Regularity Lemmas and Extremal Graph Theory

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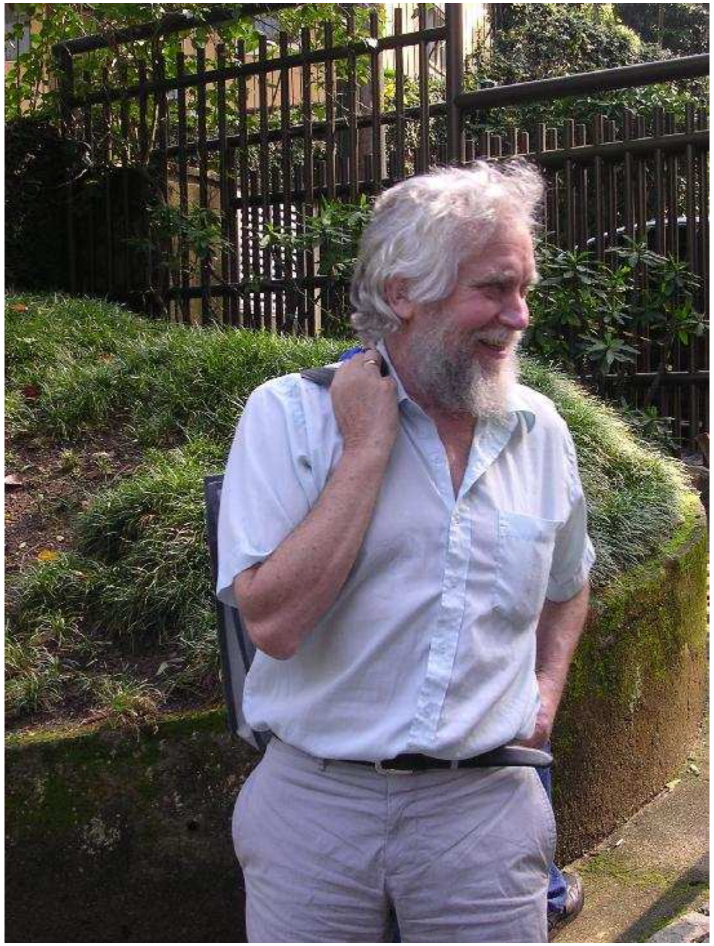
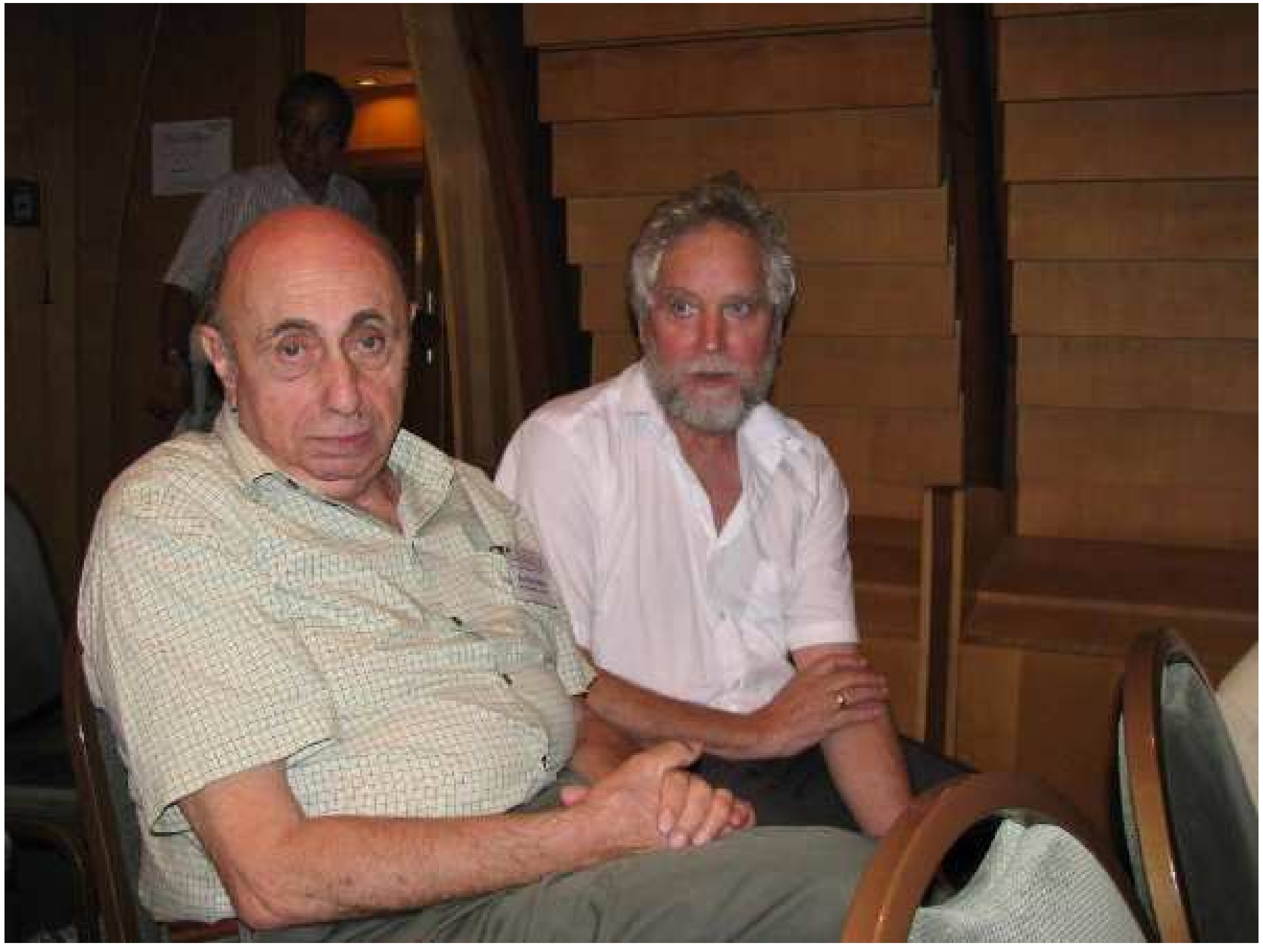
Lecture on Endre Szemerédi's 70th birthday

Streamlined version

The most important thing:

Happy birthday, Endre!





Streamlined?

Possible updated version on my homepage: www.renyi.hu/˜miki

This is basically identical with the one I used for my lecture (Endre Szemerédi's 70th birthday, Budapest, 2010 August)

The differences:

- Several misprints are corrected.
- Certain references are added.
- Certain explanations are added, often **IN BLACK.**
- Some repetitions (needed in the lecture) are eliminated
- Stepping is (mostly) eliminated.
- "Improved" colouring.

Disclaimer:

There is *no way to mention all the important results.* I do not even try here!

Extremal graph theory

Abstract

is one of the oldest areas of Graph Theory. In the 1960's it started evolving into a wide and deep, connected theory.

As soon as Szemerédi has proved his **Regularity Lemma**, several aspects of the extremal graph theory have *completely changed*.

Several deep results of *extremal graph theory* became accessible only through the application of this central result, the *Regularity Lemma*

- Also, large part of *Ramsey Theory* is very strongly connected to *Extremal graph theory*. Application of the **Regularity Lemma** in these area was also crucial.
- The first difficult result of in *Ramsey–Turán theory* was also proved using (an earlier version of) the **Regularity Lemma**, by Szemerédi.
- I will survey this area.

Map to the lecture/slides

Some references, homepages

Introduction, Extremal graph theory in general

General asymptotics

Erdos−Stone−Simonovits

Finer asymptotics, decomposition

Stability of extremal structures

Classification of problems

Szemeredi Regularity Lemma

Ramsey−Turan problems

Ramsey−Turan problems

How to use RL?

The Bollobas−Erdos construction

Conjectures

Very superficially:

Subgraphs of random graphs

Algorithmic aspects

Hypergraphs

New developments

Some references

- KOMLÓS-SIMONOVITS, Szemerédi regularity lemma, and its applications in graph theory, Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993), 295–352, János Bolyai Math. Soc., Budapest, 1996;;
- LOVÁSZ, LÁSZLÓ; SZEGEDY, BALÁZS: Szemerédi's lemma for the analyst. Geom. Funct. Anal. 17 (2007), no. 1, 252–270.
- V. RÖDL, M. SCHACHT: **Regularity Lemmas** for graphs, Bolyai volume, MS20. (Lovász Birthday)
- N. ALON, E. FISCHER, M. KRIVELEVICH, M. SZEGEDY, Efficient testing of large graphs, Combinatorica 20 (2000), 451–476.
- Kühn, Daniela and Osthus, Deryk: Embedding large subgraphs into dense graphs. Surveys in combinatorics 2009, 137–167, London Math. Soc. Lecture Note Ser., 365, Cambridge Univ. Press, Cambridge, 2009.

Some references II: end of a long list

- Yoshi Kohayakawa and Vojta Rödl: Szemerédi's regularity lemma and quasi-randomness, Recent Advances in Algorithmic Combinatorics (B. Reed and C. Linhares-Sales, eds.), CMS Books Math./Ouvrages Math. SMC, vol. 11, Springer, New York, 2003, pp. 289-351

. . .

. . .

- T.C. TAO, A variant of the hypergraph removal lemma, preprint; http://arxiv.org/abs/math.CO/0503572
- T.C. TAO, Szemerédi's regularity lemma revisited, preprint; http://arxiv.org/abs/math.CO/0504472

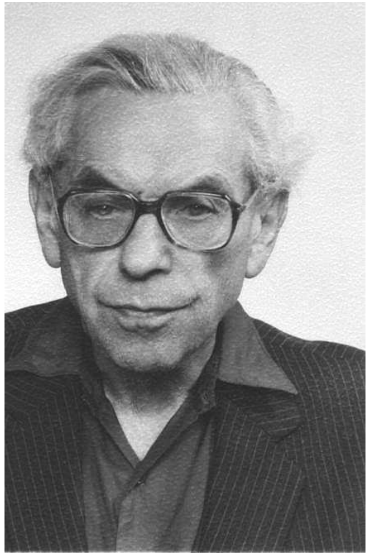
What is left out, or just mentioned?

- Sparse regularity lemma
- Many applications
- Connection to Quasi-randomness
- Hypergraph regularity

. . . and many other things

Some homepages on Regularity

- NOGA ALON: http://www.tau.ac.il/~nogaa
- YOSHI KOHAYAKAWA: http://www.ime.usp.br/~yoshi
- DERYK OSTHUS: http://web.mat.bham.ac.uk/D.Osthus/bcc09dkdo2.pdf



Erdős homepage(s), e.g. www.renyi.hu/~p_erdos

This contains Erdős' papers up to 1989

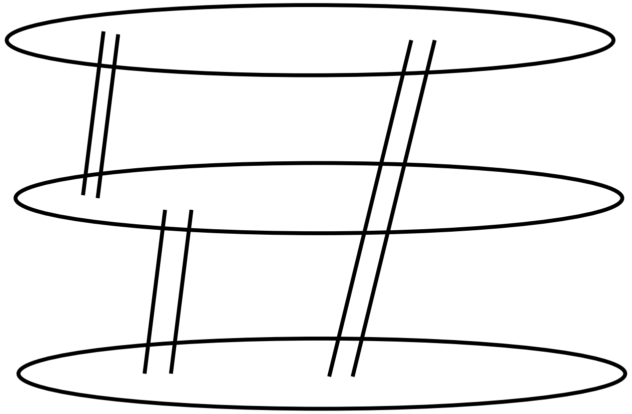
My homepage: www.renyi.hu/~miki

Some related papers,

Bollobás-Erdős-Simonovits-Szemerédi
Bollobás-Erdős-Hajnal-Sós,
Bollobás-Erdős-Hajnal-Sós-Simonovits

Extremal Graph Theory

G_n, is always a graph on n vertices. $T_{n,p}$ = Turán graph,
$K_r(m_1, \dots, m_r)$ is the complete r-partite graph with m_i vertices in its i^{th} class.



$$\mathbf{ex(n, \mathcal{L})} = \max_{\substack{L \not\subseteq G_n \\ \text{for } L \in \mathcal{L}}} e(G_n)$$

- Turán Theorem
- Determine or estimate $\mathbf{ex(n, \mathcal{L})}$.
- Describe the *structure* of extremal graphs
- Describe the structure of *almost* extremal graphs
  = *Stability Results*

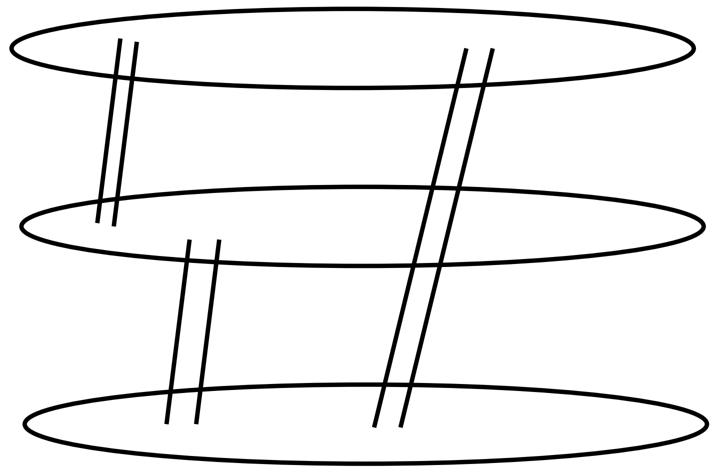
Erdős-Stone-Sim.

Put

$$p := \min_{L \in \mathcal{L}} \chi(L) - 1.$$

Then

$$\mathbf{ex(n,}\mathcal{L}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2) \quad \text{as} \quad n \to \infty.$$

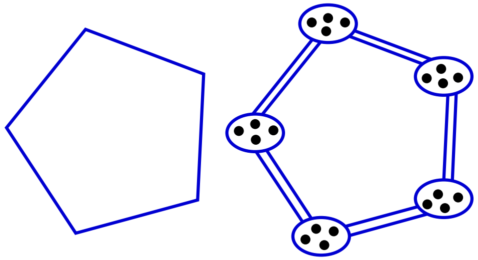


Sharpness:
The Turán graph $T_{n,p}$ provides the lower bound

Erdős-Stone-Sim. (A)

This means that the asymptotics is independent of the fine structure of the forbidden graphs, it depends only on the *minimum chromatic number*.

Another interpretation would be: the asymptotics is the same for a sample graph L and its arbitrary *blown-up* versions $L(t)$,



where blown-up means that each vertex of L is replaced by t new vertices and the new vertices are joined if the originals were joined.

- These two interpretations are the same for ordinary graphs but not in some other settings. (Not for Ramsey-Turán!)

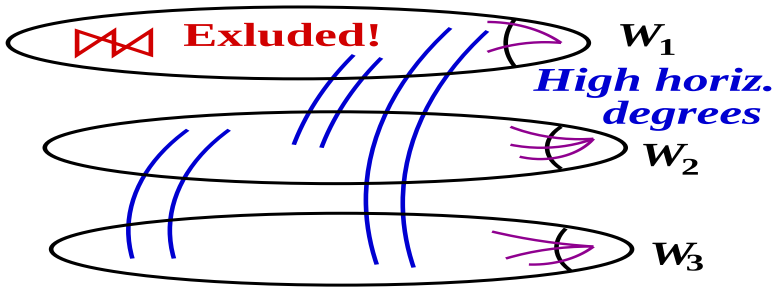
- See also W. G. BROWN AND SIM: Digraph extremal problems, hypergraph extremal problems, and the densities of graph structures. Discrete Math. 48 (1984), no. 2-3, 147–162.

Erdős-Simonovits structural description of the extremal graphs. Role of the Decomposition Class

Given $\mathcal{L}$, if S_n is $\mathcal{L}$-extremal, then it has an *optimal vertex-partition* $(U_1, \dots, U_p)$ such that

- $\sum e(U_i) = o(n^2)$, (few horizontal edges)
- the number of *vertices of horizontal degrees* $> \varepsilon n$ is $h = O_\varepsilon(1)$.

Here *optimal* means that $\sum e(U_i)$ is minimal.



The general picture:

The finer structure is governed by the *Decomposition class* $\mathcal{M}$:

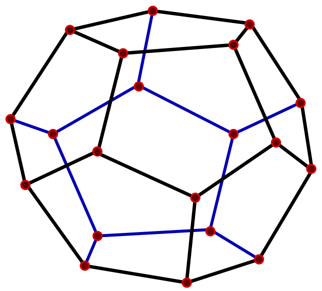
Definition of the Decomposition class $\mathcal{M}$. M is in $\mathcal{M} = \mathcal{M}(\mathcal{L})$ if there are some $L \in \mathcal{L}$ and t for which $L \subset M \otimes K_{p-1}(t, \dots, t)$.

Decomposition class explained (A)

If each $L \in \mathcal{L}$ is $p+1$-chromatic, then $\mathcal{M}$ is the family of those bipartite M that are obtained from some $L \in \mathcal{L}$ by $p+1$-colouring L and then taking two colour-classes and the bipartite subgraph defined by them.
Of course, it is enough to take the minimal M's.

If L has an edge e for which $\chi(M-e) = p$ then $\mathcal{M} = \{K_2\}$ (one edge).
Here e is called *colour-critical edge*. This is the case for K_{p+2}, $C_{2\ell+1}$, the Grötzsch-Mycelski graph, and many other graphs.

Theorem Critical edge. (Erdős for $p=2$ implicitly, Sim. in this form and for general p.) $T_{n,p}$ *is extremal for* $n > n_0(L)$ *if and only if* $\chi(L) = p+1$ *and* L *has a critical edge.*



The dodecahedron's decomposition consists of 6 independent edges.

Structure of (almost) extremal graphs

ERDŐS-SIM: Stability
The almost-extremal graphs are almost $T_{n,p}$

Distance of graphs, $\rho(G_n, H_n)$: How many edges of G_n should be changed to get a G' isomorphic to H_n?

Put

$$p := \min_{L \in \mathcal{L}} \chi(L) - 1$$

If $p > 1$ and (S_n) is an extremal sequence for $\mathcal{L}$, then

$$\rho(S_n, T_{n,p}) = o(n^2) \quad \text{as} \quad n \to \infty.$$

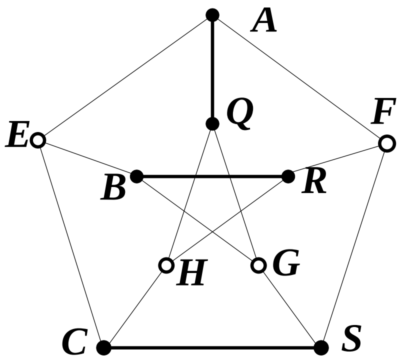
My favourite problem is:
When is S_n a p-chromatic $K(n_1, \ldots, n_p)$ + edges?
i.e. one has to add only, not to delete edges...

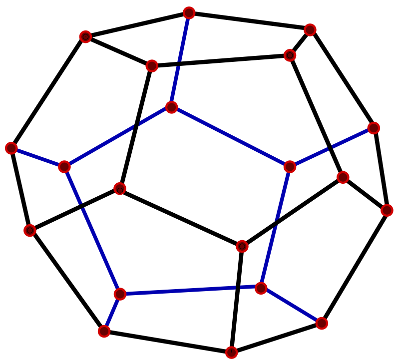
Classification of Extremal problems

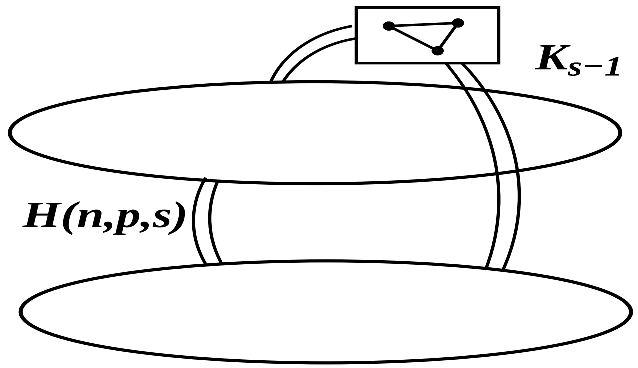
- $T_{n,p}$ is extremal: $K_2 \in \mathcal{M}$. (There is a colour-critical edge in L.)
- *Linear* error-term: $\mathcal{M}$ contains a tree (or forest)

$$\mathbf{ex}(\mathbf{n}, \mathcal{L}) = e(T_{n,p}) + O(n).$$

Example : Dodecahedron, Petersen, Icosahedron
(Askd by Turán, proved by Sim.)







- *Superlinear* error term: iff each $M \in \mathcal{M}$ has a cycle.

$$\mathbf{ex}(\mathbf{n}, \mathcal{L}) > e(T_{n,p}) + cn^{1+\alpha}.$$

Example : Octahedron

Density, ε-regularity

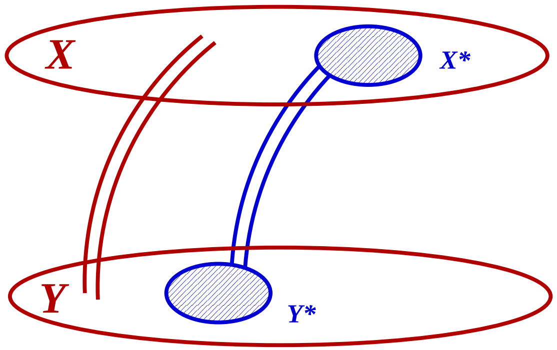
Density

$$d(X,Y) = \frac{e(X,Y)}{|X||Y|}.$$

ε-regularity

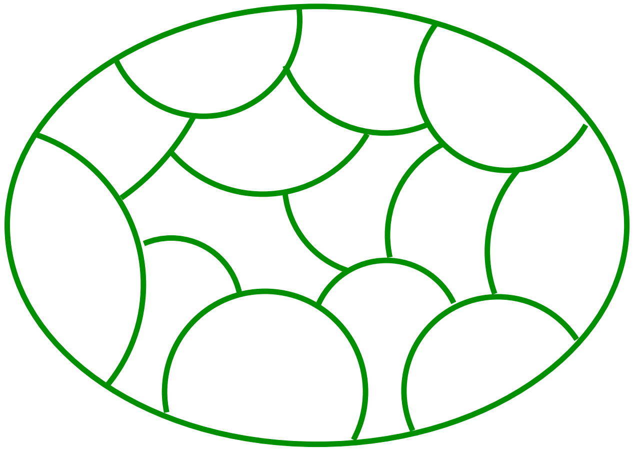
Given a graph G_n and two disjoint vertex sets $X \subseteq V$, $Y \subseteq V$, the pair (X,Y) will be called *ε–regular*, if for every $X^* \subset X$ and $Y^* \subset Y$ satisfying $|X^*| > \varepsilon|X|$ and $|Y^*| > \varepsilon|Y|$,

$$|d(X^*,Y^*) - d(X,Y)| < \varepsilon.$$



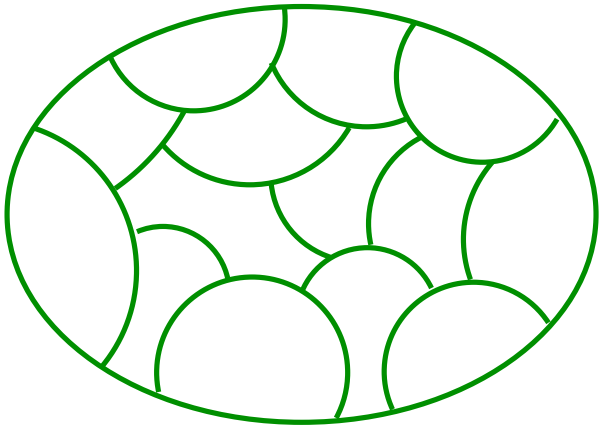
The regularity lemma

As soon as Szemerédi has proved his **Regularity Lemma**, several aspects of the extremal graph theory have completely changed.



Theorem $\approx$ (Szemerédi) *For every $\varepsilon > 0$ every graph G_n has a vertex-partition into a bounded number of classes $U_1, \dots, U_k$ of almost equal sizes so that for all but at most $\varepsilon\binom{k}{2}$ pairs i, j the bipartite graph (generated by G_n) is ε-regular*

The regularity lemma, precisely (A)



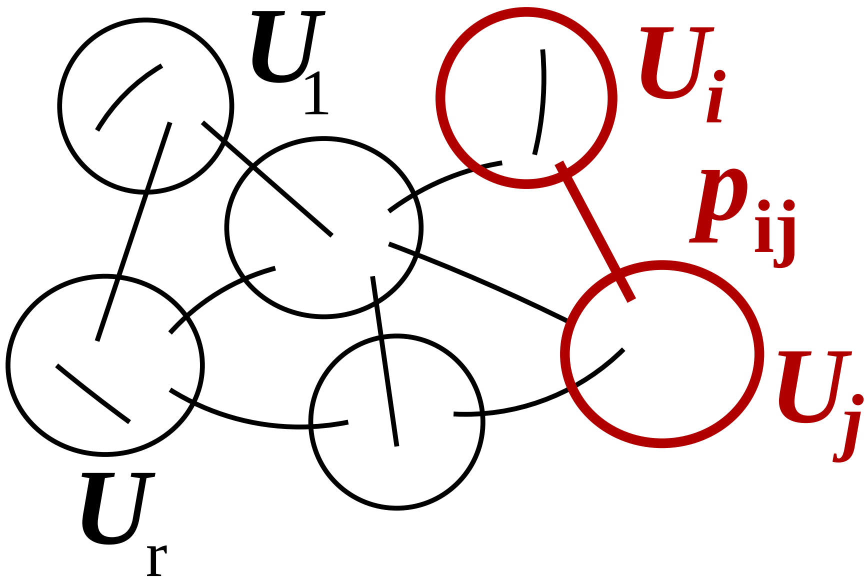
Theorem (Szemerédi) *For every $\varepsilon > 0$ and integer k every graph G_n has a vertex-partition into the classes $U_1, \dots, U_k$ of almost equal sizes, for some $\kappa < k < K(\varepsilon, \kappa)$ so that for all but at most $\varepsilon\binom{k}{2}$ pairs i, j the bipartite graphs (generated by G_n) are ε-regular.*

Originally there was an exceptional class U_0 and all the other classes had exactly the same size. The vertices of the U_0 can be distributed among the other classes, in the original version all the other classes were of exactly the same size.

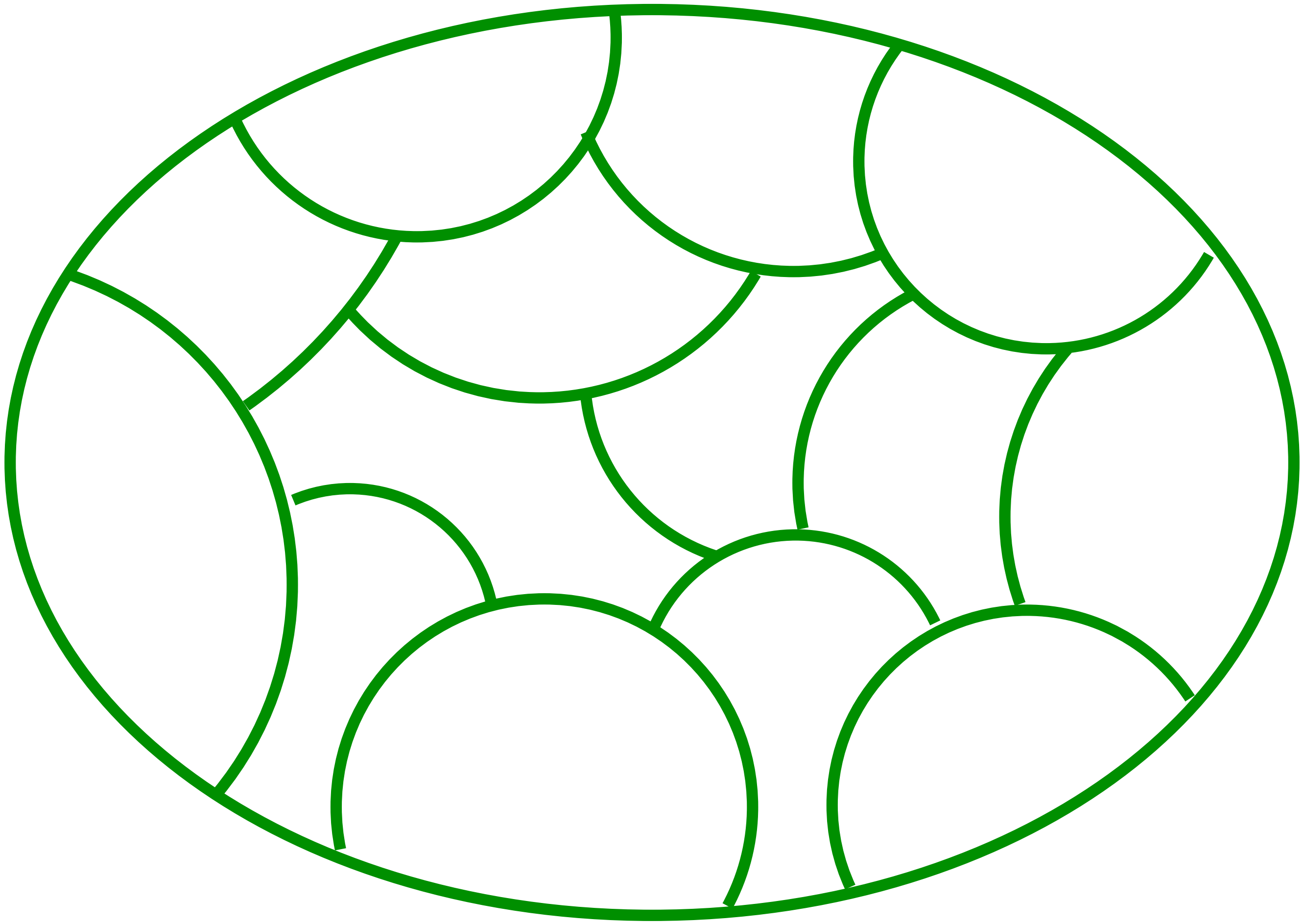
The meaning of the regularity lemma

All graphs can be *approximated by generalized random graphs* (in some sense) where

Definition of Generalized Random Graphs:
Given an $r \times r$ matrix of probabilities, $(p_{ij})_{r\times r}$ and a vector $(n_1, \ldots, n_r)$ take r groups of vertices, U_i and for each pair of vertices $x_i \in U_i$ and $x_j \in U_j$, join them independently, with probability p_{ij}.

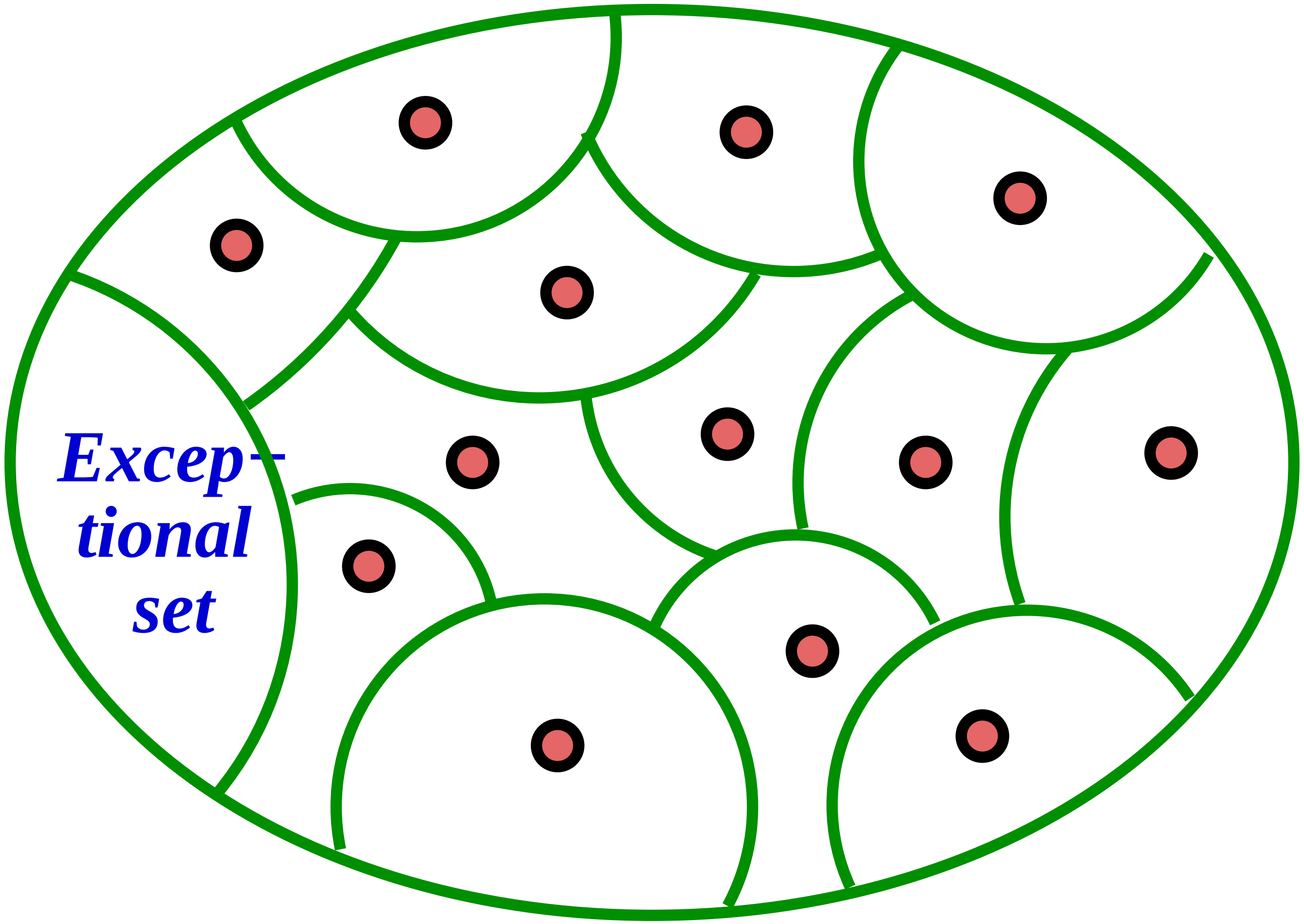


The reduced (or cluster) graph



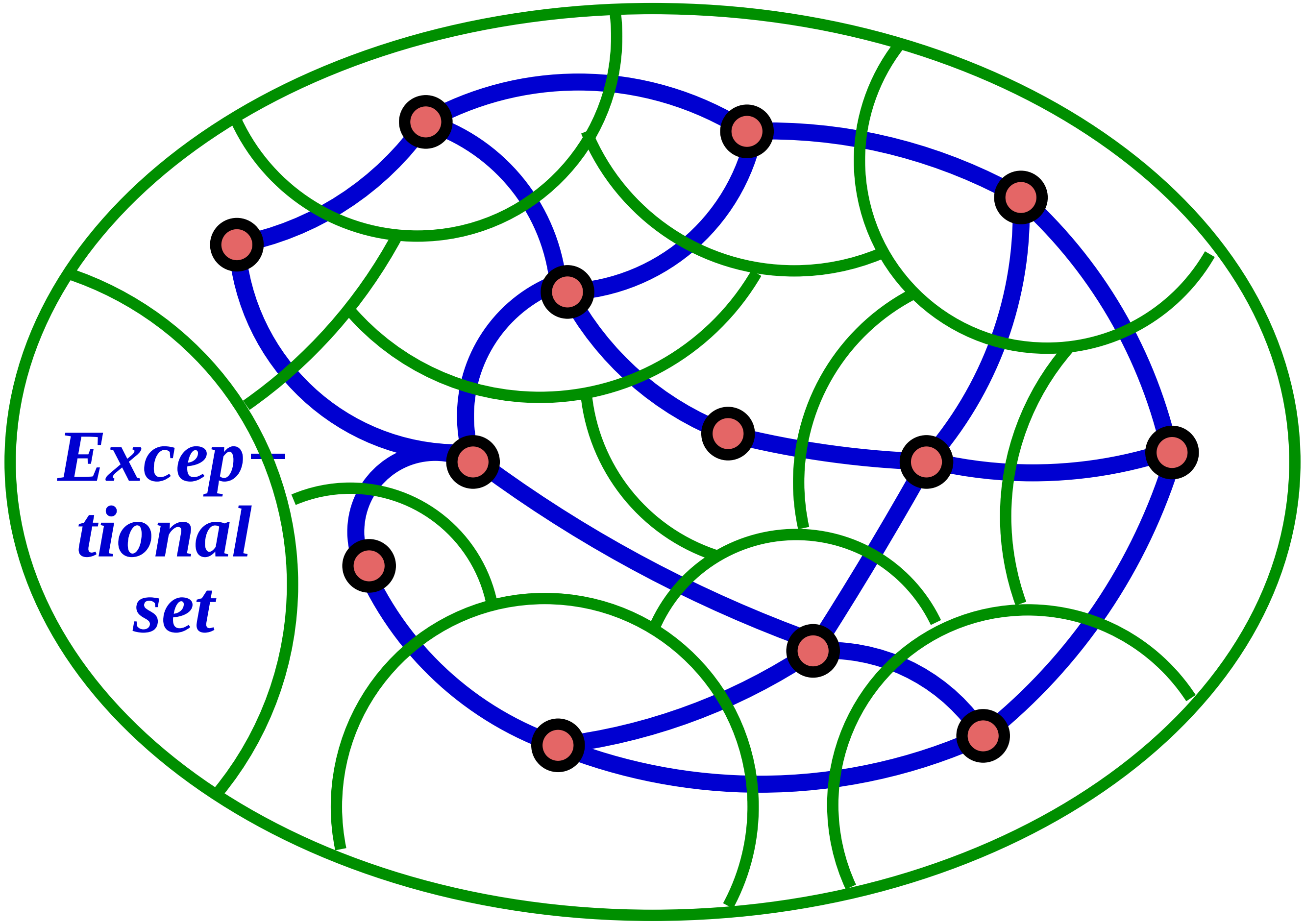
Fix two parameters: ε and $\tau \gg \varepsilon$
Start with the Szemerédi partition $U_1, \ldots, U_p$.

The reduced (or cluster) graph



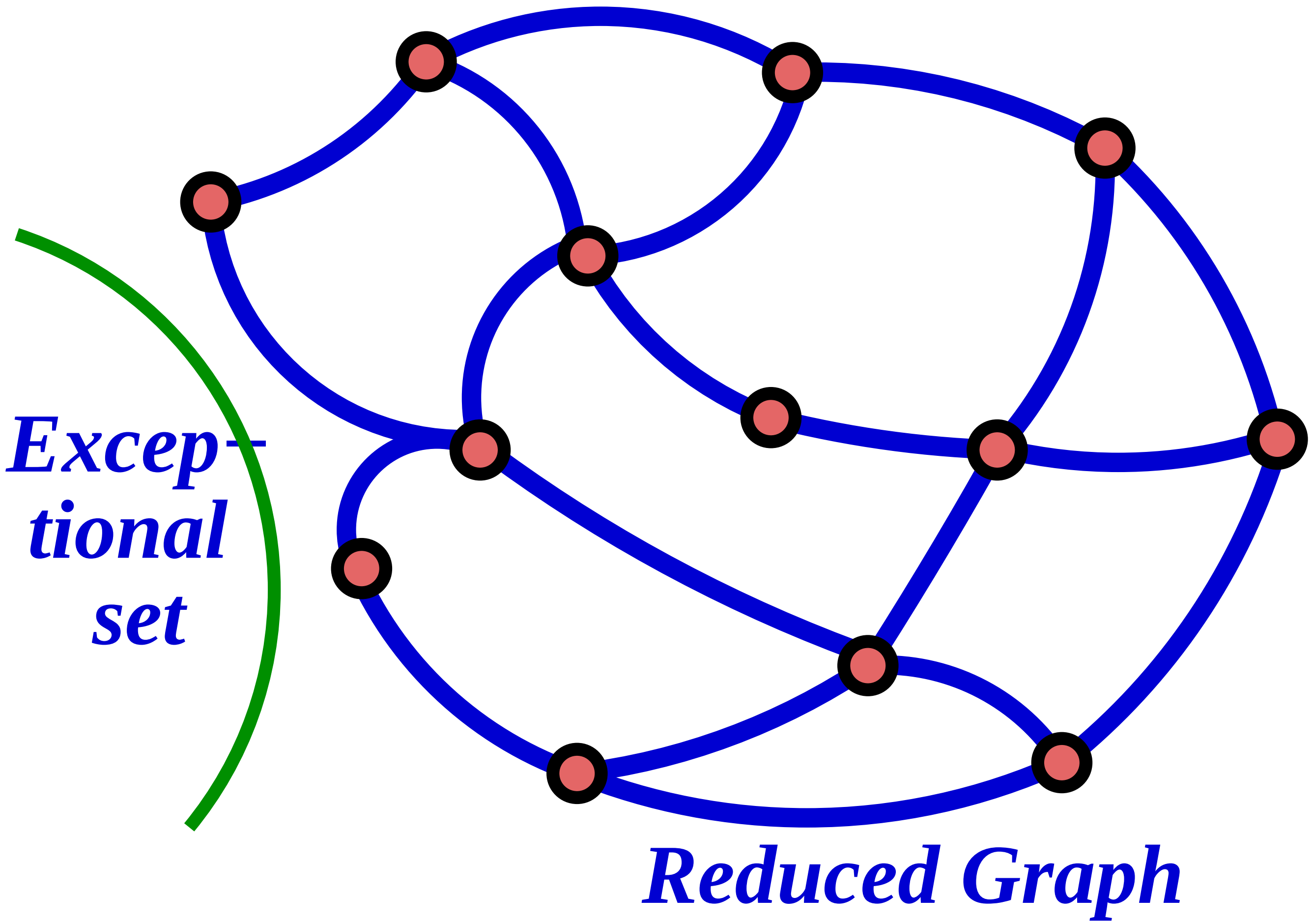
Build a graph on the classes: the vertices of H_ν are the classes

The reduced (or cluster) graph



Connect the pairs of classes (U_i, U_j) by a cluster-edge if they are classes ε-regularly connected, with density $d(U_i, U_j) > \tau$

The reduced (or cluster) graph



The vertices of U_0 are often distributed (randomly) in the others

Where does Regularity Lemma come from?

- There was an earlier "complicated" version
- The quantitative Erdős–Stone problem: Given a graph G_n with

$$e(G_n) \geq \left(1 - \frac{1}{p}\right)\binom{n}{2} + cn^2, \qquad (1)$$

define

$$m(n,p,c) = \max\{t : K_{p+1}(t,t,\ldots,t) \subset G_n \quad \text{subject to} \quad (1)\}.$$

- Bollobás-Erdős
- Bollobás-Erdős-Sim.
- Chvátal-Szemerédi: This is where Endre beautified/replaced the complicated **Regularity Lemma**.

The "complicated" version (A)

To prove the famous Szemerédi theorem *on arithmetic progressions* Endre used a more complicated **Regularity Lemma**:

- It was applied to dense bipartite graphs $G[A, B]$ where one had a partition $(U_1, \ldots, U_k)$ of A and *for each i, B had a partition* $(W_{i,1}, \ldots, W_{i,\ell})$ so that almost all pairs of classes $(U_i, W_{i,j})$ were ε-regular.
- This was enough for the famous theorem

$$r_k(n) = o(n),$$

i.e. for any fixed k,

Szemerédi: every infinite sequence of integers of positive upper density contains a k-term arithmetic progression.

- *This was* used in many early applications, not the "new" regularity lemma.

Chvátal, V.; Szemerédi, E. Notes on the Erdős–Stone theorem.

Let $m = m(c, d, n)$ be the largest natural number such that every graph with n vertices and at least $\frac{1}{2}n^2(1 - \frac{1}{d}) + cn^2$ edges contains a $K_{d+1}(m, \ldots, m))$.

- Erdős–Stone : $m(c, d, n) \to \infty$. Very weak estimate
- Erdős–Bollobás: $m \geq \eta(d, c) \log n$.

Theorem (Bollobás, Erdős, Sim.) *For some positive constant a,*

$$\frac{m(c, d, n)}{\log n} \geq \frac{a}{d \log(1/c)}.$$

Conjecture (Bollobás, Erdős, Sim.) *For some positive constant b,*

$$\frac{t(c, d, n)}{\log n} \geq \frac{b}{\log(1/c)}.$$

Chvátal, V.; Szemerédi, E.
Notes on the Erdős–Stone theorem. (cont)

- Erdős–Stone : $m(c,d,n)\to\infty$. Very weak estimate
- Erdős–Bollobás: $m \ge \eta(d,c)\log n$.

Theorem (Bollobás, Erdős, Sim.) *For some positive constant a,*

$$\frac{m(c,d,n)}{\log n} \ge \frac{a}{d\log(1/c)}.$$

Conjecture (Bollobás, Erdős, Sim.) *For some positive constant b,*

$$\frac{m(c,d,n)}{\log n} \ge \frac{b}{\log(1/c)}.$$

Chvátal Szemerédi: J. London Math. Soc. (2) 23 (1981), no. 2, 207–214;
Proves the B-E-S conjecture:

$$\lim_{n\to\infty}\frac{m(c,d,n)}{\log n} \ge \frac{1}{(500\log(1/c))}.$$

Success?

Several deep results of extremal graph theory became accessible only through the application of this central result. Some proofs are more “transparent” if we use the **Regularity Lemma**, though they can be proved also without it.

- *Ramsey-Turán* of K_4

> Let $RT(n, L, o(n))$ denotes the maximum edge-density of a graph-sequence G_n with $L \not\subseteq G_n$ and with independence number $\alpha(G_n) = o(n)$. Determine $RT(n, K_4, o(n))$.

(Many similar questions were solved by Erdős-Brown-Sós.)

- Independent matching (Ruzsa-Szemerédi), $f(n, 6, 3)$

> Brown, Erdős, and T. Sós asked (among others):
> How many triples can a 3-uniform hypergraph have without containing 6 vertices and 3 edges on this 6-tuple?

- Opens up a gate for elementary proofs of $r_k(n) = o(n)$?

The secret of success of the Regularity Lemma

- It makes possible to reduce

  embedding into deterministic structures

  to

  embedding into randomlike objects

Embedding into a random object is mostly easier.

Ramsey Theory

Also, large part of *Ramsey Theory* is very strongly connected to *Extremal Graph theory*. Application of the **Regularity Lemma** in these area was also crucial.

Stability

(Expanded)

1. *The extremal problem*
   We have a property $\mathcal{P}$, and consider the extremal problem of $G_n \notin \mathcal{P}$.
   We conjecture that S_n is an extremal graph (hypergraph, ...).

2. *What is the stability?*
   The almost extremal structures (for $\mathcal{P}$) are very similar to the extremal ones.

3. *Applying the stability method, to prove exact results*

   (a) Pick a very important, characteristic property $\mathcal{A}$ of the conjectured extremal structure S_n. (Examples: p-chromatic, ...)

   (b) Show that if a graph (hypergraph, ...) $G_n \notin (\mathcal{P} \cup \mathcal{A})$ then $e(G_n)$ is much smaller than $e(S_n)$.

   (c) So we may assume that the extremal graphs S_n have property $\mathcal{A}$.

   (d) Knowing that they have property $\mathcal{A}$, we prove the *exact* conjecture.

Füredi lecture:

The regularity lemma would immediately imply the Erdős-Simonovits Stability results if we knew the stability for K_{p+1}.

Direct proofs for this stability

Lovász-Sim.:

- On the number of complete subgraphs of a graph. II. Studies in pure mathematics, 459–495, Birkhäuser, Basel, 1983.
- On the number of complete subgraphs of a graph. Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen, 1975), pp. 431–441. Congressus Numerantium, No. XV, Utilitas Math., Winnipeg, Man., 1976.

- Füredi: His lecture here, using the Zykov symmetrization (see also Erdős, . . .) proved the stability directly for K_p. This implies the Erdős-Simonovits Stability results, via the **Regularity Lemma**

Origins of property testing?

- Bollobás-Erdős-Simonovits-Szemerédi

Is it true that if one cannot delete εn^2 edges from G_n then $C_{2\ell+1} \subseteq G_n$ for some $\ell = O_\varepsilon(1)$?

Solved in two ways:

- with **Regularity Lemma**
- without **Regularity Lemma**

This is an early application of property testing, asked by Erdős: those days property testing did not exist.

See also
Komlós: Covering odd cycles. Combinatorica 17 (1997), no. 3, 393–400.

Ramsey-Turán problems

Simplest case:

Problem (Erdős-Sós). *Given a sample graph L and we assume that*

$$L \not\subseteq G_n \quad \text{and} \quad \alpha(G_n) \le m,$$

what is the maximum of $e(G_n)$? $\mathbf{RT}(n, L, m)$

Problem (Erdős-Sós). *Given a sample graph L and and a sequence of graphs, (G_n), and we assume that*

$$L \not\subseteq G_n \quad \text{and} \quad \alpha(G_n) = o(n),$$

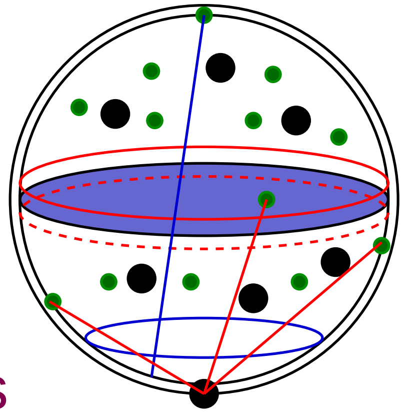
what is the maximum of $e(G_n)$? $\mathbf{RT}(n, L, o(n))$

Ramsey-Turán problems II

- Erdős-Sós: they determine $\mathbf{RT}(n, K_{2k+1}, o(n))$. (odd case)

Theorem K_4 (Szemerédi)

$$\mathbf{RT}(n, K_4, o(n)) = \frac{n^2}{8} + o(n^2).$$

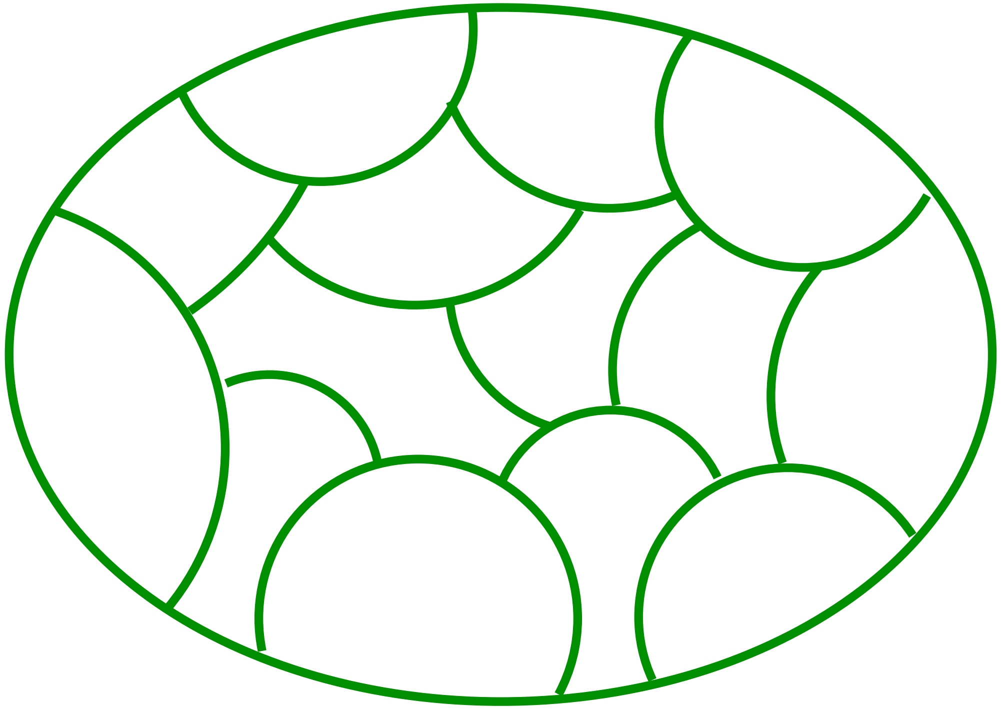


- Bollobás-Erdős

Erdős-Hajnal-Sós-Szemerédi: they determine $\mathbf{RT}(n, K_{2k}, o(n))$.

(even case)

How to prove ...



- Consider the regular partition
- take the reduced graph
- Show that it does not contain a K_3
- Show that the densities cannot (really) exceed $\frac{1}{2}$
- apply Turán's theorem

Ramsey-Turán problems IV

Continuation, among others, multigraph technique

- Erdős-Hajnal-Sim.-Sós-Szemerédi I
  - Erdős-Hajnal-Sim.-Sós-Szemerédi II

Erdős–Sós: For hypergraph questions completely new phenomena occur

Hypergraph extremal density (r-uniform):

$$\pi = \pi(L) = \limsup \left\{ \frac{e(H_n)}{\binom{n}{r}} \ : \ L \not\subseteq H_n \right\}$$

Ramsey-Turán:

$$\lambda = \lambda(L) = \limsup \left\{ \frac{e(H_n)}{\binom{n}{r}} \ : \ L \not\subseteq H_n \quad \text{and} \quad \alpha(H_n) = o(n) \right\},$$

where $\alpha(H)$ = maximum number of independent vertices in H.
Erdős and Sós asked if there exist r-uniform hypergraphs L for which $\pi(G) > \lambda(G) > 0$.

- Frankl + Rödl Combinatorica 8 (1988), no. 4, 323–332, *existence*
- Sidorenko: On Ramsey-Turán numbers for 3-graphs. J. Graph Theory 16 (1992), no. 1, 73–78. *Construction* L = 3-uniform hypergraph, $V(L) = \{1, 2, \cdots, 7\}$ and $E(L) = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 5\}, \{2, 6, 7\}, \{3, 4, 5\}, \{3, 6, 7\}, \{4, 6, 7\}, \{5, 6, 7\}\}$ satisfies $\pi(G) > \lambda(G) > 0$.
- Mubayi + Rödl Supersaturation for Ramsey-Turán problems.

Ramsey-Turán problems: open problems

Problem (Erdős-Sós). *Is it true that*

$$\mathbf{RT}(n, K_3(2,2,2), o(n)) = o(n^2)?$$

(Related constructions of Rödl)

Problem (Sim.). *Is it true, that for any L, "the"*

$$\mathbf{RT}(n, L, o(n))$$

-extremal sequence (???) can be approximated by a generalized random graph sequence *where all the probabilities are $0, \frac{1}{2}, 1$.*

Motivation: Is there always a Bollobás-Erdős type construction that is asymptotically extremal?

My meta-conjecture

- Matrix graphs

"Conjecture":
Whenever we try to prove a result where the extremal structure is described by a 0-1 matrix-graph, then the *Regularity Lemma* can be eliminated from the proof.

A counterexample?

- Ruzsa-Szemerédi: $f(n, 6, 3) = o(n^2)$
- Why is this important?
  - Füredi: Solution of the Murty-Simon (Plešnik) conjecture:

The maximum number of edges in a minimal graph of diameter 2. J. Graph Theory 16 (1992), no. 1, 81–98.

Diameter-critical if the deletion of any edge increases the diameter.

Theorem 1 (Füredi)**.** *Let G_n be a simple graph of diameter 2 on $n > n_0$ vertices, for which the deletion of any edge increases the diameter. Then $e(G_n) \leq \lfloor \frac{1}{4} n^2 \rfloor$ with equality holding if and only if $G \cong K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.*

Many open problems.

Extremal Subgraphs of random graphs

- Babai-Sim.-Spencer, J. Graph Theory 14 (1990), no. 5, 599–622.

Theorem BSS (Simplified) *There is a constant* $p_0 < \frac{1}{2}$ *such that if* R_n *is a random graph with edge-probability* $p > p_0$ *and* B_n *is the largest bipartite subgraph of it,* F_n *is the largest* K_3*-free subgraph, then* $F_n = B_n$ *(more precisely,* F_n *is bipartite!)*

- many generalizations
- Here we really needed the regularity lemma
- Generalizations to sparse random graphs, where the sparse regularity lemma is needed

What about sparse structures?

- Kohayakawa-Rödl lemma

Regularity Lemma is applied typically to dense graphs. (G_n) is sparse if $e(G_n) = o(n^2)$. Kohayakawa-Rödl extends **Regularity Lemma** to some sparse graph sequences, typically to non-random subgraphs of sparse random graph sequences.

Connection to quasi-randomness

A sequence of graphs is p-quasi-random iff it has a (sequence of) regular Szemerédi partitions, with densities tending to p.

Some of our theorems (Sim.-Sós, on quasirandomness) do not contain anything related to **Regularity Lemma** . Can one prove it without using the **Regularity Lemma** ?

Some new results

- Gyárfás-Ruszinkó-Sárközy-Szemerédi
  Ramsey, three colours, paths
- Kohayakawa-Sim.-Skokan
  Ramsey, three colours, odd cycles
- Balogh-Bollobás-Sim.
  Typical structure of L-free graphs
- Łuczak-Sim.-Skokan
  many colours, odd cycles

Property Testing?

- Bollobás-Erdős-Simonovits-Szemerédi
- Alon-Krivelevich. . .
- Alon-Schapira
-

Alon, Noga; Fischer, Eldar; Krivelevich, Michael; Szegedy, Mario: Efficient testing of large graphs. Combinatorica 20 (2000), no. 4, 451–476.

- Lovász-Balázs Szegedy: Szemerédi's lemma for analyst, Geom. Funct. Anal. 17 (2007) (1) 252–270.

- ábor Elek, . . .

It turns out that *property testing* and **Regularity Lemma** are extremely strongly connected to each other, see e.g. Alon-Shapira

Algorithmic aspects?

Alon-Duke-Leffmann-Rödl-Yuster:
The algorithmic aspects of the Regularity Lemma, Proc. 33 IEEE FOCS, Pittsburgh, IEEE (1992), 473-481.
see also J. of Algorithms 16 (1994), 80-109.

Strange situation:

- Given a partition, it is co-NPC to decide if it is ε-regular,

However,

- One can produce and ε-regular partition in polynomial time:

Theorem ADLRY (A *constructive* version of the **Regularity Lemma**) For every $\varepsilon > 0$ and every positive integer t there is an integer $Q = Q(\varepsilon, t)$ such that every graph with $n > Q$ vertices has an ε-regular partition into $k + 1$ classes, where $t \leq k \leq Q$.
For every fixed $\varepsilon > 0$ and $t \geq 1$ such a partition can be found in $O(M(n))$ sequential time, where $M(n)$ is the time for multiplying two $n \times n$ matrices with 0, 1 entries over the integers.

What about hypergraphs?

- connected to
  - Counting lemma
  - Removal lemma
- The results are much more complicated than for ordinary graphs
  - Weak hypergraph regularity lemma
  - Strong version
  - Counting lemma
  - Removal lemma
- The applications are also much more complicated
  - Rödl, Nagle, Skokan, Schacht,. . .
  - Tim Gowers, Terrence Tao
  - Ben Green

Disclaimer again: I have not tried to cover everything!

The most important thing, again:

Happy birthday, Endre!