

Now we prove Theorem 3.

PROOF. It will be useful to recall the statement which we want to prove. It states the existence of a constant M such that if G^n does not contain $T(rd, d, s)$ and $e(G^n) \geq e\big(H(n, d, s)\big) - \frac{n}{d} + M$ then G^n is "good" graph: we may omit $s - 1$ vertices of it in such a way that the obtained graph would be d-chromatic.

(A) Let $\frac{1}{2} > c > 0$ be a constant, small enough and $M_0 > 0$ be an integer, sufficiently large. The conditions on c and M_0 will not be explicitly stated here, but later it becomes clear, which conditions must c and M_0 satisfy. However, it must be remarked, that c is fixed first and the conditions on M_0 may depend on c.

Denote by S^n an extremal graph of the considered stability problem, i.e. let S^n be a "bad" graph not containing $T(rd, d, s)$ and having the maximum number of edges among bad graphs not containing $T(rd, d, s)$.

Write

$$\Delta(n) = e(S^n) - e(H(n, d, s)) + \left[\frac{n}{d}\right]. \tag{5}$$

As it will be seen later, $\Delta(n)$ is a bounded functon of n. Theorem 3 states only that $\Delta(n)$ is bounded from above. To show this it will be enough to prove that there is an n_1 such that if $n > n_1$, then either $\Delta(n) < 0$ or there is an n' such that $\Delta(n') \geq \Delta(n)$ and $n' < n$.

Suppose that $\Delta(n) \geq 0$. Then according to the theorem of ERDŐS and STONE S^n contains a $T^{(M_0+2s)d,d}$ if $n > n_0(M_0)$. In each class of it there are maximally s independent edges. Hence we may omit $2s$ vertices of each class of $T^{(M_0+2s)d,d}$ so that the remaining $T^{M_0 d,d}$ is a spanned subgraph of S_n

(B) Suppose that $T^{\nu d,d}$ is a spanned subgraph of S^n satisfying the following conditions:

Write $\widetilde{S} = S^n - T^{\nu d,d}$ and denote by $B_1, \ldots, B_d$ the classes of $T^{\nu d,d}$ Suppose that the vertices of $\widetilde{S}$ can be partitioned into $d+2$ classes $C_1, \ldots, C_d, D, E$ where

(i) every $x \in E$ is joined with every vertex of $T^{\nu d,d}$;

(ii) if $x \in C_i$ then x is joined to at least $(1-c)\nu$ vertices of $B_j (j \neq i)$ and at most $\frac{1}{2} c\nu$ vertices of B_i;

(iii) if $x \in D$, then there are two different classes of $T^{\nu d,d}$: $B_{i(x)}$ and $B_{j(x)}$ such that x is joined to less than $(1-c)\nu$ vertices of $B_{j(x)}$ and less than $\frac{1}{2} c\nu$ vertices of $B_{i(x)}$. Further if D is not empty then it contains at least one $x_0 \in D$ which is not joined with two suitable classes B_{i_0} and B_{j_0} of $T^{\nu d,d}$ at all.

(iv) $\widetilde{S}$ is a "bad" graph;

(v) $\frac{n}{3d} \geq \nu > c^{3s} M_0 > r$.

If we know the existence of such a $T^{\nu d,d}$, we can finish the proof in the following way:

From (5) we have

$$\Delta(n-\nu d) - \Delta(n) = \{e(S^{n-\nu d}) - e(S^n)\} + \{e(H(n,d,s)) - e(H(n-\nu d, d, s))\} +$$

$$+ \left\{\left[\frac{n-\nu d}{d}\right] - \left[\frac{n}{d}\right]\right\} \geq$$

$$\geq \{e(\widetilde{S}) - e(S^n)\} + e(H(n,d,s)) - e(H(n-\nu d, d, s)) - \nu \tag{6}$$

since $\widetilde{S}$ is a bad graph not containing $T(rd, d, s)$ and thus

$$e(S^{n-\nu d}) \geq e(\widetilde{S}) .$$

If e_S denotes the number of edges joining $\widetilde{S}$ and $T^{\nu d,d}$ in S^n then

$$e(S^n) = e(\widetilde{S}) + e_S + e(T^{\nu d,d}) . \tag{7}$$

Select a $T^{\nu d,d} \leq H(n, d, s)$ not containing the vertices of valence $n - 1$ of $H(n, d, s)$, then clearly

$$H(n, d, s) - T^{\nu d,d} = H(n - \nu d, d, s) .$$

Denote by e_H the number of vertices joining $T^{\nu d,d}$ and $H(n - \nu d, d, s)$ in $H(n, d, s)$. Since the vertices of $H(n - \nu d, d, s)$ are joined to $(d - 1)\nu$ vertices of $T^{\nu d,d}$, except the vertices of K_{s-1}

$$e_H = (n - \nu d) \cdot \nu \cdot (d - 1) + \nu(s - 1) . \tag{8}$$

Further

$$e\big(H(n, d, s)\big) = e\big(H(n - \nu d, d, s)\big) + e_H + e\,(T^{\nu d,d}). \tag{9}$$

We have from (6), (7) and (9)

$$\Delta(n - \nu d) - \Delta(n) \geq (e_H - \nu) - e_S . \tag{10}$$

Distinguish the following two cases:

(a) $|E| \leq s - 2$. Then $e_S \leq e_H - \nu = (n - \nu d) \cdot \nu \cdot (d - 1) + \nu \cdot (s - 2)$. Indeed, the vertices of E are joined to νd vertices of $T^{\nu d,d}$, the vertices of $\bigcup_1^d C_i$ to at most $(d - 1)\nu$ vertices of $T^{\nu d,d}$ and the vertices of D are joined to less than $(d - 1) \cdot \nu - \frac{1}{2} c\nu$ vertices of $T^{\nu d,d}$. Thus

$$e_S \leq (n - \nu d - |E| - |D|)\, \nu \cdot (d - 1) + E \cdot d\nu + |D| \cdot (d - 1)\, \nu - |D| \cdot \frac{1}{2} c\, \nu \leq$$
$$\leq (n - \nu d)\, \nu \cdot (d - 1) + (s - 2)\, \nu = e - \nu .$$

In this case we have from (10): $\Delta(n - \nu d) \geq \Delta(n)$, which was to be proved.

Remark A. *Equality holds* (i.e. $\Delta(n - \nu d) = \Delta(n)$)*only if D is empty and $|E| = s - 1$, further, all the vertices of C_i are joined to all the vertices of B_j $(i \neq j)$.*

(b) If $|E| \geq s - 1$ then $|E| = s - 1$ otherwise s vertices of E and $rd - s$ suitable vertices of $T^{\nu d,d}$ would determine a $T(rd, d, s)$ in S^n.

In this case $B_i \cup C_i$ does not contain edges, otherwise a $T(rd, d, s)$ would be contained by S^n: $r - s - 1$ vertices of B_i, the vertices of E and the endpoints of the considered edge (x, y) would form the first class of it and r vertices of each B_j $(j \neq i)$ would form the other class of our $T(rd, d, s) \subseteq S^n$ (where these $r(d - 1)$ vertices must be joined with the endpoints of the considered edge). Thus $B_i \cup C_i$ does not contain edges. From the fact that S^n is "bad" and from $|E| \leq s-1$ follows that D is not empty. According to (iii) the number of edges joining D and $T^{\nu d,d}$ is at most $|D|\cdot(d-1)\cdot\nu-\nu-$ $-\frac{1}{2}c\nu(|D|-1)$. Thus

$$e_S \leq (n-\nu\cdot d)\cdot\nu\cdot(d-1)+(s-1)\,\nu-\nu-(|D|-1)\cdot\frac{1}{2}c\,\nu \leq e_H-\nu$$

and thus $\Delta(n-\nu\, d) \geq \Delta(n)$ in this case, too.

Therefore if we construct a $T^{\nu d,d}$ having the properties (i)–(v), the proof of Theorem 3 will be completed.

REMARK B. *In the case* (b) $\Delta(n)=\Delta(n-\nu d)$ *only if* $|E|=s-1$, $|D|=1$ *and the vertices of* C_i *are joined to all the vertices of* B_j $(i \neq j)$ *and* $x_0 \in D$ *is joined to all the vertices of* $T^{\nu d,d}$ *except the* 2ν *vertices of the two considered classes* B_{i_0}, B_{j_0}.

(C) First a $T^{\nu d,d}$ which will be constructed which may not satisfy (iv), i.e. may be $\widetilde{S}=S^n-T^{\nu d,d}$ is "good".

Let $T_0=T^{hd,d}$ be a spanned subgraph of S^n such that $M_0 \leq h \leq \frac{n}{3d}$. As we have seen in (A), there exists such a $T^{hd,d}$.

Notation. Let t be a real number, then $\{t\}$ denotes the "upper *entier*" of t, i.e.: $\{t\}=\min\,(n\colon n$ is integer, $n \geq t)$.

(C_1) If there is an $x_1 \in S^n$ joined to all the classes of $T^{hd,d}=T_0$ by more than c^2h vertices, then T_0 contains a $T_1=T^{\{c^2h\}\cdot d,d}$ each vertex of which is joined to x_1; . . . If there is an x_i joined to at least $c^{2i}h$ vertices of each class of T_{i-1}, then there is a $T_i=T^{\{c^{2i}h\}\cdot d,d} \subseteq T_{i-1}$ each vertex of which is joined to all the vertices $x_1,\ldots,x_i$. Thus we may define recursively a sequence of graphs. However, this process stops at last after the construction of T_{s-1}, since if we could find a $T_s \subseteq S^n$ then $rd-s$ suitable vertices of it and the vertices $x_1,\ldots,x_s$ would determine a $T(rd,\,d,\,s)$ in S^n. If this process stops after the j-th step, consider whether there is any vertex $u \in S^n$ for which besides the fact that it is joined to a class of T_j by less than $\{c^{2j+2}h\}$ edges, is joined to another class of T_j by less than $\left(1-\frac{1}{2}c\right)c^{2j}h$ edges, or not. If there is no such u, the algorithm stops. If there is such a u, then T_j contains a $T_j^*=T^{\{c^{2j+2}h\}d,d}$, two suitable classes of which are not joined by edges to u at all.

Now continue the original algorithm with T_j^*. When this algorithm stops, we obtain a $T_k^*=T^{\bar{\nu}d,d}$ such that

(α) each vertex of T_k^* is joined to each vertex of $\{x_1,\ldots,x_k\}$ where $0 \leq k \leq s-1$.

(β) $$\frac{n}{3} > \bar{\nu} > 2\,M_1 = 2\,c^{2s}\cdot M\,.$$

(γ) No vertex of S^n is joined to more than $c^2\bar{\nu}$ vertices of each class of $T^{\bar{\nu}d,d}$

(C_2) Denote by $\bar{B}_i$ the i-th class of $T^{\bar{\nu}d,d}$, and write $\widetilde{\widetilde{S}}=S^n-T^{\bar{\nu}d,d}$, $E=\{x_1,\ldots,x_k\}$. If $x \in \widetilde{\widetilde{S}}$ is joined to a $\bar{B}_i$ by less than $c^2\bar{\nu}$ edges, to another

$\bar{B}_j$ by less than $\left(1-\frac{1}{2}c\right)\bar{\nu}$ vertices, then let $x \in D$. If $x \in \tilde{\tilde{S}} - D - E$ then there is an $i = i(x)$ such that $\bar{B}_{i(x)}$ is joined to x by less than $c^2\bar{\nu}$ edges but to every other $\bar{B}_j$ by more than $\left(1-\frac{1}{2}c\right)\bar{\nu}$ edges. Thus $i(x)$ is uniquely determined by x. Let $x \in C_i$ in this case.

(C_3) Now, it will be proved that there are at most $s-1$ independent edges in $\bar{B}_i \cup \bar{C}_i$. Suppose the contrary: let (x_l, y_l) be independent edges in $\bar{B}_i \cup \bar{C}_i$, $l = 1, \ldots, s$. Then the vertices x_l, y_l and $r-2s$ other vertices of $\bar{B}_i$ together with r suitable vertices of each $\bar{B}_j$ determine a $T(rd, d, s)$ in S^n. (The expression "suitable" means that the r selected vertices of $\bar{B}_j$ must be joined to each x_l and y_l. But they actually can be selected in this way, since B_j contains at least $\bar{\nu} - dc\bar{\nu}$ vertices joined to each x_l and y_l, and if c is small enough, and M_0 is large enough, then $\bar{\nu} - dc\bar{\nu} \geq r$. Thus $\bar{B}_i \cup \bar{C}_i$ does not contain s independent edges.

(C_4) Consider the edges joining $\bar{B}_i$ and $\bar{C}_i$ and select a maximal set of independent edges among them: (x_l, y_l), $l = 1, \ldots, z_i$, $x_l \in \bar{B}_i$, $y_l \in \bar{C}_i$. Clearly if $x \in \bar{B}_i$ is joined to $y \in \bar{C}_i$, then there is an l such that either $x = x_l$ or $y = y_l$.

The number of vertices of $\bar{B}_i$ joined to at least one of $y_1, \ldots, y_{z_i}$ is less than $c^2\bar{\nu}(s-1)$. Omit from $\bar{B}_i$ $[c^2\bar{\nu}s]$ vertices and let be among them all the vertices joined to at least one y_l. Trivially $\approx c^2\bar{\nu}$ vertices of $\bar{B}_i$ were selected arbitrarily, what will be useful later. Add these vertices to $\bar{C}_i$, thus we obtain $B_i \subseteq \bar{B}_i$ and $C_i \supseteq \bar{C}_i$. Easy to see that B_i and C_i are not joined by edges. Suppose the contrary: $x \in B_i$ is joined to $y \in C_i$. Then $x \in \bar{B}_j$ and $y \notin \bar{B}_i$, since B_i does not contain edges. Thus $y \in \bar{C}_i$, hence (x, y) connects $\bar{B}_i$ and C_i: either $x = x_l$ or $y = y_l$ and thus x is joined to an y_m, so x was omitted from $\bar{B}_i : x \notin B_i$. This contradicts the original condition.

(C_5) The classes B_i determine a $T^{\nu d,d}$ having the following properties: If $\tilde{S} = S^n - T^{\nu d,d}$ then the vertices of $\tilde{S}$ can be divided into the classes $C_1, \ldots, C_d$, D, E so that

(i)* If $x \in E$ then x is joined to all the vertices of $T^{\nu d,d}$ (since $T^{\nu d,d} \subseteq T^{\bar{\nu}d,d}$).

(ii)* Let $x \in C_i$. Then x is not joined to less than $^1/_2\, c\bar{\nu}$ vertices of $\bar{B}_j$ so it is not joined to less than $\frac{1}{2}c\bar{\nu} \approx \nu \cdot \frac{c}{2(1-c^2)}$ edges of B_j thus x is joined to at least $\left(1-\frac{1}{2}\cdot\frac{c}{1-c^2}\right)\nu$ vertices of $B_j (j \neq i)$. Similarly x is joined to at most $\frac{c^2}{1-c^2}\nu$ vertices of B_i.

(iii)* Let $x \in D$. Then there are $\bar{B}_i$ and $\bar{B}_j$ such that x is joined to less than $c^2\bar{\nu}$ vertices of B_i and to less than $\left(1-\frac{1}{2}c\right)\bar{\nu}$ vertices of B_j. Therefore, x is joined to B_i by less than $c^2\nu \approx \frac{c^2}{(1-c^2)^2}\nu$ edges and to B_j by less than $\left(1-\frac{1}{2}c\right)\tilde{\nu} \approx \left(1-\frac{1}{2}c\right)\cdot\frac{1}{1-c^2}\nu$ edges. Further, if $|D| \neq 0$ then there are $x \in D$ and two classes B_{i_0} and B_{j_0} such that x is not joined to any vertices of $B_{i_0} \cup B_{j_0}$. Thus we obtained a graph $T^{\nu d,d}$ and the classes $C_1, \ldots, C_d, D, E$ which satisfy the conditions (i), (ii), (iii) and (v) not in their original form but in a little bit modified form: the constants are others in it. However, if c is small, these differences make no change in our proof, thus we need not notice this difference.

The only problem is that generally $\tilde{S}$ does not satisfy (iv), i. e. $\tilde{S}$ is "good". Now it will be shown that if $T^{hd,d}$ is selected in a suitable way, $\tilde{S}$ will be "bad". If we knew this, we would have proved the Theorem.

We will find our $T^{\nu d,d}$ in three steps.

Heuristically:

First we select a $T^{\nu_1 d,d}$ in S^n with $\nu_1 = O(1)$. If $\tilde{S}_1 = S^n - T^{\nu_1 d,d}$ is "bad", we are ready with our proof. If it is "good", then just because of this $\tilde{S}_1$ contains a $T^{\nu_2 d,d}$ where $\nu_2 \geq c_2 n$ and $c > 0$ is a constant. Denote by D_2 the class D corresponding to $\tilde{S}_2 = S^n - T^{\nu_2 d,d}$. If $|D_2|$ is great, then e_S is small and, since $\Delta(n) \geq 0$, we obtain $e(\tilde{S}_2) \geq e\,(H(n-\nu_2 d, d, s)$. Since $H(n-\nu_2 d, d, s)$ has more edges, than any other "good" $G^{n-\nu_2 d}$ has, thus $\tilde{S}_2$ is "bad" and this completes the proof. If $|D_2|$ is small, then we try to find another graph $T^{\nu_3 d,d}$ for which $|D_3|$ is great, but if we cannot do it, then we can modify (C_4) in the method (C) so that though $|D_2|$ is small, $\tilde{S}_2 = S^n - T^{\nu_2 d,d}$ is "bad". In details:

Select a $T^{M_0 d,d} \subseteq S^n$ and construct a $T^{\nu_1 d,d}$ from it using (C). If $\tilde{S}_1 = S^n - T^{\nu_1 d,d}$ is "bad", then we can apply (B) which completes the proof. If it is not, then by omitting $s-1$ suitable vertices of $\tilde{S}_1$ we obtain a G^{n_1} $(n_1 = n - \nu_1 d - s + 1)$ with $\chi(G^{n_1}) = d$. Since $e(G^{n_1}) \geq e(S^n) - O(n) = e(T^{n_1 d,d}) + O(n_1)$ we may apply Lemma 2: G^{n_1} contains a $T^{hd,d}$ where $h \geq c_1 n$ (and $c_1 > 0$ is a constant). Construct a $T^{\nu_2 d,d}$ from $T^{hd,d}$ using (C). Then $\nu_2 \geq c^{3S} c_1 n = c_2 n$, $(c_2 > 0)$. Put $M_2 = \frac{2}{c c_2 d}$. If D_2 is the class D corresponding to $T^{\nu_2 d,d}$ and it has more than M_2 vertices, then $\tilde{S}_2$ is "bad". It will be proved indirectly:

Suppose that $\tilde{S}_2$ is "good". Then $e(\tilde{S}_2) \leq e(H(n-\nu_2 d, d, s)$ since $H(n, d, s)$ has the maximum number of edges among the "good" graphs. Thus

$$(11)\qquad \Delta(n) = e(S^n) - e\big(H(n,d,s)\big) + \left[\frac{n}{d}\right] = e(\tilde{S}_2) + e_S + e(T^{\nu_2 d,d}) -$$

$$- e\big(H(n,d,s)\big) + \left[\frac{n}{d}\right] \leq \left[\frac{n}{d}\right] + e(T^{\nu_2 d,d}) - e\big(H(n,d,s)\big) -$$

$$- e\big(H(n-\nu_2 d, d, s)\big) + e_S.$$

From (9) and (11) it follows that

$$(12)\qquad \Delta(n) \leq \left[\frac{n}{d}\right] - (e_H - e_S)\,.$$

On the other hand

(13) $$e_S \leq (d-1)\,\nu_2 \cdot (n - \nu_2 d) - \frac{1}{2}|D_2|\,c\,\nu_2 = e_H - \frac{1}{2}|D_2|\,c\,\nu_2 .$$

Since $\Delta(n) \geq 0$ we have from (12) and (13) that

$$0 \leq \left[\frac{n}{d}\right] - \frac{1}{2}|D_2|\,c\,\nu_2 \leq \frac{n}{d} - \frac{1}{2}|D_2|\,cc_2\,n .$$

Thus $|D_2| \leq \dfrac{2}{cc_2\,d} = M_2$, which gives a contradiction.

Suppose now that $|D_2| \leq M_2$. Apply (C) to $T^{hd,d}$ slightly modifying it in (C_4):

Applying (C_1), (C_2), (C_3) to $T^{hd,d}$, we obtain a $T^{\bar{\nu}_2 d,d}$ and the classes E_2, D_2, $\bar{C}_{2i}$, $\bar{B}_{2i}$ $(i = 1, \ldots, d)$. Now we omit first only $[(s-1)c^2\,\bar{\nu}_2]$ vertices from $\bar{B}_{2i}$ and put them into $\bar{C}_{2i}$.

Thus we obtain the classes B_i^* and C_i^*. We do it so that the vertices of B_i^* are not joined to C_i^*. Now omit from B_i^* and put into C_i^* $\left[\frac{1}{2}c^2\,\bar{\nu}\right]$ other vertices. The obtained classes are denoted by B_i^{**} and C_i^{**}. Now we define the classes R_i:

Let $x \in R_i$ if $x \in C_i^{**}$ and there is a $j(x)$ such that x is joined to less than s vertices of $B_{j(x)}^* - B_{j(x)}^{**} = C_{j(x)}^{**} - C_{j(x)}^*$.

The following two cases will be distinguished:

Either $\left|\bigcup_i R_i\right| > \dfrac{C_2}{8s}\bar{\nu}_2$ or $\left|\bigcup_i R_i\right| \leq \dfrac{C^2}{8s}\bar{\nu}_2$.

If $\left|\bigcup_i R_i\right| \geq \dfrac{C^2}{8s}\bar{\nu}_2$ then we forget $T^{\nu_2 d,d}$ and construct a new graph as follows:

The classes $B_i^* - B_i^{**}$ determine a $T^{\left[\frac{1}{2}c^2\bar{\nu}_2\right]d,d}$. Apply (C) to this graph. The obtained $T^{\nu_3 d,d}$ will satisfy our conditions: $\tilde{S}_3$ is "bad". Since $\nu_3 \geq c_5 n$, if we knew $|D_3| > \dfrac{2}{cc_5\,d}$ as we have seen above, we should know that $\tilde{S}_3$ is "bad".

To show $|D_3| > \dfrac{2}{cc_5\,d}$ notice that $\bigcup_i R_i \subseteq D_3$ and $\left|\bigcup_i R_i\right| > \dfrac{C_2}{8s}\bar{\nu}_2 \geq$ $\geq C_6\,n$. $\bigcup_i R_i \subseteq D_3$ can be proved as follows: Let $B_{3,1}, \ldots, B_{3,d}$ be the classes of $T^{\nu_3 d,d}$. It may be supposed that $B_{3i} \subseteq C_i^{**} - C_i^*$. If $x \in R_k$ then x is not joined to $C_k^{**} - C_k^*$ and it is joined to less than s vertices of $C_{j(x)}^{**} - C_{j(x)}^*$. Thus $x \in D_3$, therefore $|D_3| \geq c_6\,n > \dfrac{2}{cc_5\,d}$ and from this we have that $\tilde{S}_3$ is "bad".

Lastly we must investigate the case $\left|\bigcup_i R_i\right| \leq \dfrac{C^2}{8s}\bar{\nu}_2$. In this case we apply the following trick:

If $x \in R = (\bigcup_i R_i) \cup E \cup D$ and x is joined to less than s vertices of B_i^{**} then let us put all these vertices from B_i^{**} into C_i^{**}. If it is joined to more than s vertices of B_i^{**} then put s arbitrary vertices, joined to x from B_i^{**} into C_i^{**}. Do this for every $x \in R$. Since $|R| > \frac{C^2}{8s}\bar{v}_2 + \frac{2}{cc_2 d} + S$, it can be done. After this put some other vertices of B_i^{**} into C_i^{**} so that if B_{2i}, C_{2i} denote the obtained classes, then $|\bar{B}_{2,i} - B_{2,i}| = |C_{2,i} - \bar{C}_{2,i}| = [c^2 \bar{v} s]$. These classes determine a $T^{v_2 d,d}$ and it will be shown that $\tilde{S}_2 = S^n - T^{v_2 d,d}$ is "bad". This will be shown by an indirect proof:

Suppose that $\chi(\tilde{S}_2 - \{u_1, \ldots, u_{s-1}\}) = d$ where $u_1, \ldots, u_{s-1}$ are suitable vertices of $\tilde{S}_2$. Then colour $\tilde{S}_2 - \{u_1, \ldots, u_{s-1}\}$ by "1", ..., "d" so that the colour of the vertices of $C_i^{**} - \bar{C}_{2,i}$ be "i". Notice that if $x \in C_{2,i} - R_i$ its colour is also i. Indeed, x is joined to at least s vertices of $\tilde{S}_2$, thus it is joined to at least one vertex of $C_k^{**} - C_k^* - \{u_1, \ldots, u_{s-1}\}$ having the colour "i". Thus its colour differs from "k". This is true whenever $k \neq i$, thus x has the colour "i". Colour now the vertices of B_{2i} by "i". Thus each vertex of $S^n - \{u_1, \ldots, u_{s-1}\}$ has a uniquely determined colour. It will be shown that this colouring is good colouring of $S^n - \{u_1, \ldots, u_{s-1}\}$. If we knew this, the proof should be complete: we should obtain that $\chi(S^n - \{u_1, \ldots, u_{s-1}\}) = d$, thus S^n is "good", which contradiction should prove that $\tilde{S}_2$ is "bad". This is just the statement to be proved.

Thus we show now that the considered colouring of $S^n - \{u_1, \ldots, u_{s-1}\}$ is a good colouring. Let x and y be two vertices in $S^n - \{u_1, \ldots, u_{s-1}\}$. It must be shown, that if both x and y have the colour "i", then they are not joined. Since $T^{v_2 d,d}$ and $\tilde{S}_2 - \{u_1, \ldots, u_{s-1}\}$ are well-coloured, we may assume that $x \in T^{v_2 d,d}$ and $y \in \tilde{S}_2 - \{u_1, \ldots, u_{s-1}\}$. If $y \notin R$, then $y \in C_{2,i} - R_i$ since its colour is "i". Thus x and y are not joined (a vertex of $C_{2,i}$ can not be joined to a vertex of $B_{2,i}$). The other case is, when $y \in R$. In this case according to the modification of (C_4) if y were joined to x then it were joined to at least s vertices of $C_i^{**} - C_i^*$ and, consequently, to at least one vertex of $C_i^{**} - C_i^* - \{u_1, \ldots, u_{s-1}\}$ which is also of the i-th colour. Since $\tilde{S}_2 - \{u_1, \ldots, u_{s-1}\}$ is well-coloured, this would be a contradiction, from which follows, that x and y are not joined. As we have remarked already, from this follows that $S^n - \{u_1, \ldots, u_{s-1}\}$ is well-coloured by S^n colours, thus it is a "good" graph and this contradiction gives the desired result: $\tilde{S}_2$ is "bad". Qu. e. d.

The structure of S^n in Theorem 3

Proving Theorem 3 we have eliminated all the difficulties of the stability problem of $T(rd, d, s)$. Our next purpose is to determine the structure of S^n. First we investigate some candidates for it.

Let $\Gamma(n, d, s, r)$ be the following graph. Join each vertex of a K_{s-1} to each vertex of a $T^{n-s+1,d}$. Thus we obtain an $H(n, d, s)$. Let $A_1, \ldots, A_d$ be the classes of $T^{n-s+1,d}$. We may suppose that $|A_1| = \left[\frac{n-s+1}{d}\right]$. Let now $x_1, \ldots, x_{r-1}, x$ be the vertices of A_2, $y_1, \ldots, y_{r-1}$ the vertices of A_1 and join

x to $x_1, \ldots, x_{r-1}$ further omit the edges joining x and vertices of A_2, except the edges (x, y_i) $i = 1, \ldots, r-1$. The graph obtained will be denoted by $\Gamma(n, d, s, r)$ or if d, s, r are fixed, denote it shortly by Γ^n. Clearly $e(\Gamma^n) = e(H(n, d, s)) - \left[\frac{n-s+1}{d}\right] + 2r - 2$.

It is easy to see that $\chi(\Gamma^n - K_{s-1}) = d + 1$, therefore $\chi(\Gamma^n) = d + s$ and consequently *Γ^n is a "bad" graph.*

Now it will be shown that *Γ^n does not contain $T(rd, d, s)$*. Suppose, it does. Omit K_{s-1} and x from it, then the remaining graph G^* is d-chromatic. Since $T(rd, d, s)$ is not contained by G^*, we omitted s vertices of $T(rd, d, s)$ such that the graph T^* obtained from $T(rd, d, s)$ is d-chromatic. This fact determines T^*: we had to omit s endpoints of the s (different) extra edges. The remaining graph is a complete d-partite graph, having $d - 1$ classes of r vertices and one class, containing r—3 vertices. Each class of G^* contains just one class of T^*. Thus either A_1 or A_2 contains a class of T^* having r vertices. These two cases do not differ essentially. Consider e.g. when the class of x contains a class of T^* containing r vertices. Since x is contained in $T(rd, d, s)$, x must be joined to this r vertices of T^*. This is impossible, since x is joined only to $r - 1$ vertices of A_2. This proves that Γ^n does not contain $T(rd, d, s)$.

Since Γ^n is a "bad" graph not containing $T(rd, d, s)$, therefore $e(\tilde{S}^n) \geq e(\Gamma^n)$ and consequently $\Delta(n) > 0$. (This completes the proof of the statement that $\Delta(n) = O(1)$. Thus Theorem 3 cannot be improved essentially.) After having this construction, one may conjecture that $\Gamma(n, d, s, r)$ is an extremal graph. But generally it is not true and this follows immediately from the properties of the graph $\Sigma(n, d, s, r) = \Sigma^n$ defined as follows:

Consider an $H(n, d, s-1)$ obtained from a $T^{n-s+2,d}$ and a K_{s-2}. Let x_{ij} be r vertices in the i-th class of $i = 1, \ldots, d$ $j = 1, \ldots, r$ and join x_{i1} to $x_{i2}, \ldots, x_{ir}$. Denote by Σ^n the obtained graph.

Trivially, $e(\Sigma^n) = e(H(n, d, s-1)) + d \cdot (r-1)$. Here $d \geq 2$ and thus *Σ^n is "bad"*. This statement can be proved easily. It is a little more difficult to show that *Σ^n does not contain $T(rd, d, s)$*. If $d + s - 2 < r$, the method used to prove that Γ^n does not contain $T(rd, d, s)$ works also, but it breaks down if $d + s - 2 \geq r$.

This statement can be proved in the general case in the following indirect way. Suppose that $T(rd, d, s) \subseteq \Sigma^n$. Let be $A_1, \ldots, A_d$ the classes of the $T^{n-s+2,d}$ of Σ^n. A_i may contain at most r vertices of $T(rd, d, s)$: either a whole class B_j, or a vertex of B_j which coincide with x_{i1} and less than r vertices from another B_k. This statement is the trivial consequence of the fact that all the edges joining two vertices of A_i contain x_{i1} as endpoint, and that if $x \in B_j$, $y \in B_k$, then x and y are joined. (A_i cannot contain vertices from 3 different B_j-s, since it does not contain any triangle.)

Hence we may r estrict our investigation to the case of Σ^{rd+s-1} i.e. when $|A_i| = r$. Denote by E the class consisting of the vertices of K_{s-2} and of the vertices x_{i1} $(i = 1, \ldots, d)$. It will be shown that by our hypothesis each B_j contains at least one vertex of E and if B_1 is the class containing the extraedges of $T(rd, d, s)$, then $|B_1 \cap E| \geq s$. Thus we shall have $|E| \geq d - 1 + s$ and this contradiction will prove our assertion.

Suppose that B_1 contains t vertices of K_{s-2} $(t \geq 0)$. Then $B_1 - (K_{s-2} \cap B_1)$ must contain at least $s - t \geq 2$ independent edges.

(α) Suppose, that there exist A_i and A_j each of which contains at least 2 vertices of B_1. Then both A_i and A_j contain at most one vertex from the other B_k's. Thus $A_i \cup A_j$ contains at most 2 other vertices of $T(rd, d, s)$ and from this follows that $(s - 2) + (d - 2)r + 2$ vertices of Σ^{rd+s-1} must contain $\geq r\,(d - 1)$ other vertices of $T(rd, d, s)$. Clearly this is impossible since $s \leq \frac{r}{2} < r$.

(β) Now it may be supposed that there is only one A_i containing at least two vertices of B_1. Moreover, it may be supposed that the other vertices of B_1 are certain x_{i1}-s, otherwise we should have the same contradiction as in (α). We obtain from this, that B_1 contains at least s vertices of E since it contains s independent edges.

Consider now another B_j, $(j \geq 2)$. If there is an A_i containing B_j, then $A_i = B_j$ and thus B_i contains just one vertex of E. If there is not such an A_i, then it can be shown by the method used in (α) that there is an A_k containing just one vertex of B_i, moreover $A_k \cap B_i = x_{i_1} \in E$. Thus we have proved that each B_i contains at least one vertex of E and B_1 contains at least s vertices of E. As we have seen already this is a contradiction which completes the proof of our statement.

Thus we obtained a second counter-example showing that $\Delta(n) = e(S^n) - e\big(H(n, d, s)\big) + \left[\frac{n}{d}\right]$ is bounded from below. Sometimes $e(\Sigma^n) > e(\Gamma^n)$.

The following modification of Σ^n will also be needed in the special case $s = 2$, $r = 4$:

Instead of putting 3 edges (x_{i1}, x_{il}) $l = 2, 3, 4$ into B_i put a triangle (x_{i1}, x_{i2}, x_{i3}) in it (for certain values of i). It is easy to verify that *the graph obtained* $\tilde{\Sigma}^n$ *is "bad"* and the method used above gives that *it does not contain* $T(4d, d, 2)$. The only new idea of this proof is that three classes may exist e.g. B_1, B_2, B_3, such that A_i contains vertices from each of them. But then, clearly, these vertices must be the vertices of a triangle in A_i. Thus A_i does not contain other vertices from $T(rd, d, s)$. From this we have the following contradiction: $4d - 4$ vertices of $\tilde{\Sigma}^n$ contain $4d - 3$ vertices of $T(4d, d, 2)$.

Now we have seen all the candidates for S^n and it will be proved that S^n is really one of them (where in the case of Γ^n S^n may differ from Γ^n having classes which contain more than $\left[\frac{n}{d}\right] + 1$ or less than $\left[\frac{n}{d}\right]$ vertices.)

In the proof of Theorem 3 we used mathematical induction to show that $\Delta(n)$ is bounded from above. It is known already that $\Delta(n)$ is also bounded from below, moreover, that in the proof of Theorem 3 $\Delta(n) \geq 0$ always holds. Thus the proof gives us the existence of an n' such that $\frac{n}{2} < n' < n$ and $\Delta(n') \geq \Delta(n)$. Clearly, if n is sufficiently large, $\Delta(n') = \Delta(n)$ must hold just because of $\Delta(n') \geq \Delta(n)$ and $\Delta(n) = O(1)$. This means that S^n have either the property described in Remark A or the property described in Remark B: If $T^{rd,d}$ is a suitable subgraph of S^n and $C_1, \ldots, C_d, D, E$ are the well-known classes of $\tilde{S} = S^n - T^{rd,d}$, then each vertex of C_i is joined to each vertex of B_j, whenever $i \neq j$ and either

(A) $|E| = s - 2$ and D is empty, or

(B) $|E| = s - 1$ and $D = \{x_0\}$, where x_0 is not joined to B_1 and B_2 but it is joined to all the other vertices of $T^{rd,d}$.

Consider first (B) and prove that in this case S^n has the same structure as Γ^n.

When we proved that if $|E| = s - 1$, then D was not empty, we also saw that in this case $B_i \cup C_i$ did not contain edges. Since S^n is "bad", x must be joined to each C_i', otherwise $S^n - E$ would be d-chromatic. Denote by $x_1, \ldots, x_l$ the vertices of C_1, by $y_1, \ldots, y_n$ the vertices of C_2 joined to x, respectively.

If $u \in B_i \cup C_i$ $V \in B_j \cup C_j$, but v does not coincide with any x_k or y_k, and $u \neq x_0$, then u and v are joined. In order to show this suppose the contrary and join them. The new graph S^* must contain a $T(rd, d, s)$ and this $T(rd, d, s)$ is not contained by S^n. Thus it contains the edge (u, v). But change v on a $v^* \in B_j$, Since v^* is joined to all the vertices which are joined to v, we obtain a new subgraph $G^{rd} \subseteq S^n$ which contains a $T(rd, d, s)$. This contradiction shows that u and v are joined in S^n. Similarly, $u \in E$ is joined to all the vertices of S^n except may be to x_0. Easy to see now that *y_k is joined at most to $r - 1$ x_k*: if y_1 were joined to $x_1, \ldots, x_r$ then x_0, y_1 and the vertices of E with $r - s - 1$ arbitrary vertices of B_1 and the vertices $y_1, \ldots, y_d$ from C_2 and finally r arbitrary vertices of B_j $j = 3, \ldots, d$ would determine a $T(rd, d, s) \subset S^n$. Similarly: *an x_h is joined at most to $r - 1$ y_h*. A short computation shows that S^n has a maximum number of edges if *x_0 is joined just to $r - 1$ vertices of B_1 and $r - 1$ vertices of B_2 and all the vertices of E are joined to x_0*. Thus we have proved that S^n has the same structure as Γ^n. In the case of the original problem, when $s = 1$ then $|E| = s - 2$ is impossible. Thus S^n has really the structure of Γ^n. In general it is also a possible version that $|E| = s - 2$ and $|D| = 1$.

Then by the method used in the proof of Theorem 3 it can be proved that $B_i \cup C_i$ does not contain two independent edges. It is known from Remark A that each vertex of C_i is joined to each vertex of B_j, $(i \neq j)$.

(i) First suppose, that only one class $B_i \cup C_i$ contains edges. Since D is empty and $\tilde{S}$ is "bad", these edges cannot have a common endpoint. This fact and the fact that $B_i \cup C_i$ does not contain 2 independent edges, give that the edges in $B_i \cup C_i$ form a triangle.

Thus S^n has less edges than Σ^n what disproves that S^n is an extremal graph. This is a contradiction.

(ii) Therefore we may assume that $B_1 \cup C_1$ and $B_2 \cup C_2$ contain edges. Denote by Q_i the class of the vertices of $B_i \cup C_i$ joined to another vertex of $B_i \cup C_i$. Then *all the vertices of $C_i \cup B_i$ are joined to all the vertices of $(C_j \cup B_j) - Q_j$, $(j \neq i)$*. To show this suppose the contrary: suppose that $u \in (C_j \cup B_j) - Q_j$ and $v \in C_j \cup B_j$ are not joined. Join them. The obtained S^* contains a $T(rd, d, s)$ since S^n is extremal graph and S^* trivially "bad". This $T(rd, d, s)$ contains (u, v). Change u on a $u^* \in B_i$ in $T(rd, d, s)$. Since

u^* is joined to all the vertices joined to u thus we obtain a $T(rd, d, s) \subseteq S^n$. This contradiction shows that u and v are joined in S^n. Similarly, *if* $v \in E$, $u \in (C_i \cup B_i) - Q_i$, *then* u *and* v *are also joined.*

(iii) Now we prove that $e(S^n) \leq e(\Sigma^n)$.

Suppose the contrary. From $e(S^n) > e(\Sigma^n)$ easily follows that there are an $i \leq d$ and $r + 1$ vertices, $v, v_1, \ldots, v_r \in B_i \cup C_i$, such that v is joined to $v_1, \ldots, v_r$. According to (ii) there is a $B_j \cup C_j$ which contains an edge (u_1, u_2). If both u_1 and u_2 were joined to all the v_i's then u_1, u_2, v, E and $r - s - 1$ vertices of B_j, $v_1, \ldots, v_r$ and r vertices of $B_k (k \neq i, k \neq j)$ would determine $T(rd, d, s) \subseteq S^n$. Therefore it can be assumed that u_1, and v_1 are not joined. Omit (v_1, v) from S^n and join u_1 to v_1. The argument used in (ii) shows also that the obtained S^n does not contain $T(rd, d, s)$. Since $S^n - (v_1, v)$ does not contain $T(rd, d, s)$ clearly S_1^n is "bad". Since $e(S_1^n) \geq (S^n) > e(\Sigma^n)$ we may construct an S_2^n from S_1^n repeating the argument (iii), and then the graphs $S_3^n, \ldots, S_k^n$ such that $e(S_i^n) > e(\Sigma^n)$. This process does not stop. But on the other hand the sum of edges contained in $B_i \cup C_i$ $(i = 1, \ldots, d)$ is greater in the case of S_k^n than in the case of S_{k+1}^n, which gives that the sequence of $S_1^n, \ldots, S_k^n$ must be finite. This contradiction shows that $e(S^n) \leq$ $\leq e(\Sigma^n)$. Thus if $|E| = s - 2$ then Σ^n is an extremal graph for the stability problem of $T(rd, d, s)$.

Now we must only decide, whether Σ^n of Γ^n has more edges. If $d \geq 3$, $e(\Sigma^n) > e(\Gamma^n)$ but in the case $d = 2$ $e(\Gamma^n) = e(\Sigma^n)$. An easy discussion of our proof shows that if $d \geq 3$, $r \geq 5$, Σ^n is the only extremal graph S^n. If $d \geq 3$ but $r = 4$ (and consequently $s = 2$), the $\widetilde{\Sigma^n}$ is also an extremal graph, but there are no other extremal graphs. If $d = 2$, there are also many other extremal graphs.

The problem of s independent K_p

PROBLEM. *What is the maximum number of edges a graph can have if it does not contain s independent K_p?*

Put $d = p - 1$. We have seen, that the "good" graphs do not contain s independent K_{d+1}. J. W. MOON has proved [3], generalizing some results of P. ERDŐS and T. GALLAI, [4], [5], that $H(n, p - 1, s)$ *is the extremal graph for the problem of s independent K_{d+1} if n is large enough.*

This result is an easy consequence of Theorem 3, moreover:

THEOREM 4. Suppose that n is sufficiently large. *Each graph having more than $e(\Gamma^*(n, d, s, t)) + 2$ edges, and not containing s independent K_{d+1} is a "good" graph, where $\Gamma^*(n, d, s, t)$ is the following graph:*

Consider an $H(n, d, s)$ and let x, y be two vertices in its $T^{n-s+1, d}$ belonging to the same class and be $z_1, \ldots, z_k$ the vertices of another (minimal) class of it. Omit the edges joining $z_1, \ldots, z_t$ to x and $z_{t+1}, \ldots, z_{t+2}$ to y, lastly join x and y.

REMARKS. 1. This graph $\Gamma^*(n, d, s, t)$ is the generalization of the graph Γ^n introduced in the last part of the third paragraph, about which it is known that it is $d + 1$-chromatic but does not contain K_{d+1}.

2. Since $e(\Gamma^*(n, d, s, t)) = e(H(n, d, s)) - \left[\frac{n-s+1}{d}\right] + 1$, trivially, it follows from Theorem 4 that if G^n does not contain s independent K_{d+1}, then $e(G^n) \leq e(H(n, d, s))$.

PROOF. (A) Clearly $\chi(\Gamma^*(n, d, s, t)) = \chi(\Gamma^*(n, d, s, t) - K_{s-1}) + s - 1 = d + s$. Thus Γ^* *is a "bad" graph.* If it contained s ,independent K_{d+1}, then $\Gamma^* - K_{s-1}$ would contain at least one K_{d+1}, but it does not contain, thus Γ^* does not contain s independent K_{d+1}. Thus Γ^* shows that Theorem 4 is sharply apart from the constant 2. Now we prove Theorem 4. Notice, that in the proof of Theorem 2 we used only that S^n is an extremal graph of a given problem, and that it does not contain $T(rd, d, s)$. Let S^n denote now the extremal graph for the problem of s independent K_{d+1}, then S^n does not contain $T(rd, d, s)$ either. Therefore the proof of Theorem 2 remains also valid in this case. (It remains also valid in every case when $F_1, \ldots, F_l$ are such that $\Gamma(rd, d, s)$ contains at least one of them, but $H(n, d, s)$ does not contain any F_i. However, in this case $\Delta(n)$ may tend to $-\infty$, i.e. generally this result will not be the best possible.)

Since the proof of Theorem 2 remains valid for s independent K_{d+1} and Γ^* proves that $\Delta(n) = e(S^n) - e(H(n, d, s)) + \left[\frac{n}{d}\right] > 0$, we may determine the extremal graphs in the stability-problem of s independent K_{d+1} by the same method, as we did in the case of $T(rd, d, s)$.

Using the well-known notations:

(a) Suppose that $|D| = 0$, $|E| = s - 2$ and each vertex of B_i is joined to each vertex of C_j, $(i \neq j)$.

In this case $B_i \cup C_i$ does not contain two independent edges. This has been proved already (see the case of $T(rd, d, s)$). However, here we know much more. *If $B_i \cup C_i$ contains an edge, and $j \neq i$, then $B_j \cup C_j$ does not contain edges*, otherwise S^n would contain s independent K_{d+1}. Since S^n is "bad", there is just one class $B_{i_0} \cup C_{i_0}$ containing edges. We cannot omit any vertex x of $B_{i_0} \cup C_{i_0}$ such that x is the endpoint of all the edges contained in $B_{i_0} \cup C_{i_0}$. Thus $B_{i_0} \cup C_{i_0}$ contains 3 edges forming a triangle. It is easy to see that in this case *S^n can be obtained from an $H(n, d, s-1)$ by putting a K_3 into a class of its $T^{n-s+2, d}$*. This graph Σ^* has $e(\Gamma^*) + 2$ edges, thus in this case Theorem 4 is true and is best possible.

(b) The other case which must be investigated is when $|E| = s - 1$, $D = \{x_0\}$ and x_0 is joined to all the vertices of $T^{vd, d} - B_1 - B_2$ but it is not joined to $B_1 \cup B_2$. Further, each vertex of C_i is joined to each vertex of B_j.

Denote by $x_1, \ldots, x_l$ the vertices of B_1 by $y_1, \ldots, y_m$ the vertices of B_2 joined to x_0, respectively. Then x_k and y_h *are not joined.* An easy argument shows that this graph S^n has at most as many edges as a Γ^* and the equality holds only if $S^n = \Gamma^*(n, d, s, t)$ for a suitable t or it has the same structure as $\Gamma^*(n, d, s, t)$ only, maybe, $S^n - K_{s-1}$ has also classes of more than $\left[\frac{n}{d}\right] + 1$ and less than $\left[\frac{n}{d}\right]$ vertices. This proves completely our statement.

REMARK. It can be asked that if $e(\Gamma^*) = e(\Sigma^*) - 2$, why is $e(\Gamma^*) + 2$ in Theorem 4 stated instead of $e(\Sigma^*)$. The answer is that in one of the most important cases, i.e. in the case $s = 1$ Σ^* does not exist.

The problem of $Q(r, d)$

This paragraph contains the solution of an extremal graph problem similar to the problem of $T(rd, d, s)$.

$Q(r, d)$ denotes the graph of $rd + 1$ vertices such that omitting a suitable vertex of it having valency rd there remains a graph $T^{rd, d}$. The omitted vertex is uniquely determined by $Q(r, d)$. It will be called the extra vertex.

PROBLEM. *Determine the maximum number of edges a graph G^n can have if it does not contain $Q(r, d)$.*

This problem was posed by P. ERDŐS in connection with a geometrical problem of the four-dimensional Euclidean space. To solve this geometrical problem ERDŐS needed the problem above in the special case $r = 3$, $d = 2$. An unpublished result of ERDŐS states that the extremal graph for $Q(3, 2)$ can be obtained from $T^{rd,d}$ adding edges to it so that each vertex is joined with 2 other vertices of its class.

In connection with my method ERDŐS asked me whether it works in the case of $Q(3, 2)$. I solved this problem and not only in this special case but in the general case too.

For the sake of simplicity this paper contains only the solution of the case when r is odd. The case when r is even makes non-essential difficulties only because there do not exist regular graphs of order $r - 1$ of $n = 2k + 1$ vertices. (In the case, when r is even, the regular graphs of order $r - 1$ must sometimes be replaced by graphs having vertices of valence $r - 1$ except one vertex which has valency $r - 2$. Then all our results remain valid.)

Let r be a given odd integer, d arbitrary and denote by $U^n = U(n, d)$ the following graph.

Put edges into each class of a $T^{n, d}$ so that any vertex of $T^{n, d}$ be joined just to $r - 1$ other vertices of the same class. The graph U^n obtained thus is not uniquely determined. $\mathcal{U}_n$ denotes the class of these graphs U^n. If the edges are put into $T^{n, d}$ so that no class of $T^{n, d}$ contains a triangle, then let $U^n \in \mathcal{U}_n^*$.

THEOREM 5. *Let n be large enough. Then $\mathcal{U}_n^*$ is not empty and all the graphs of $\mathcal{U}_n^*$ are extremal graphs of the problem of $Q(r, d)$. On the other hand, all the extremal graphs of $Q(r, d)$ (having n vertices) belong to $\mathcal{U}_n$.*

PROOF. (A) Let m be large enough, and let r be an odd positive integer. There exist regular graphs of order $r - 1$ without triangle and having m vertices.

(A_1) If $m = 2k$ let $x_1, \ldots, x_k, y_1, \ldots, y_k$ be $2k$ different vertices. Join x_i with $y_i, y_{i+1}, \ldots, y_{i+r-2}$ (here $y_{j+r} \equiv y_k$). The resulting graph is 2-chromatic regular graph of order $r - 1$ and trivially, without triangles.

(A_2) If $m = 2k + 3$, consider the graph of (A_1). Denote it by G^{2k}. It contains $3/2(r - 1)$ independent edges such that the endpoints of these edges are independent vertices of G^{2k}. Split these edges into $1/2\,(r - 1)$ disjoint classes each of which contains just 3 edges: e_{i1}, e_{i2}, e_{i3}. Let u, v, w be 3 new vertices. Join u to an endpoint of e_{i1} and to an endpoint of e_{i2} ($i = 1\ 2, \ldots, 1/2\ (r-1)$). Then do the same with v, e_{i2}, e_{i3} and with w, e_{i3}, e_{i1}. Thus we obtain a regular graph of order $r-1$ and it does not contain any triangles. Indeed, if (x, y, z) were a triangle in it, then omitting u, v and w we would obtain a 2-chromatic graph G^{2k} (without triangles). Thus $\{u, v\ w\}$ must contain at least one of x, y and z. It may be supposed that $u = x$. Then y and z are vertices joined to u but all the vertices joined to u are independent: y and z are independent. Hence (x, y, z) is not a triangle. This contradiction proves (A). Thus $\mathcal{U}_n^*$ is not empty.

(B) It will be proved now that *the graphs of* $\mathcal{U}_n^*$ *do not contain* $Q(r, d)$. Apply mathematical induction on d. If $d = 1$, (B) is trivial. (Here $d = 1$ is allowed, in other parts it is prohibited.) It will be shown that if the statement is not true for d, then it is not true for $d - 1$, either. Suppose that $U_n \in \mathcal{U}_n^*$ contains a $Q(r, d)$. It may be supposed that the extra vertex $x^* \in Q(r, d)$ is in the first class A_1 of the $T^{n,\,d} \subseteq U^n$ Because of the definition of U_n x is joined to $r-1$ other vertices of its class. Denote them by $x_1, \ldots, x_{r-1}$. These are independent vertices since the classes of $T^{n,\,d}$ do not contain any triangle. Thus they belong to the same class B_i of $T^{rd,d} \subseteq Q(r, d)$ (Maybe not all of them belong to $Q(r, d)$.) At least one vertex of this class B_i is not contained in A_1. Thus the other classes $A_2, \ldots, A_d$ of U^n contain $d-1$ classes B_i ($j \neq i$) of $Q(r, d) - \{x^*\}$ and a vertex $x' \in B_i - A_1$ joined to each vertex of these B_j-*s*. But this proves just the existence of an U' which contains a $Q(r, d-1)$ where $U' = U'\left(n - \left[\frac{n}{d}\right],\ d-1, r\right)$. Thus the lemma is not true for $d - 1$ either. This proves our statement (B).

(C) Let $c = \frac{1}{5r}$, $M = 10r^2$. Denote by V^n an extremal graph of n vertices. We want to prove that if n is sufficiently large, then $V^n \in \mathcal{U}_n$. This will complete the proof, since all the graphs of $\mathcal{U}_n$ have the same number of edges, thus the graphs of $\mathcal{U}_n^*$ are all extremal graphs, indeed.

This statement will be proved by progressive induction.

Let $\tilde{U}^n$ be a special graph of $\mathcal{U}_n^*$ the classes of which contain disconnected regular graphs of order $v-1$, each of which has a component of M vertices. Thus $\tilde{U}^n \in \mathcal{U}_n^*$ contains a $U^{n-Md} \in \mathcal{U}_{n-Md}^*$ such that $\tilde{U}^n - U^{n-Md} = U^{Md} \in \mathcal{U}_{Md}^*$. (Since M is even, there are regular graphs of order $r-1$ having M vertices. Thus $\tilde{U}^n$ exists if n is large enough.)

Clearly the subgraphs U^{Md} and U^{n-Md} are joined by

$$e_u = (n - Md) \cdot (d-1) \cdot M \tag{16}$$

edges in $\tilde{U}^n$ and

$$e(U^n) = e(U^{Md}) + e_u + e(U^{n-Md})\,. \tag{17}$$

Since $e(V^n) \geq e(\tilde{U}^n) > e(T^{n,\,d})$ thus V^n contains a $T^{Md,\,d}$ if n is large enough. Put $\tilde{V} = V_u - T^{Md,\,d}$ and divide the vertices of $\tilde{V}$ into the following classes:

B_i denotes the i-th class of $T^{Md,d}$. If $x \in \tilde{V}$ then there is a $B_{i(x)}$ such that x is joined to less than r vertices of $B_{i(x)}$. Indeed, if x were joined to at least r vertices of each B_j then x and rd vertices of B_i-s joined to x would form a $Q(r, d)$ in V^n. This contradicts the definition of V^n. If the considered x is joined by more than $(1 - c)M$ edges to B_j whenever $j \neq i(x)$, then let $x \in C_i$. (In this case $i = i(x)$ is uniquely determined by x). In the other cases, when there is at least one B_j, $(j \neq i(x))$ such that x is joined to at most $(1 - c)M$ vertices of B_j then let be $x \in D$. Hence $\tilde{V}$ is the disjoint union of $C_1, \ldots, C_d, D$. Denote by c_v the number of edges joining $\tilde{V}$ and $T^{Md,d}$. If V^* is the subgraph of V^n spanned by the vertices of $T^{Md,d}$, then $e(V^*) \geq$ $\geq e(T^{Md,d})$ and

$$(18) \qquad e(V^n) = e(V^*) + e_v + e(\tilde{V}) \geq e(\tilde{V}) + e_v + e(T^{Md,d}) .$$

Put $\Delta(n) = \Delta(V^n) - \Delta(\tilde{U}^n)$. $\Delta(n)$ depends only on n and it is a non-negative integer. If $V^n \in U_n$ then $\Delta(n) = 0$. From (17) and (18) it follows that

$$\begin{aligned}\Delta(n) - \Delta(n - Md) &= \big(e(V)^n - e(\tilde{U}^n)\big) - \big(e(V^{n-Md}) - e(\tilde{U}^{n-Md})\big) = \\ &= \big(e(V^n) - e(V^{n-Md})\big) - \big(e(\tilde{U}^n) - e(U^{n-Md})\big) = \\ &= \big(e(\tilde{V}) - e(V^{n-Md})\big) + e_v + \big(e(V^*) - e(U^{Md})\big) - e_u .\end{aligned}$$

Thus

$$(19) \ \Delta(n) - \Delta(n - Md) = \big(e(\tilde{V}) - e(V^{n-Md})\big) + \big(e(V^*) - e(U^{Md})\big) + + (e_v - e_u)$$

There is an M_1 such that

$$(20) \qquad |e(V^*) - e(V^{Md})| \leq M_1 .$$

Further, since $\tilde{V}$ does not contain $Q(r, d)$

$$(21) \qquad e(\tilde{V}) - e(V^{n-Md}) \leq 0 .$$

It will be proved that if n is large enough

(a) either $\Delta(n) < \Delta(n - 1)$,
(b) or $\Delta(n) < \Delta(n - Md)$,
(c) or $V^n \in \mathcal{U}_n$.

This will complete our progressive induction.

(a) If there is a vertex $x \in V^n$ having valency less than $\frac{n}{d}(d - 1)$, then $\Delta(n) < \Delta(n - 1)$:

Let $V^{**} = V^n - \{x\}$. It does not contain $Q(r, d)$, thus $e(V^n) -$ $- \sigma(x) = e(V^{**}) \leq e(V^{n-1})$ and from this

$$e(V^n) - e(V^{n-1}) \leq \sigma(x) < \frac{n}{d}(d - 1) < e(\tilde{U}^n) - e(U^{n-1})$$

$$\left(= n - \left[\frac{n}{d}\right] + \frac{r - 1}{2}\right).$$

This inequality gives just the required result.

Suppose now that neither (a) nor (b) hold: each $x \in V^n$ has valency at least $\frac{n}{d}(d-1)$ and $\Delta(n) \geq \Delta(n - Md)$. From this and from (19), (20), (21) it follows that

$$0 \leq \Delta(n - Md) - \Delta(n) \leq (e_u - e_v) + M_1 . \tag{22}$$

We shall prove in five steps that $V^n \in \mathcal{U}_n$.

(i) (22) gives possibility to estimate $|D|$. First we remark that $B_i \cup C_i$ *does not contain such a vertex which is joined to r other vertices of it.* If $B_i \cup C_i$ contained an x and $x_1, \ldots, x_r$ such that x is joined to each x_i, then these vertices and r suitable vertices each of B_j $(j \neq i)$ would determine a $Q(r, d) \subseteq \subseteq V^n$. ("Suitable" means that it is joined to each x_l and to x, too.) Thus the number of edges joining B_i and C_i less than $M \cdot (r-1)$ and

$$e_v \leq (n - d \cdot M) \cdot (d-1) \cdot M + M \cdot (r-1)\, d - |D| \cdot r = \\ = e_u + M \cdot (r-1) \cdot d - |D| \cdot r \tag{23}$$

since a vertex of D is joined to less than $(d-2)M + r + (1-c)M = (d-1)\, M - r$ vertices of $T^{Md,d}$. With the help of (23) and (22) we obtain

$$|D| \leq \frac{1}{r}\big(e_v - e_u + M(r-1) \cdot d\big) \leq \frac{1}{r}\big(M_1 + M(r-1)\, d\big) = M_2 . \tag{24}$$

Thus $|D|$ *is bounded.*

(ii) We have also proved that *a vertex belonging to $B_i \cup C_i$ is joined at most to r other vertices of $B_i \cup C_i$.*

(iii) $|B_i \cup C_i| = \frac{n}{d} + O(\sqrt{n})$. In order to show this omit the edges joining two vertices of the same $B_i \cup C_i$ $(i = 1, \ldots, d)$ and the edges of D. Thus there remains a $G^{n-|D|}$ which is d-chromatic and has $e(T^{n,d}) - O(n)$ edges. Applying Lemma 1 to $G^{n-|D|}$ we obtain the required result. Thus there is a constant M_3 such that

$$\left| |B_i \cup C_i| - \frac{n}{d} \right| \leq M_3 \sqrt{n} .$$

(iv) There is a constant M_4 such that every $x \in B_i \cup C_i$ is joined to all the vertices of $V^n - (B_i \cup C_i)$ except less than $M_4 \sqrt{n}$ vertices. This follows immediately from the fact that x is not joined at least to $\frac{n}{d} - M_2 \sqrt{n} - r$ vertices of $B_i \cup C_i$ but $\frac{n}{d}(d-1) \leq \sigma(x) \leq n$ $\big(\sigma(x)$ denotes the valency of $x\big)$.

(v) Now we prove that $V^n \in \mathcal{U}_n$ (which was to be shown). The vertices of V^n will be partitioned into d classes such that each vertex will be joined to less than r other vertices of its class. Suppose, it has been done already. Then trivially $e(V^n) \leq e(\widetilde{U}^n)$ and $e(V^n) = e(\widetilde{U}^n)$ if and only if $V^n \in \mathcal{U}_n$. But $e(V^n) \geq e(\widetilde{U}^n)$ since V^n is an extremal graph for $Q(r, d)$ and $\widetilde{U}^n$ does not contain $Q(r, d)$. Thus $V^n \in \mathcal{U}_n$.

Let us see now the partition mentioned above.

The classes $B_i \cup C_i$ are not good for our purpose only because $\bigcup_i (B_i \cup C_i)$, maybe, does not contain all vertices of V^n. Therefore we classify only the vertices belonging to D. Let D_i be the class of those vertices, which are joined to $B_i \cup C_i$ by less than r edges.

First it will be shown that D is the disjoint union of $D_1, \ldots, D_d$.

(α) $D_i \cap D_j$ is empty since from $x \in D_i \cap D_j$ would follow $\sigma(x) = (d-2)\frac{n}{d} + O(\sqrt{n}) < (d-1)\frac{n}{d}$.

(β) $D = \bigcup_i D_i$. Indeed, let $x \in D$ and n_i be the number of vertices of $B_i \cup C_i$ joined to x. It may be supposed that $n_1 \leq n_i$. Under this assumption it will be proved that $x \in D_1$. Suppose the contrary. Then $n_1 \geq r$. From $\sigma(x) \geq \frac{n}{d}(d-1)$ follows that $n_i > \frac{1}{3}\frac{n}{d}$ if $i \neq 1$ otherwise $\sigma(x) \approx \sum n_i < (d-2)\frac{n}{d} + O(\sqrt{n}) + \frac{2}{3}\cdot\frac{n}{d}$ would hold. Now we select $rd+1$ vertices from V^n determining a $Q(r, d)$ in it: x be the extra vertex of it, and select r vertices of $B_1 \cup C_1$ joined to x. Then select r vertices in $B_2 \cup C_2$ joined to x and to the r vertices considered in $B_1 \cup C_1$. Let us continue this selection and lastly select r vertices of $B_d \cup C_d$ joined to x and all the $r(d-1)$ vertices selected from $B_1 \cup C_1, \ldots, B_{d-1} \cup C_{d-1}$. (It is always possible to do this since each vertex selected from $B_1 \cup C_1, \ldots, B_i \cup C_i$ is joined to at least $\frac{n}{d} - O(\sqrt{n})$ vertices of $B_{i+1} \cup C_{i+1}$ and x is joined to at least $\frac{1}{3}\frac{n}{d}$ vertices of $B_{i+1} \cup C_{i+1}$). These $rd+1$ vertices determine a $Q(r, d)$ in V^n contradicting the definition of V^n. Thus, indeed, D is the disjoint union of $D_1, \ldots D_d$. Consequently V^n is the disjoint union of the classes $E_i = B_i \cup C_i \cup D_i$. The only thing needed to be shown is that if $x \in E_i$, then x is joined to less than r other vertices of E_i. But it is known that $x \in D_i$ is joined to less than $r + M_2$ vertices of $B_i \cup C_i$ and $\sigma(x) \geq \frac{n}{d}(d-1)$, thus x is joined to all except maybe to $O(\sqrt{n})$ vertices of $V^n - B_i \cup C_i$. Therefore supposing that a vertex $x \in E_i$ is joined to r other vertices of E_i it is easy to construct a $Q(r, d)$ in V^n. This contradiction proves that every $x \in E_i$ is joined to at most $r-1$ other vertices of E_i. As we have remarked already, it follows from this result that $V^n \in U_n$.

The stucture of the extremal graphs in the general case.
The stability theorem of the general problem

Let us consider the following problem: $F_1, \ldots, F_l$ are given graphs. Determine the maximum number of edges a graph can have if G^n which does not contain an F_i.

We have solved this problem for some special F_i-s and many other problems of this type have also been solved in other papers. The cases, when the extremal graphs are known are such that the extremal graphs can be obtained from a $T^{n,d}$ omitting some edges from it and adding some new edges to it where the number of the omitted and added edges is small. According to this I posed the following conjecture:

If $F_1, \ldots, F_l$ are given graphs and K^n is the extremal graph for them, then K^n can be obtained from a $T^{n,d}$ adding less than $o(n^2)$ edges to it.

Since that I have known that this conjecture is not true in such a general form. However, it is almost true:

THEOREM 6. *If $F_1, \ldots, F_l$ are given graphs and K^n is the extremal graph for them then there is a constant $c > 0$ such that K^n can be obtained from a $T^{n,d}$ omitting less than n^{2-c} edges from it and adding less than n^{2-c} edges to it, where $d = \min \chi(F_i) - 1$.*

Moreover, the following general stability theorem also holds:

THEOREM 7. Using the notations of Theorem 6: *There is a constant C such that if $\varepsilon > 0$ is arbitrary, $n > n_0(\varepsilon)$ and $e(G^n) \geq e(K^n) - C\,\varepsilon\, n^2$ and G^n does not contain any F_i then we may omit less than εn^2 edges of K^n so that the obtained graph be d-chromatic.*

Clearly, if G^n is d-chromatic, since $d = \min \chi(F_i) - 1$, G^n does not contain any F_i and according to Theorem 6 we may omit n^{2-c} edges from the extremal graphs so that the new graph is d-chromatic. Thus Theorem 6 is a general stability theorem.

REMARK. The first result in connection with my conjecture is due to ERDŐS, who has noticed that there follows a part of the conjecture from theorem of ERDŐS—STONE [6]: $|e(K^n) - e(T^{n,d})| \leq n^{2-c}$, where c is a suitable positive constant. In the proof of this assertion there was used theorem ERDŐS—STONE in the following form: If r, d are given integers, there is a $c > 0$ such that from $e(T^{n,d}) - n^{2-c} < e(G^n)$ follows that G^n contains a $T^{r(d+1),d+1}$. To obtain Theorem 6 we needed the following sharpening of the ERDŐS—STONE theorem.

If $e(T^{n,d}) \leq e(G^n)$ and G^n does not contain $T^{r(d+1),d+1}$, then it is possible to omit n^{2-c} edges from it so that the resulting graph is d-chromatic.

Looking for such a theorem ERDŐS *and I proved Theorem 6 independently (where my statement contained instead of $C\,\varepsilon n^2$ only δn^2,* where δ is a suitable positive constant depending on ε. However, this is not an essential difference).

ERDŐS mentioned this result in [7] but without proof, thus I give here a complete proof of it.

First we consider only the problem of $T^{r(d+1),d+1}$, i.e. we generalize the theorem of ERDŐS—STONE.

THEOREM 8. (a) *If $r \geq 2$, $d \geq 2$ are given positive integers, ε is a positive constant, then there exists a $\delta > 0$ and an n_0 such that if $n > n_0$ and G^n does not contain $T^{r(d+1),d+1}$, further if $e(G^n) \geq e(T^{n,d}) - \delta\, n^2$, then we may omit $[\varepsilon\, n^2]$ edges of G^n so that the resulting graph is d-chromatic.*

(b) Denote by K^n an extremal graph for the problem of $T^{r(d+1),d+1}$. We may omit $o(n^2)$ edges of K^n so that the resulting graph is d-chromatic.

REMARKS. 1. ERDŐS and independently T. KŐVÁRI, T. V. SÓS and P. TURÁN have proved [8] that if $e(G^n) \geq Cn^{2-\frac{1}{r}}$ (where $C > 0$ is a suitable constant), then G^n contains a $T^{2r,2}$ which shows that Theorem 8 remains also valid for $d = 1$.

2. Apply Theorem 8 (a) on the extremal graph K^n. Since K^n does not contain $T^{r(d+1),d+1}$ and $e(K^n) \geq (T^{n,d})$, Theorem 8 (a) gives just Theorem 8 (b). Thus it is enough to prove Theorem 8 (a).

We need the following

LEMMA 3. If $F_1, \ldots, F_l$ are given graphs and K^n is the extremal graph of their problems, then $\dfrac{e(K^n)}{\binom{n}{2}}$ converges.

(This lemma is contained in [9] and in [2] also. [2] proves it in a more general form using the theorem of ERDŐS—STONE. But this (trivial) lemma is needed just to avoid the use of the original ERDŐS—STONE theorem.)

PROOF. It is enough to show that $\dfrac{e(K^n)}{\binom{n}{2}}$ is a strictly decreasing sequence:

$$\frac{e(K^n)}{\binom{n}{2}} \leq \frac{e(K^{n-1})}{\binom{n-1}{2}}.$$

This is equivalent to

$$(n-2)\cdot e(K^n) \leq n \cdot e(K^{n-1}).$$

Let $G_1^{n-1}, \ldots, G_n^{n-1}$ be the spanned subgraphs of K^n having $n-1$ vertices. Clearly

$$(n-2)\, e(K^n) = \sum_{i=1}^{n} e(G_i^{n-1}) \leq \sum_{i=1}^{n} e(K^{n-1}) = n \cdot e(K^{n-1}),$$

since (a) each edge of K^n is contained just in $n-2$ G_i^{n-1} and since (b) G_i^{n-1} does not contain any F_i, from what follows $e(G_i^{n-1}) \leq e(K^{n-1})$. Q. e. d.

PROOF OF THEOREM 8. As we know from the mentioned result of ERDŐS, KÖVÁRI, SÓS and TURÁN, Theorem 8 is true for $d = 1$. Thus we use mathematical induction on d.

Let now $\varepsilon > 0$ be fixed and put $c = \dfrac{1}{10r}$, $\eta = \varepsilon \cdot c \cdot \dfrac{1}{10d}$ further $\sqrt{n} \geq M \geq 2 \cdot (10r)^{r+1} = 20r \cdot c^{-r}$. Here M is also a constant, but it will be fixed only later. Lastly, let G^n be any graph of n vertices not containing $T^{r(d+1),d+1}$ and such that

$$e(G^n) > e(T^{n,d-1}) + \frac{n^2}{2d^2}.$$

According to the inductional hypothesis, if n is sufficiently large, G^n contains a $T^{Md,d}$. Without loss of generality it may be supposed that $T^{Md,d}$ is a spanned subgraph of G^n. (Apply for e.g. theorem of Ramsey to a $T^{Kd,d}$ where $K >> M$). If there is an x_1 joined to each class of $T^{Md,d}$ by more, than cM edges, then $T^{Md,d}$ contains a $T^{c\,Md,d}$, each vertex of which is joined to x_1. Similarly there can be constructed recursively $T_2, \ldots, T_j$: if there is an x_i joined to each class of $T_{i-1} = T^{\{c^{i-1}M\}d,d}$ by more than $\{c^iM\}$ edges, then T_{i-1} contains a $T_i = T^{\{c^iM\}d,d}$ each vertex of which is joined to each of $x_1, \ldots, x_i$. This procedure stops in less than r steps, since if we obtained T_r by it, then certain vertices of T_r and $x_1, \ldots, x_r$ would determine a $T^{r(d+1),\,d+1}$ in G^n. This contradicts our assumption that G^n does not contain $T^{r(d+1),d+1}$.

Now let T_j be the graph obtained in the last step. Let E denote the class consisting of $x_1, \ldots, x_j$, $\widetilde{G} = G^n - T_j$ and let the classes of $T_j = T^{M_0 d,d}$ be denoted by $\bar{B}_1, \ldots, \bar{B}_d$. We split the vertices of G^n into classes $\bar{C}_1, \ldots, \bar{C}_d, D, E$. D is the class of those vertices of $\widetilde{G}$, which are joined to less than cM_0 vertices of a $\bar{B}_i$ and to less than $(1 - 2c)\, M_0$ vertices of another B_j (where i and j depend on x). Further, let $x \in C_i$ if there is a $\widetilde{B}_{i(x)}$ such that x is joined to less than cM_0 vertices of $\widetilde{B}_{i(x)}$ but x is joined at least to $(1 - 2c)M_0$ vertices of B_j if $j \neq i(x)$. Since all the vertices of $\bar{G} - E$ are joined by less than cM_0 edges to at least one $\bar{B}_i$ (just because of the algorithm used to select T_j), thus each vertex of $\bar{G}$ belongs just to one of $\bar{C}_1, \ldots \bar{C}_d$, D, E.

Now we show that $\bar{B}_i \bigcup \bar{C}_i$ does not contain $T^{2r,\,2}$. Suppose the contrary: $T^{2r,\,2}$ is contained in $\bar{B}_1 \bigcup \bar{C}_1$. (Without loss of generality may be assumed that $i = 1$.) Now we may select $r - r$ vertices in $\bar{B}_2, \ldots, \bar{B}_d$ so that the r considered vertices of $\bar{B}_l$ are joined to each of the $2r$ vertices of $T^{2r,\,2}$. These $r(d+1)$ vertices determine a $T^{r(d+1),\,d+1} \subseteq G^n$ disproving that G^n does not contain $T^{r(d+1),\,d+1}$. Hence $\bar{B}_i \bigcup \bar{C}_i$ does not contain $T^{2r,\,2}$ indeed.

Two cases will be distinguished:

(a) $|D| \leq 8d\eta c^{-1}$. In this case we need not go further in our proof: omit the vertices of $D \bigcup E$ and omit the edges in $\bar{B}_i \bigcup \bar{C}_i$ $(i = 1, \ldots, d)$. The remaining graph is d-chromatic and the number of the omitted edges is less than $8d\eta c^{-1}n^2 + |E|n + O\left(n^{1-\frac{1}{r}}\right)$.

The other case is when

(b) $|D| > 8d\eta c^{-1}n$.

The difficulty is that $\bar{B}_i$ and $\bar{C}_i$ are joined by many edges. This case can be eliminated in the following way: Let t be the number of those vertices of $\bar{B}_i$ which are joined at least to ηn vertices of C_i. Consider r arbitrary vertices of $\bar{C}_i$. It will be said, the multiplicity of this r-tuple is h if there are just h vertices in $\bar{B}_i$ each of which is joined to each vertex of the considered r-tuple. Then the multiplicity of an r-tuple is less than r, otherwise a $B_i \bigcup C_i$ would contain a $T^{2r,2}$. Thus the sum of the multiplicities of the r-tuples contained in $\bar{C}_i$ is less than $r\dbinom{n}{r}$. On the other hand it is at least $t\binom{[\eta n]}{r}$. Hence

$$t\binom{[\eta n]}{r} \leq r\binom{n}{r}.$$

From this it follows that $t \leq \frac{r}{\eta^r} + o(1)$ i.e. there exists an M_1 such that $t \leq M_1$ for every n. Now we fix M so that be $M_0 \geq \max\{c^{-2}M_1, 20r\}$.

Omit M_1 vertices from each $\bar{B}_i$ so that the vertices joined to more than ηn edges of C_i be among them. Put them into $\bar{C}_i$. The obtained classes will be denoted by B_i and C_i, respectively. Clearly $|B_i| = |\bar{B}_i| - M_1 = M_2$ and $M_2 \geq M_0 - c^2 M_0$ from what $M_0 \leq \frac{M_2}{1-c^2}$.

The classes $B_1, \ldots, B_d$ determine a $T^* = T^{M_2 d, d} \subseteq G^n$. Let $G^{n-M_2 d} = G^n - T^*$. Then the decomposition of $G^{n-M_2 d}$ into $C_1, \ldots, C_d, D, E$ has essentially the same properties as $\bar{G}_1, \bar{C}_1, \ldots, \bar{C}_d, D, E$ had. We need only the following properties of it:

(i) Each vertex of E is joined to all the vertices of T^*.

(ii) The classes B_i and C_i are joined by less than $2\eta n M_2$ edges.

(iii) If $x \in D$, then x is joined to a $B_{i(x)}$ by less than $\frac{c}{1-c^2} M_2$ edges and to a $B_{j(x)} (i \neq j)$ by less than $\frac{1-2c}{1-c^2} M_2$ edges.

The number of edges joining T^* and $G^{n-M_2 d}$ in G^n will be denoted by e_G. From (i), (ii) and (iii) follows that

$$e(G) \leq (n - M_2 d - |E| - |D|) \cdot (d-1) M_2 + |E| \cdot d \cdot M_2 + 2d\eta n M_2 +$$

$$+ |D| \cdot (d-2) \cdot M_2 + |D| \cdot \frac{1-2c}{1-c^2} \cdot M_2 + |D| \frac{c}{1-c^2} \cdot M_2 . \tag{25}$$

Here the terms of the sum on the right hand side estimate the number of edges joining

(a) the vertices of B_i to the vertices of B_j $(i \neq j)$,

(b) the vertices of E to the vertices of T^*,

(c) the vertices of B_i to the vertices of C_i and, finally

(d)–(e) estimate the number of edges joining an $x \in D$ to T^* $B_{i(x)} - B_{j(x)}$ to $B_{j(x)}$ and to $B_{i(x)}$, respectively.

By (25) we have

$$e_G \leq (n - M_2 d) \cdot (d-1) \cdot M_2 + |E| \cdot M_2 + 2d\eta n M_2 - |D| M_2 \cdot \frac{c-c^2}{1-c^2} <$$

(26)

$$< (n - M_2 d) \cdot (d-1) \cdot M_2 + r M_2 + 2d\eta n \cdot M_2 - M_2 \cdot 8d\eta \cdot c^{-1} \cdot \frac{c}{1+c}$$

since $|E| < r$ and $|D| > 8d\eta c^{-1}n$.

Clearly $\frac{c}{1+c} \cdot c^{-1} = \frac{1}{1+c} \geq \frac{10}{11}$, thus from (26) we have

$$e_G \geq (n - M_2 d)(d-1) - 5\eta n M_2 . \tag{27}$$

Put $\mu = d \cdot M_2$. Clearly, from $G^{n-\mu} = G^n - T^{\mu,d}$ we have

$$e(G^n) = e(G^{n-\mu}) + e_G + e(T^{\mu,d}) \tag{28}$$

and

$$e(T^{n,d}) = e(T^{n-\mu,d}) + e_T + e(T^{\mu,d}), \tag{29}$$

where

$$e_T = (n - \mu) \cdot (d-1) \cdot M. \tag{30}$$

Hence. if $\Delta(G^n) = e(G^n) - e(T^{n,d})$ then

$$\Delta(G^n) - \Delta(G^{n-\mu}) = \{e(G^n) - e(G^{n-\mu})\} - \{e(T^{n,d}) - e(T^{n-\mu,d})\} =$$
$$= \{e_G + e(T^{\mu,d})\} - \{e_T + e(T^{\mu,d})\} = e_G - e_T .$$

Here we have used (28) and (29). From (27) and (30) it follows that

$$\Delta(G^n) - \Delta(G^{n-\mu}) < -5\mu\eta n .$$

Put $k_0 = \frac{\eta}{\mu} \cdot n$. Since $\Delta(G^{n-\mu}) > \Delta(G^n)$, if n is sufficiently large, $G^{n-\mu}$ contains also a $T^{Md,d}$. Apply our method to it: either we may omit less than εn^2 edges from it so that the resulting graph is d-chromatic or we obtain a $G^{n-2\mu}$ such that $\Delta(G^{n-\mu}) - \Delta(G^{n-2\mu}) < -5\mu\eta(n-\mu)$. Let us continue this procedure, thus we obtain recursively the graphs $G^{n-3\mu}, \ldots, G^{n-j\mu}$. If this procedure finishes in less than k_0 steps, then omitting all the edges of G^n at least one endpoint of which occurs in a $G^n - G^{n-j\mu}$ and omitting less than εn^2 edges from, $G^{n-j\mu}$, we obtain a d-chromatic graph. The number of the omitted edges is less than

$$\varepsilon n^2 + k_0 \mu \cdot n = \varepsilon n^2 + \eta n^2 = \varepsilon n^2 \left(1 + \frac{1}{100rd}\right)$$

(which is greater than εn^2 but it does not matter.) In this case we are ready with our proof.

If the procedure does not finish in k_0 steps, we have the graphs $G^{n-i\mu}$ $(i = 1, \ldots, k_0)$ such that

$$\Delta(G^{n-(i-1)\mu}) > \Delta(G^{n-i\mu}) + 5\mu\eta(n - i\mu) .$$

From this we obtain

$$\Delta(G^{n-k_0\mu}) > \Delta(G^n) + 5\mu \cdot \eta \cdot \sum_{i=0}^{k_0-1} (n - i\mu) > \Delta(G^n) + 2\mu\eta n \cdot k_0 =$$
$$= \Delta(G^n) + 2\eta^2 n^2 .$$

If we know the ERDÖS—STONE theorem, we can finish the proof in the following way: Let be $\delta = \eta^2$. Then from $e(G^n) > e(T^{n,d}) - \delta n^2$ follows that $\Delta(G^n) > -\delta n^2$, thus $\Delta(G^{n-k_0 d}) > \delta n^2$ and consequently $e(G^{n-k_0 d}) > e(T^{n,d}) + \delta n^2$. Apply theorem ERDŐS—STONE: $G^{n-k_0 d}$ contains a $T^{r(d+1),d+1}$ and therefore G^n contains also this $T^{r(d+1),d+1}$. This contradiction shows that if $e(G^n) > e(T^{n,d}) - \delta n^2$ and G^n does not contain $T^{r(d+1),d+1}$, then the procedure finishes in less than k_0 steps. Hence it is possible to omit $\varepsilon n^2 \left(1 + \frac{1}{100\,rd}\right)$ edges from G^n so that the obtained graph be d-chromatic. This is just the statement to be proved.

But we may prove our theorem avoiding the use of the ERDŐS—STONE theorem More exactly: we may prove the ERDŐS—STONE theorem easily using our results above and apply it only thereafter. Let $\{K^n\}$ be the sequence of extremal graphs for $T^{r(d+1),d+1}$. According to Lemma 3 $\frac{e(K^n)}{n^2}$ converges to a non-negative constant α. Applying our method to K^n we obtain that if n is large enough, then it is possible to omit $2\varepsilon n^2$ edges from K^n so that the obtained graph be d-chromatic. This will be shown by an indirect proof: Suppose that there are infinitely many n_i such that it is impossible to delete less than $2\varepsilon n_i^2$ edges from it so that the obtained graph be d-chromatic. Then from these graphs construct by the method described above the graph $\widetilde{K}^{n_i^*}$ where $n_i^* = n_i - \mu\left(\frac{\eta}{\mu}\cdot n_i\right) = n_i\,(1-\eta)$.

Thus we have

$$\Delta(\widetilde{K}^{n_i^*}) \geq e(K^{n_i}) + 2\eta^2 n_i^2. \text{ From } e(K^{n_i^*}) \geq e(\widetilde{K}^{n_i^*})$$

we obtain

$$\Delta(K^{n_i^*}) \geq \Delta(K^{n_i}) + 2\eta^2 n_i^2\,. \tag{31}$$

Clearly $\frac{\Delta(K^n)}{n^2}$ converges since $\frac{e(K^{n_i})}{n^2}$ and $\frac{e(T^{n,d})}{n^2}$ converge. But from (31) we have

$$\frac{\Delta(K^{n_i^*})}{(n_i^*)^2} \geq \frac{\Delta(K^{n_i})}{(n_i)^2} + 2\eta^2 \text{ i.e. } \overline{\lim}\,\frac{\Delta(K^n)}{n^2} > \underline{\lim}\,\frac{\Delta(K^n)}{n^2}\,.$$

This contradiction proves that if $\varepsilon > 0$ is an arbitrary given constant, and n is large enough, then we may delete $2\varepsilon n^2$ edges from K^n so that the obtained graph be d-chromatic. Thus $e(K^n) = e(T^{n,d}) + o\,(n^2)$ what is just the ERDŐS—STONE theorem. Now we may use it already and thus we have proved Theorem 8 completely.

The following sharpening of Theorem 8 (b) is also true:

THEOREM 9. *Let $r \geq 2$ and $d \geq 2$ given integers and denote by K^n an extremal graph of n vertices for the problem of $T^{r(d+1),d+1}$. Then we may omit $O\left(n^{2-\frac{1}{r}}\right)$ vertices of K^n so that the resulting graph is d-chromatic.*

REMARK. Denote by $f(n)$ the number of edges of the extremal graph in the problem of $T^{2r,2}$. We know that $f(n) = O(n^{2-\frac{1}{r}})$ and because of this we shall prove just that we may delete $O(n^{2-\frac{1}{r}})$ vertices from K^n so that the resulting graph is d-chromatic. As a matter of fact, our proof gives, that it is possible to omit less than

$$d \cdot f\left(\frac{n}{d} + o(n)\right)$$

edges from K^n so that the resulting graph is d-chromatic. This result is the best possible apart from the factor d.

Let H^n be the extremal graph for $T^{2r,2}$ and write a H^n into a class of $T^{n,d}$. The resulting graph does not contain $T^{r(d+1),d+1}$ and it is impossible to omit $f\left(\left[\frac{n}{d}\right]\right)$ edges of it so that the obtained graph should be d-chromatic.

LEMMA 4. *Let M be a given positive integer and $c > O$ be an arbitrary constant. Then there exist an M' and a $c' < O$ such that if a set A of n elements contains M' subsets, $A_1, \ldots, A_{M'}$ each of which contains at least cn elements, then there are M given subsets $A_{i1}, \ldots, A_{i_M}$ whose intersection contains more than $c'n$ elements.*

(This almost trivial lemma is contained in a lemma of ERDŐS [10]).

PROOF OF LEMMA 4. It is enough to prove this lemma when $M = 2^m$. This will be proved by induction on m. If $M = 2$, it is almost trivial. It may be supposed that $c = \frac{1}{t}$, where t is an integer. Consider the set A and 3 subsets of it $A_1, \ldots, A_{3t}$. Put $B_i = A_i - \bigcup_{j \neq i} A_j$. Then B_i-s are disjoint sets and at least one of them has less than $\frac{n}{3t}$ elements. If for e.g. $|B_1| \leq \frac{n}{3t}$ then there is an A_j, which contains at least $\frac{2}{3} \frac{n}{t} : 3t = \frac{2n}{9t^2}$ elements of A_1. This proves the lemma in the case $m = 1$. If we knew the lemma for $M = 2^m$ we could prove it for 2^{m+1} as follows: there are an M_1 and a $c_1 > 0$ such that if A is a set of n elements and $A_1, \ldots, A_{M_1}$ are subsets of it such that $|A_i| \geq cn$ then there are 2^m subsets among them, the intersection of which contains at least $c_1 n$ elements. It may be supposed that $c_1 = \frac{1}{q}$ where q is an integer.

Now let $A_1, \ldots, A_{3M_1q}$ be given subsets of a set A and let $|A| = n$, $|A_i| \geq cn$. We make $3q$ groups of the subsets A_i:

$$\{A_1, \ldots, A_{3qM_1}\} = \{B_{i,j} : i = 1, \ldots, M_1,\ j = 1, \ldots, 3q\}.$$

Applying the inductional hypothesis for $B_{i,1}, \ldots, B_{i,M_1}$ we obtain that there is a subset $C_i \subseteq A$ contained by at least 2^m of $B_{i,1}, \ldots, B_{i,M_1}$ and having at least $c_1 n$ elements. Apply now our result concerning the case $M = 2$ to the subsets $C_1, \ldots, C_{3q}$: there is a $D \subseteq A$ contained by two C_i and having at least $\frac{2}{9q^2} \cdot n$ elements.

Trivially D is contained in at least 2^{m+1} subsets A_i. This completes the proof of our lemma.

Proof of Theorem 9. Let K^n be an extremal graph for $T_0 = T^{r(d+1), d+1}$ and let us colour its vertices with d colours so that the number of edges, joining vertices of the same colour be minimal. According to Theorem 7 (b) this number is $o(n^2)$. The set of vertices of the i-th colour will be denoted by A_i. Clearly if $x \in A_i$, then x is joined to less vertices of A_i than of A_j $(j \neq i)$. This follows from the minimality-condition of the colouring.

For the sake of simplicity the edges joining two vertices of different classes will be called black edges, the edges joining two vertices from the same A_i will be called green edges, finally if $x \in A_i$, $y \in A_j$ are not joined and $i \neq j$, then x and y will be said to be joined by a red edge. The number of green edges is $o(n^2)$. From this and from $e(K^n) \leq e(T^{n,d})$ it follows that the number of red edges of K^n is also $o(n^2)$. Now we prove an important property of the green edges. Let $c > 0$ be an arbitrary but fixed constant. Then the number of those vertices of K^n, which are the endpoints of at least cn green edges, is bounded. In order to show this let us determine $M_1, \ldots, M_d$ and $c_1, \ldots, c_d$ recursively so that if $|B| \leq n$, and $B_1, \ldots, B_{M_i}$ are the subsets of B containing at least cn elements of B, then there exist M_{i-1} subsets B_{i_k} such that $|\bigcup B_{i_k}| \geq c_i n$. According to Lemma 4, if we put $M_0 = r$, then we may determine such constants $M_1, \ldots, M_d$; $c_1, \ldots, c_d > 0$.

Now it will be shown that the number of vertices of A_i joined at least to cn other vertices of A_i (i.e. the number of those edges which are the endpoints of at least cn green edges) is less than M_d.

Suppose the contrary: let $x_1, \ldots, x_{M_d}$ be vertices of A_1 each of which is joined to at least cn other vertices of A_i. Let us denote by B_{ij} the vertices of A_j joined to x_i. From Lemma 4 we have that there are M_{d-1} vertices among $x_1, \ldots, x_{M_{d-1}}$ and a set $C_1 \subseteq A_1$ such that $|C_1| \geq c_d n$ and each of the considered x_k-s is joined to each vertex of C_1. We may assume that these x_k-s are just $x_1, \ldots, x_{M_{d-1}}$. Apply Lemma 4 to $B_{1,2}, \ldots, B_{M_{d-2},2}$. There are M_{d-2} x_k-s and a subset C_2 of A_2 such that each considered x_k is joined to each vertex of C_2. Then we select M_{d-3} vertices from these M_{d-2} vertices and a $C_3 \subseteq A_3$, so that each considered x_k is joined to each vertex of A_3, and so on. Thus we obtain d subsets $C_1, \ldots, C_d$ and r vertices $x_1, \ldots, x_r$ so that $C_i \subseteq A_i$, $|C_i| \geq c_{d-1} n$ and each vertex of C_i is joined to each x_k. Put $c' = \min c_d$ and let C_i^* be a subset of C_i containing $[c'n]$ edges. G^* denotes the subgraph of K^n spanned by the vertices of $\bigcup C_i^*$. It contains $o(n^2)$ red edges and from this follows that it contains a $T^{rd,d}$. The vertices of this $T^{rd,d}$ are joined to each x_k, thus $x_1, \ldots, x_r$ and $T^{rd,d}$ determine a $T^{r(d+1),d+1}$ contained in K^n. This contradiction shows that the number of the vertices of A_i joined to at least cn other vertices of A_i is less than M_d.

A second interesting result (conjectured by ERDŐS and me) is that if $\varepsilon > 0$ is arbitrary and $n > n_0(\varepsilon)$, then each vertex of K^n has valency greater than $\frac{n}{d}(d-1) - \varepsilon n$. (This is also true in the general case when $T^{r(d+1),d+1}$ is replaced by $F_1, \dots, F_l$). This can be proved by the following argument:

There are $r(d+1) = t$ vertices $x_1, \dots, x_t$ in A_1 each of which is the endpoint of less than $\frac{1}{t} \cdot \varepsilon n$ green edges. Let now x^* be a new vertex (i.e. a vertex not contained in K^n). Join it to all the vertices of $A_2 \cup A_3 \cup \dots A_d$ except to those which are not joined to all x_k $(k = 1, \dots, t)$. We assert that the resulting graph $\widetilde{K}^{n+1}$ does not contain $T^{r(d+1),d+1}$. Let us suppose the contrary: $T^{r(d+1),d+1}$ is contained in K^n. Then clearly this $T^{r(d+1),d+1}$ contains x^*, otherwise $T^{r(d+1),d+1}$ would also be contained in K^n. But it does not contain all the vertices $x_1, \dots, x_t$. It may be assumed that $x_1 \notin T^{r(d+1),d+1}$. x_1 is joined to all the vertices which are joined to x^*. Therefore changing x^* on x_1 in $T^{r(d+1),d+1}$ we obtain an other $T^{r(d+1),d+1}$ not containing x^* and thus contained in K^n. This contradiction shows that $\widetilde{K}^{n+1}$ does not contain $T^{r(d+1),d+1}$. Hence if $|A_1| \le \left[\frac{n}{d}\right]$ which may be assumed we have

$$e(K^{n+1}) \ge e(\widetilde{K}^{n+1}) \ge e(K^n) + \frac{n}{d} \cdot (d-1) - \varepsilon n \,. \tag{32}$$

On the other hand let $x \in K^{n+1}$, then $\widetilde{K}^n = K^{n+1} - \{x\}$ does not contain $T^{r(d+1),d+1}$ either, and thus

$$e(K^n) \ge e(\widetilde{K}^n) = e(K^{n+1}) - \sigma(x) \,. \tag{33}$$

(32) and (33) imply that $\sigma(x) \ge \frac{n}{d}(d-1) - \varepsilon n$.

Thus each vertex of K^{n+1} is of valency greater than $\frac{n}{d}(d-1) - \varepsilon n$. Replace n by $n-1$: each vertex of K^n is of valency greater than $\frac{n-1}{d} \cdot (d-1) - \varepsilon(n-1)$ which is essentially the desired result.

Let now be $c = \frac{1}{10rd}$. Omit those vertices of A_i which are joined to at least cn other vertices of A_i. The obtained class will be denoted by A_i^*. Clearly $|A_i^*| \ge |A_i| - M_d$. A_i^* does not contain $T^{2r,2}$ as a subgraph. In order to show this suppose the contrary and fix a $T^{2r,2}$ in A_1. (We may assume $i = 1$.) The vertices of A_j^* are joined to less than cn vertices of A_j, thus if $x \in A_j^*$ then A_k $(k \ne 1)$ contains less than $2cn$ vertices joined to x by red edges. Really, $|A_i^*| = \frac{n}{d} + O(\sqrt{n})$, $\sigma(x) \ge \frac{d-1}{d} n - o(n)$ and x is joined to less than cn vertices of A_1, from what follows the statement.

Now we select r vertices of A_2^* each of which is joined to all the vertices of $T^{2r,2} \subseteq A_1^*$, then select r vertices of A_3^* each of which is joined to all the vertices of $T^{2r,2}$, and to all the vertices that have been selected from A_1^*, and so on. Thus we obtain r vertices in each A_k^* $k = 2, \dots, d$ which together with the vertices of $T^{2r,2}$ determine a $T^{r(d+1),d+1}$. This selection is possible since c is small enough. Thus we obtained a contradiction which proves that A_i^* does not contain $T^{2r,2}$. Thus A_i^* contains $O(n^{2-\frac{1}{r}})$ edges. Since $|A_i - A_i^*| = O(1)$, A_i contains also $O(n^{2-\frac{1}{r}})$ edges. Thus we may omit $O(n^{2-\frac{1}{r}})$ edges of K^n so that the resulting graph is d-chromatic. This completes the proof of Theorem 9.

REMARK. *Applying our proof with* $e(K^n) = e(T^{n,d}) + O(n^{2-\frac{1}{r}})$ *we obtain that:*

(a) $|A_i| = \frac{n}{d} + O(n^{1-\frac{1}{2r}})$.

(b) *The number of the red and green edges is* $O(n^{2-\frac{1}{r}})$.

(c) *Each vertex of* K^n *has valency greater than* $\frac{n}{d}\cdot(d-1) + O(n^{1-\frac{1}{r}})$.

PROOF OF THEOREM 6. Since $d = \min \chi(F_i) + 1$, we may assume that $\chi(F_1) = d + 1$. There is an r such that $F_1 \subseteq T^{r(d+1),d+1}$. Thus, if a graph does not contain any F_i, then it does not contain $T^{r(d+1),d+1}$ either. In the proof of Theorem 8 we have used only the fact that K^n is an extremal graph for a TURÁN type problem and that K^n does not contain $T^{r(d+1),d+1}$. But this remains valid in our case, too, thus the proof of Theorem 9 remains valid for the general case, too. Moreover, if there is an F_{i_0} such that F_{i_0} can be coloured by "1", "2", . . . , "d", further the number of vertices of the colour "1" is at most r, then, maybe $T^{r(d+1),d+1}$ does not contain F_{i_0} however $O(n^{2-\frac{1}{r}})$ vertices of the extremal graph K^n can be omitted so that the resulting graph is d-chromatic. (From this it follows that $e(K^n) = e^{(n,d)} + O(n^{2-\frac{1}{r}})$, too.)

PROOF OF THEOREM 7. We prove only the existence of a $\delta(\varepsilon)$ such that if $e(G^n) \geq e(T^{n,d}) - \delta(\varepsilon)\cdot n^2$ and it does not contain any $T^{r(d+1),d+1}$, then we may omit $[\varepsilon n^2]$ vertices of G^n so that the resulting graph is d-chromatic.

Since there is an F_{i_0} having the chromatic number $d + 1$ and there is a $T^{r(d+1),d+1}$ which contains F_{i_0} as a subgraph, if G^n does not contain any F_i then it does not contain $T^{r(d+1),d+1}$ either. Thus Theorem 7 is the trivial consequence of Theorem 8.

Here we finish our investigation.

Summarq of our results concerning the general problem

Let $F_1, \ldots, F_l$ be given graphs. Denote by K^n an extremal graph of the problem of $F_1, \ldots, F_l$.

1. $e(K^n) = e(T^{n,d}) + O\left(n^{1-\frac{1}{r}}\right)$, where $d = \min \chi(F_i) - 1$ and r is a positive integer such that there is an F_i and a suitable colouring of it by $d+1$ colours so that at most r vertices of F_{i_0} have the first colour.

2. We may colour K^n by d colours so that the number of the edges having endpoints of the same colour is $O\left(n^{2-\frac{1}{r}}\right)$. The number of the "red edges" is also $O\left(n^{2-\frac{1}{r}}\right)$.

3. Each vertex of K^n has valency $\frac{n}{d}(d-1) + O\left(n^{1-\frac{1}{r}}\right)$ (i.e. the vertices of K^n have essentially the same valency as the vertices of $T^{n,d}$). If $\varepsilon > 0$ is a positive constant, then the number of vertices having valency greater than $\frac{n}{d}(d-1) + \varepsilon n$ is bounded.

4. Each of the given colouring of K^n contains $\frac{n}{d} + O\left(n^{1-\frac{1}{r}}\right)$ vertices (i.e. the classes have almost the same number of vertices).

5. We may omit $O\left(n^{2-\frac{1}{2r}}\right)$ edges of K^n and add $O\left(n^{2-\frac{1}{2r}}\right)$ new edges to it so that the obtained graph is just $T^{n,d}$. (This result follows from the others trivially.)

6. All the graphs having almost as many edges as K^n has, are almost of the same structure:

If $\varepsilon > 0$ is arbitrary, there exist a $\delta > 0$ and an n_0 such that if $n > n_0$ and $e(G^n) > e(T^{n,d}) - \delta n^2$ then either G^n contains an F_i, or we may delete $[\varepsilon n^2]$ edges of it so that the resulting graph is d-chromatic.

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