Chapter 1

Paul Erdős' Influence on Extremal Graph Theory

Dedicated to Paul Erdős on the occasion of his 80th birthday

Summary. Paul Erdős is 80[1] and the mathematical community is celebrating him in various ways. Jarik Nešetřil also organized a small conference in Prague in his honour, where we, combinatorists and number theorists attempted to describe in a limited time the enourmous influence Paul Erdős made on the mathematics of our surrounding (including our mathematics as well). Based on my lecture given there, I shall to survey those parts of Extremal Graph Theory that are connected most directly with Paul Erdős's work.

In *Turán type extremal problems* we usually have some sample graphs $L_1, \dots, L_r$, and consider a graph G_n on n vertices not containing any L_i, We ask for the maximum number of edges such a G_n can have. We may ask similar questions for hypergraphs, multigraphs and digraphs.

We may also ask, how many copies of forbidden subgraphs L_i, must a graph G_n contain with a given number of edges superseding the maximum in the corresponding extremal graph problems. These are the problems on *Supersaturated Graphs*.

We can mix these questions with Ramsey type problems, (Ramsey-Turán Theory). This topic is the subject of a survey by V. T. Sós [162].

These topics are definitely among the favourite areas in Paul Erdős's graph theory.

Keywords: graphs, extremal graphs, graph theory.

1.1 Introduction

Extremal graph theory is a wide and fast developing area of graph theory. Having many ramifications, this area can be defined in a broader and in a more

[1]This refers to the time of the conference, not to when this volume appears.

P. Erdős: On sequences of integers no one of which divides the product of two others and related problems, Mitt. Forsch. Institut Mat. und Mech. Tomsk 2 (1938) 74-82.

restricted sense. In this survey we shall restrict our considerations primarily to **"Turán Type Extremal Graph Problems"** and some closely related areas.

Extremal graph theory is one of the wider theories of graph theory and – in some sense – one of those where Paul Erdős's profound influence can really be seen and appreciated.

What is a Turán Type Extremal Problem?

We shall call the **Theory of Turán type extremal problems** the area which – though being much wider – still is originated from problems of the following type:

Given a family $\mathcal{L}$ of sample graphs, what is the maximum number of edges a graph G_n can have without containing subgraphs from $\mathcal{L}$.

Here "subgraph" means "not necessarily induced". In Section 1.11 we shall also deal with the case of "excluded induced subgraphs", as described by Prömel and Steger.

Below K_t, C_t, and P_t will denote the complete graph, the cycle and the path of t vertices and $e(G)$ will be the number of edges of a graph G. G_n will be a graph of n vertices, $G(X,Y)$ a bipartite graph with colour classes X and Y.

The first result in our field may be that of Mantel [130] back in 1907, asserting that if a graph G_n contains no K_3, then

$$e(G_n) \leqslant \left[\frac{n^2}{4}\right].$$

Mantel's result soon became forgotten. The next extremal problem was the problem of C_4.

The C_4-Theorem and Number Theory

In 1938 Erdős published a paper [43]

In this paper Erdős investigated two problems:

(A) Assume that $n_1 < \ldots < n_k$ are positive integers such that n_i does not divide $n_h n_\ell$, except if either $i = h$ or $i = \ell$. What is the maximum number of such integers in $[1, n]$? Denote this maximum by $A(n)$.

Let $\pi(x)$ denote the number of primes in $[2, x]$. Clearly, the primes in $[2, n]$ satisfy our condition, therefore $A(n) \geqslant \pi(n)$. One could think that one can find much larger sets of numbers satisfying this condition. Surprisingly enough, the contrary is true: Erdős has proved that $A(n) \approx \pi(n)$. More precisely,

$$\pi(n) + \frac{n^{2/3}}{80 \log^2 n} \leqslant A(n) \leqslant \pi(n) + O\left(\frac{n^{2/3}}{\log^2 n}\right).$$

For us the other problem of [43] is more important:

(B) Assume that $n_1 < \ldots < n_k$ are positive integers such that $n_i n_j \neq n_h n_\ell$ unless $i = h$ and $j = \ell$ or $i = \ell$ and $j = h$. What is the maximum number of such integers in $[1, n]$? Denote this maximum by $B(n)$.

Here Erdős proved that

$$\pi(n) + \frac{cn^{3/4}}{(\log n)^{3/2}} \leqslant B(n) \leqslant \pi(n) + O(n^{3/4}).$$

Later Erdős improved the upper bound to

$$B(n) \leqslant \pi(n) + O\left(\frac{n^{3/4}}{\log^{3/2} n}\right),$$

see [55]. It is still open if, for some $c \neq 0$,

$$B(n) = \pi(n) + (1 + o(1)) \frac{cn^{3/4}}{(\log n)^{3/2}}$$

or not.

Solving this unusual type of number theoretical problem, Erdős (probably first) applied Graph Theory to Number Theory. He did the following: Let $\mathcal{D}$ be the set of integers in $[1, n^{2/3}]$, IP be the set of primes in $(n^{2/3}, n]$ and $\mathcal{B} = \mathcal{D} \cup IP$.

Lemma (Erdős). *Every integer $a \in [1, n]$ can be written as*

$$a = bd : b \in \mathcal{B}, d \in \mathcal{D}.$$

Let $\mathcal{A}$ be a set satisfying the condition in (B). Let us represent each $a \in \mathcal{A}$ as described in the Lemma: $a_i = b_j(i) d_j(i)$. We may assume that $b_i > d_i$. Build a bipartite graph $G(\mathcal{B}, \mathcal{D})$ by joining b_i to d_j if $a = b_i d_j \in \mathcal{A}$. Thus we represent each $a \in \mathcal{A}$ by an edge of a bipartite graph $G(\mathcal{B}, \mathcal{D})$. Erdős observed that the number theoretic condition in (B) implies that $C_4 \nsubseteq G(\mathcal{B}, \mathcal{D})$.

Indeed, if we had a 4-cycle $(b_1 d_1 b_2 d_2)$ in our graph, then $a_1 = b_1 d_1$, $a_2 = d_1 b_2$, $a_3 = b_2 d_2$ and $a_4 = d_2 b_1$ all would belong to $\mathcal{A}$ and $a_1 a_3 = a_2 a_4$ would hold, contradicting our assumption. So the graph problem Erdős formulated was the following:

Given a bipartite graph $G(X, Y)$ with m and n vertices in its colour classes. What is the maximum number of edges such a graph can have without containing a C_4?

Erdős proved the following theorem:

Theorem 1. *If $C_4 \nsubseteq G(X,Y)$, $|X| = |Y| = k$, then*

$$e(G(X,Y)) \leqslant 3k^{3/2}.$$

Here the constant 3 is not sharp (see Section 1.4). Basically this theorem implied the upper bound on $B(n)$. To get the lower bound Erdős used finite geometries. Erdős writes:

"...Now we prove that the error term cannot be better than $O\left(\frac{cn^{3/4}}{(\log n)^{3/2}}\right)$. First I prove the following lemma communicated to me by Miss E. Klein.[2]

Lemma. *Given $p(p+1)+1$ elements, (for some prime p) we can construct $p(p+1)+1$ combinations, taken $(p+1)$ at a time[3] having no two elements in common."*

Clearly, this is a finite geometry, and this seems to be the first application of Finite Geometrical Constructions in proving lower bounds in Extremal Graph Theory. Yet, Erdős does not speak here of finite geometries, neither of lower bounds for the maximum of $e(G(X,Y))$ in Theorem 1.

In the last years Erdős, András Sárközy and V. T. Sós started applying similar methods in similar number theoretical problems, which, again, led to new extremal graph problems, [80]. I mention just one of them:

Let $F_k(N)$ be the maximum number of integers $a_1 < a_2 < \ldots < a_t$ in $[1,N]$ with the property that the product of k different ones is never a square.

Theorem 2. (Erdős-A. Sárközy-T. Sós, [80]). *There exists a positive absolute constant $c > 0$ and for every $\varepsilon > 0$ an $N_0(\varepsilon)$ such that for $N > N_0(\varepsilon)$ we have*

$$\frac{(\sqrt{2}-\varepsilon)N^{2/3}}{\log^{4/3} N} < F_6(N) - \pi(N) - \pi(N/2) < cN^{7/9}\log N. \tag{1.1}$$

Taking all the primes of $[2,N]$ and all the numbers $2p$ where p is a prime in $[1,N/2]$ we get $\pi(N)+\pi(N/2)$ such numbers and the above theorem suggests that this construction is almost the best.

The solution of this problem depends on extremal graph theorems connected to excluding C_6. Analogous theorems hold for the even values of k, and somewhat different ones for odd values of k. The question which was asked is:

What is the maximum number of edges a bipartite graph $G(U,V)$ with u RED and v BLUE vertices can have if $G(U,V)$ contains no C_6 and $uv \leqslant N$?

In [80] the following conjecture was formulated:

Conjecture 3. *If $G(U,V)$ is a bipartite graph with $u = |U|$ RED vertices and $v = |V|$ BLUE ones, and $G(U,V)$ contains no C_6, and $v \leqslant u \leqslant v^2$, then $e(G(U,V)) \leqslant c(uv)^{2/3}$.*

The above upper bound of [80] has been improved first by Gábor Sárközy [146]. Then E. Győri proved the above conjecture, which in turn brought down the upper bound of (1.1) to the lower bound, apart from some log-powers.

[2] Eszter Klein, later Mrs. Szekeres.

[3] Meaning: $p+1$-tupes .... The text here does not tell us if the role of Mrs. Szekeres was important here or not, but somewhere else Erős writes: "Mrs. Szekeres and I proved ..."

As related references, [80], [32], [146], [122], [123] should be mentioned.

Theorem 4. ***(Győri).*** *If $C_{2k} \nsubseteq G(m,n)$, then $e(G(m,n)) \leqslant (k-1)n + O(m^2)$.*

Conjecture 5. ***(Győri).*** *There exists a $c > a$ for which, if $C_{2k} \nsubseteq G(m,n)$, then $e(G(m,n)) \geqslant (k-1)n + O(m^{2-c})$.*

Perhaps even $e(G(m,n)) \leqslant (k-1)n + O(m^{3/2})$ could be proved for C_6.

"How Did Crookes Miss to Invent the X-Ray?"

Erdős feels that he "should have invented" Extremal Graph Theory, back in 1938. He has failed to notice that his theorem was the root of an important and beautiful theory. 2-3 years later Turán proved his famous theorem and right after that he posed a few relevant questions, thus initiating a whole new branch of graph theory. Erdős often explains his "blunder" by telling the following story.

Crookes observed that leaving a photosensitive film near the cathod-ray-tube causes damage to the film: it becomes exposed. He concluded that

"Nobody should leave films near the cathod-ray-tube." Röntgen observed the same phenomenon a few years later and concluded that this can be used for filming the inside of various objects. His conclusion changed the whole Physics.[4] "It is **not enough** to be in the right place at the right time. You should also have an open mind at the right time," Paul concludes his story.

Erdős's influence on the field is so thorough that we do not even attempt to describe it in its full depth and width. We shall neither try to give a very balanced description of the whole, extremely wide area. Instead, we pick a few topics to illustrate Erdős's role in developing this subject, and his vast influence on others.

Also, I shall concentrate more on the new results, since the book of Bollobás [10], or the surveys of myself, [154], [156], [157], or the surveys of Füredi [99] and Sidorenko [149] provide a lot of information on the topic and some problem-papers of Erdős, e.g., [51] [59], [61], [62] are also highly recommended for the reader wishing to learn about the topic. Also, wherever it was possible, I selected newer results, or older but less known theorems (partly to avoid *unnecessary* repetition compared to the earlier surveys).

"The Complete List of Theorems"

If one watches Erdős in work, beside of his great proving power and elegance, one surprising feature is, how he poses his conjectures. This itself would deserve a separate note. Sometimes one does not immediately understand the importance of his questions. Slightly mockingly, once his friend, Hajnal told to him: "You

[4] When I asked Paul, why did he think that Röntgen's discovery changed the whole Physics, he answered that Röntgen's findings had led to certain results of Curie and from that point it was only a short step to the A-bomb.

I had to leave out quite a few very interesting topics. Practically, I skipped all the hypergraph theorems, [99], [149] the covering problems connected to the Erdős–Goodman-Pósa theorem, applications of finite geometrical methods in extremal graph theory, see, e.g. [161], [156], ... application of Lazebnik-Ustimenko type constructions, [122] and many more...Among others, I had to leave out that part of Ramsey Theory, which is extremely near to Extremal Graph Theory, (see [94]) ...and for many other things see Bollobás [10], [13], [14] ...

would like to have a Complete List of Theorems". I think there is some truth in this remark, still one modification should be made.

Erdős does not like to state his conjectures immediately in their most general forms. Instead, he picks very special cases and attacks first these ones. Mostly he picks his examples "very fortunately". Therefore, having solved these special cases he very often discovers whole new areas, and it is difficult for the surrounding to understand how can he be so "fortunate". So, the reader of Erdős and the reader of this survey should keep in mind that Erdős's method is to attack always **important special cases**.

Notation

We shall primarily consider simple graphs: graphs without loops and multiple edges but later there will be paragraphs where we shall consider digraph- and hypergraph extremal problems.

Given a family $\mathcal{L}$ of – so called – **excluded** or **forbidden** subgraphs, $\mathrm{ex}(n, \mathcal{L})$ will denote the maximum number of edges a graph G_n can have without containing forbidden subgraphs. (Containment does not assume "induced subgraph" of the given type.) The family of graphs attaining the maximum will be denoted by $\mathrm{EX}(n, \mathcal{L})$. If $\mathcal{L}$ consists of a single L, we shall use the notation $\mathrm{ex}(n, L)$ and $\mathrm{EX}(n, L)$ instead of $\mathrm{ex}(n, \{L\})$ and $\mathrm{EX}(n, \{L\})$.

For a set Q, $|Q|$ will denote its cardinality. Given a graph G, $e(G)$ will denote the number of its edges, $v(G)$ the number of its vertices, $\xi(G)$ and $a(G)$ its chromatic and independence numbers, respectively. For graphs the (first) subscript will mostly denote the number of vertices: G_n, S_n, $T_{n,p}, \dots$ denote graphs on n vertices. There will be one exception: speaking of excluded graphs $L_1, \dots, L_r$ we use superscripts just to enumerate these graphs. Given two disjoint vertex sets, X and Y, in a graph $G_n, e(X, Y)$ denotes the number of edges joining X and Y. Given a graph G and a set X of vertices of G, the number of edges in a subgraph spanned by a set X of vertices will be denoted by $e(X)$, the subgraph of G spanned by X is $G(X)$.

Special graphs. K_p will denote the complete graph on p vertices, $T_{n,p}$ is the so called Turán graph on n vertices and p classes: n vertices are partitioned into p classes as uniformly as possible and two vertices are joined iff they belong to different classes. This graph is the (unique) p-chromatic graph on n vertices with the maximum number of edges among such graphs. $K_p(n_1, \dots, n_p)$ (often abbreviated to $K(n_1, \ldots, n_p)$) denotes the complete p-partite graph with n_i vertices in its i^{th} class, $i = 1, 2, \ldots, p$.

We shall say that X is **completely joined** to Y if every vertex of X is joined to every vertex of Y. Given two vertex-disjoint graphs, G and H, their **product** $G \otimes H$ is the graph obtained by joining each vertex of G to each one of H.

Quoting. Below sometimes I quote some paragraphs from other papers, but the references and occasionally the notations too are changed to comply with mines.

1.2 Turán's Theorem

Perhaps Turán was the third to arrive at this field. In 1940 he proved the following theorem, [173] (see also [174], [172]):

Theorem 1. (Thrán). *(a) If G_n contains no K_p, then $e(G_n) \leqslant e(T_{n,p-1})$. In case of equality $G_n = T_{n,p-1}$.*

Turán's original paper contains much more than just this theorem. Still, the main impact coming from Turán was that he asked the general question:

> What happens if we replace K_p with some other forbidden graphs, e.g., with the graphs coming from the Platonic polyhedra, or with a path of length ℓ, etc.

Turán's theorem also could have sunk into oblivion. However, this time Erdős was more open-minded. He started proving theorems, talked to people about this topic and people started realizing the importance of the field.

Turán died in 1976. The first issue of Journal of Graph Theory came out around that time. Both Paul [60] and I were asked to write about Turán's graph theory [154]. (In the introductory issue of the journal Turán himself wrote a Note of Welcome, also mentioning some historical facts about his getting involved in graph theory [176].) Let me quote here some parts of Paul Erdős's paper [60].

> "In this short note I will restrict myself to Turán's work in graph theory, even though his main work was in analytic number theory and various other branches of real and complex analysis. Turán had the remarkable ability to write perhaps only one paper in various fields distant from his own; later others would pursue his idea and new subjects would be born.
>
> In this way Turán initiated the field of extremal graph theory. He started this subject in [1941], (see [173], [174]) He posed and completely solved the following problem ..."

Here Erdős describes Turán Theorem and Turán's hypergraph conjecture, and a result of his own to which we shall return later. Then he continues:

> "Turán also formulated several other problems on graphs, some of which were solved by Gallai and myself [66]. *I began a systematic study of extremal graph theory in 1958 on the boat from Athens to Haifa and have worked on it since then.* The subject has a very large literature; Bollobás has written a comprehensive book on extremal problems in graph theory which will appear soon." (Paul meant [10].)

One final remark should be made here. As I stated in other places, Paul Turán' s role was crucial in the development of Extremal Graph Theory. Still, even here there is a point, where Erdős's influence should be mentioned again. More precisely, the influence of an Erdős–Szekeres paper. As today it is already well known, Erdős and Szekeres tried to solve a problem in convex geometry and rediscovered Ramsey Theorem [91]. They informed Turán about their theorem, according to which either the graph or the complementary graph contains a large complete graph. Turán regarded this result as a theorem where one ensures the existence of a large complete subgraph in G_n by assuming something about the complementary graph. So Turán wanted to change the condition and still arrive at the same conclusion. This is why he supposed that a lower bound is given on the number of edges and deduced the existence of a large complete subgraph of G_n. Turán writes in [173]:

> "Theorem I gives a condition to guarantee the existence of a complete subgraph on k vertices in a graph on a finite number of vertices. The only related theorem – as far as I know – can be found in a joint paper of Pál Erdős and György Szekeres [91] and essentially states that if a graph A on n vertices is such that its complement $\bar{A}$ contains only complete subgraphs having "few" vertices, then the graph A contains a complete subgraph on "many" vertices. Their theorem contains only bounds in the place of the expressions "few" and "many", in fact it gives almost only the existence; the exact solution seems to be very interesting but difficult . . .

Some (Further) Historical Remarks

(a) Turán's paper contains an infinite Ramsey theorem. I quote:

Theorem II. *Let us suppose that for the infinite graph A containing countably many vertices $P_1, P_2, \ldots$ there is an integer $d > 1$ such that if we choose arbitrary d different vertices of A, there will be at least two among these vertices joined by an edge in A. Then A has at least one complete subgraph of infinitely many vertices.*

This theorem is weaker than the one we usually teach in our courses, nevertheless, historically it is interesting to see this theorem in Turán's paper.

(b) Turán's theorem is connected to the Second World War in two ways. On the one hand, Turán, sent to forced labour service and deprived of paper and pencil, started working on problems that were possible to follow without writing them down. Also he made his famous hypergraph conjecture, thinking that would he have paper and pencil, he could have easily proved it.

On the other hand it is worth mentioning that Turán's Theorem was later rediscovered by A. A. Zykov [[181], 1949] who (because of the war) learned too late that it had already been published.

(c) As to Mantel's result, I quote the last 4 lines of [173]: "**Added in proof.** ... Further on, I learned from the kind communication of Mr. József Krausz that the value of $d_k(n)(= \text{ex}(n, K_3))$ is given on p438, for $k = 3$ was found already in 1907 by W. Mantel, (Wiskundige Opgaven, vol. **10**, p60-61).

I know his paper only from the reference of Fortschritte d. Math., vol **38** p.270."

(d) During the war Turán was trying to prove that either a graph G_n or its complementary graph contains a complete graph of order $[\sqrt{n}]$. He writes in [176]:

"I still have the copybook in which I wrote down various approaches by induction, all they started promisingly, but broke down at various points. I had no other support for the truth of this conjecture, than the symmetry and some dim feeling of beauty: ... In one of my first letters to Erdős after the war I wrote of this conjecture to him. In his answer he proved that my conjecture was utterly false ..."

Of course, all we know today, what Erdös wrote to Turán: the truth is around $c\ \log n$. This was perhaps the first application of probability to Graph Theory, though many would deny that Erdős's elegant answer uses more than crude counting. Probably this is where the Theory of Random Graphs started. (To be quite precise, one should mention, that T. Szele had a similar proof for Rédei's theorem on directed Hamiltonian cycles in tournaments, [166], already in 1943, however, Erdős's proof was perhaps of more impact and it was the first where no other approach could replace the counting argument. Another early breakthrough of the Random Graph Method was when Erdős easily answered the following problem of Schütte [49]: Is there a tournament where for every k players there is a player which beats all of them?)

I would suggest the reader to read also the beautiful paper of Turán [176], providing a lot of information on what I have described above shortly.

(e) For a longer account on the birth of the Erdős-Szekeres version of Ramsey theorem see the account of Gy. Szekeres in the introduction of the Art of Count ing, [57].

1.3 Erdős-Stone Theorem

Setting out from a problem in topology, Erdős and A. H. Stone proved the following theorem in 1946 [90]:

Theorem 1. ***(Erdős-Stone).*** *For every fixed p and m*

$$\mathrm{ex}(n, K_{p+1}(m, \dots, m)) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2). \qquad (1.2)$$

Moreover, if p is fixed and $m := \sqrt{\ell_p(n)}$ where $\ell_p(x)$ denotes the p times iterated logarithm of x, (1.2) still holds.

Here $m = \sqrt{\ell_p(n)}$ is far from being the best possible. The sharp order of magnitude is $c \log n$. Let $m = m(n, \varepsilon)$ be the largest integer such that, if $e(G_n) > e(T_{n,p}) + \varepsilon n^2$, then G_n contains the regular $(p+1)$-partite graph $K_{p+1}(m, m, \dots, m)$. One can ask how large is $m = m(n, \epsilon)$, defined above? This was determined by Bollobás, Erdős, Simonovits, [15], [17] Chvátal and Szemerédi [37].

The first breakthrough was that the p-times iterated log was replaced by $K \log n$, where K is a constant [15]. In the next two steps the dependence of this constant on p and ε were determined.

Theorem 2. (Bollobás-Erdős-Simonovits [17]). *There exists an absolute constant $c > 0$ such that every G_n with*

$$e(G_n) \geqslant \left(1 - \frac{1}{p} + \varepsilon\right)\binom{n}{2}$$

contains a $K_{p+1}(m, m, \dots, m)$ with

$$m > \frac{c \log n}{p \log(1/\varepsilon)}.$$

The next improvement, essentially settling the problem completely is the result of Chvátal and Szemerédi, providing the exact dependence on all the parameters, up to an absolute constant.

Theorem 3. (Chvátal-Szemerédi [37]).

$$\frac{\log n}{500\ \log(1/\varepsilon)} < m(n, \varepsilon) < \frac{c \log n}{\log(1/\varepsilon)}.$$

One could have thought that the problem is settled but here is a nice result of Bollobás and Kohayakawa, improving Theorem 3.

Conjecture 4. (Bollobás-Kohayakawa [18]). *There exists an absolute constant $\alpha > 0$ such that for all $r \geqslant 1$ and $0 \leqslant \varepsilon \leqslant 1/r$ every G_n of sufficiently large order satisfying*

$$e(G_n) \geqslant \left(1 - \frac{1}{r} + \varepsilon\right)\binom{n}{2}$$

contains a $K_{r+1}(s_0, m_0, \ldots, m_0)$, where

$$s_0 = s_0(n) = \left\lfloor \frac{\alpha \log n}{\log(1/\varepsilon)} \right\rfloor \qquad \text{and} \qquad m_0 = m_0(n) = \left\lfloor \frac{\alpha \log n}{\log r} \right\rfloor.$$

Bollobás and Kohayakawa [18] succeded in proving that under the above conditions, if $0 < \gamma < 1$, then G_n contains a $K_{r+1}(s_1, m_1, \ldots, m_1, \ell)$, where

$$s_1 = \left\lfloor \frac{\alpha(1-\gamma)\log n}{r\log(1/\varepsilon)} \right\rfloor, \qquad m_1 = \left\lfloor \frac{\alpha(1-\gamma)\log n}{\log r} \right\rfloor, \qquad \text{and} \qquad \ell = \lfloor \alpha\varepsilon^{1+\gamma/2} n^\gamma \rfloor.$$

Observe that this result is fairly near to proving Conjecture 4: the first class is slightly smaller and the last class much larger than in the conjecture.

The Kővári-V. T. Sós-Turán Theorem

The Kővári-T. Sós-Turán theorem [121] solves the extremal graph problem of $K_2(p, q)$, at least, provides an upper bound, which in some cases proved to be sharp.[5] This theorem is on the one hand a generalization of the C_4-problem, since $C_4 = K(2, 2)$, and, on the other hand, is a special case of the Erdős-Stone theorem, apart from the fact that we get sharper estimates.[6],[7]

Theorem 5. (Kővári-T. Sós-Turán). *Let $2 \leqslant p \leqslant q$ be fixed integers. Then*

$$\operatorname{ex}(n, K(p, q)) \leqslant \frac{1}{2}\sqrt[p]{q-1}\, n^{2-1/p} + O(n).$$

The exponent $2-(1/p)$ is conjectured to be sharp but this is known only for $p = 2$ and $p = 3$, (see Erdős, Rényi, V. T. Sós, [78] and independently W. G. Brown [25]). Random graph methods [89] show that

$$\operatorname{ex}(n, K(p, p)) > c_p n^{2-\frac{2}{p+1}}.$$

Recently Füredi [101] improved the constant in the upper bound, showing that

$$\operatorname{ex}(n, K(2, t+1)) = \frac{1}{2}\sqrt{t}\, n^{3/2} + O(n^{4/3}).$$

and that the constant provided by Brown's construction is sharp. While one conjectures that $\operatorname{ex}(n, K(4, 4))/n^{7/4}$ converges to a positive limit, we know only, by the Brown construction, that $\operatorname{ex}(n, K(4, 4)) > \operatorname{ex}(n, K(3, 3)) > cn^{5/3}$. It is unknown if

$$\frac{\operatorname{ex}(n, K(4, 4))}{n^{5/3}} \to \infty.$$

[5] A footnote of [121] tells us that the authors have received a letter from Erdős in which Erdős informed them that he also had proved most of the results of [121].

[6] Estimates, sharper than in the original Erdős-Stone.

[7] Very recently J. Kollár, L. Rónyai and T. Szabó showed that if $q > p!$, then $\operatorname{ex}(n, K_2(p, q)) > c_p n^{2-(1/p)}$. The paper: "Norm-graphs and Bipartite Turán numbers" will appear in Combinatorica, 1996.

The Matrix Form

The problem of Zarankiewicz is to determine the maximum integer $k_p(n)$ such that if An is a matrix with n rows and n columns consisting exclusively 0's and 1's, and the number of 1's is at least $k_p(n)$, then An contains a minor B_p of p rows and columns so that all the entries of B_p are 1's.

One can easily see that this problem is equivalent with determining the maximum number of edges a bipartite graph $G(n,n)$ can have without containing $K(p,p)$.

In [121] the authors remark that the problem can be generalized to the case of general matrices: when A has m rows and n columns and B has p rows and q columns. Denote the maximum by $k(m,n,p,q)$. There are many results on estimating this function but we shall not go here into details. Rather, we explain the notion of symmetric and asymmetric bipartite graph problems.

As Erdős pointed out,

Theorem 6. *Every graph G_n has a bipartite subgraph $H(U,V)$ in which each vertex has at least half of its original degree: $d_H(x) \geqslant \frac{1}{2}d_G(x)$, and therefore $e(H(U,V)) \geqslant \frac{1}{2}e(G_n)$.*

One important consequence of this (almost trivial) fact is that (as to the order of magnitude), it does not matter if we optimize $e(G_n)$ over *all* graphs or only over the *bipartite* graphs. Another important consequence is that some matrix extremal problems are equivalent to graph extremal problems. Conversely, many extremal graph problems with bipartite excluded subgraphs have equivalent matrix forms as well:

As usually, having a bipartite graph, $G(U,V)$ we shall associate with it a matrix A, where the rows correspond to U, the columns to V and $a_{ij} = 1$ if the i^{th} element of U is joined to the j^{th} element of V, otherwise $a_{ij} = 0$.

Given a bipartite graph $L = L(X,Y)$ and another bipartite graph $G(U,V)$ $|U| = m$ and $|V| = n$, take the $m \times n$ adjacency matrix A of C and the adjacency matrix B of L. Assume for a second that the colour-classes of L are symmetric (in the sense that there is an automorphism of L exchanging the two colour-classes). Then the condition that $L \nsubseteq G(U,V)$ can be formulated by saying that the matrix A has no submatrix equivalent to B, where equivalency means that they are the same apart from some but same permutation of the rows and columns. So Turán type problems lead to problems of the following forms:

Given an $m \times n$ $0-1$ matrix, how many 1's ensure a submatrix equivalent to B?

If, on the other hand, the forbidden graph $L = L(X,Y)$ has no automorphism exchanging X and Y, then the matrix-problem and the graph-problem may slightly differ. Excluding the submatrices equivalent to B means that we exclude that $G(U,V)$ contains an L with $X \subseteq U$ and $Y \subseteq V$, but we do not exclude $L \subseteq G(U,V)$ in the opposite position. Denote by $\mathrm{ex}^*(n,L)$ the maximum in this asymmetric case. Clearly, $\mathrm{ex}^*(n,L) \geqslant \mathrm{ex}(n,L)$, and they are equal if L has a colour-swapping automorphism.

Conjecture 7. ***(Simonovits).*** *If L is bipartite, then* $\mathrm{ex}^*(n, L) = O(\mathrm{ex}(n, L))$.

We do not know this even for $K(4, 5)$. The difficulty in disproving such a conjecture is partly that in all the proofs of upper bounds on degenerate extremal graph problems, we use only "one-sided" exclusion. Therefore the upper bounds we know are always upper bounds on $\mathrm{ex}^*(n, L)$.

Conjecture 8. ***(Erdős-Simonovits).*** *For every $\mathcal{L}$ with a bipartite $L \in \mathcal{L}$ there is a bipartite $L^* \in \mathcal{L}$ for which* $\mathrm{ex}(n, \mathcal{L}) = O(\mathrm{ex}(n, L^*))$.

We close this part with a beautiful but probably difficult problem of Erdős.

Conjecture 9. $\mathrm{ex}(n, \{C_3, C_4\}) = \frac{1}{2\sqrt{2}} n^{3/2} + o(n^{3/2})$.

The meaning of this conjecture is that excluding C_3 beside C_4 has the same effect as if we excluded all the odd cycles. If we replace C_3 by C_5, then this is true, see [84]. Erdős risks the even sharper conjecture that the exact equality may hold:

$$\mathrm{ex}(n, \{C_3, C_4\}) = \mathrm{ex}(n,\ \{C_3, C_4, C_5, C_6, C_7, C_9, C_{11} \ldots\}).$$

For some further information, see a survey paper of Richard Guy, [103] and also a paper of Guy and Znam [104] on $K(p, q)$ and the results of Lazebnik, Ustimenko and Woldar on cycles [122], [123].

Applications of Kővári-T. Sós-Turán Theorem

It is interesting to observe that the C_4-theorem and its immediate generalizations (e.g. the Kővári-T. Sós–Turán theorem) have quite a few applications. Some of them are in geometry. For example, as Erdős observed, if we have n points in the plane, and join two of them if their distance is exactly 1, then the resulting graph contains no $K(2, 3)$. So the number of unit distances among n points of the plane is $O(n^{3/2})$. Similarly, the unit-distance-graph of the 3-dimensional space contains no $K(3, 3)$, therefore the number of unit distances in the 3-space is $O(n^{5/3})$. There are deeper and sharper estimates on this subject, see Spencer, Szemerédi and Trotter [163] or Clarkson, Edelsbrunner, Guibas, Sharir and Welzl [38].

Conjecture 10. ***(Erdős).*** *For every $\varepsilon > 0$, the number of unit distances among n points of the plain is $O(n^{1+\varepsilon})$.*

We mention one further application: the **chromatic number of the product** hyper-graph. Claude Berge was interested in calculating the chromatic number of the product of two graphs, $\mathcal{H}$ and $\mathcal{H}'$. Generally there are various ways to define the product of r-uniform hypergraphs. **This** product is defined as the r^2-uniform hypergraph whose vertex-set is the Cartesian product $V(\mathcal{H}) \times V(\mathcal{H}')$ and the edgeset is

$$\{H \times H' : H \in E(\mathcal{H}) \qquad \text{and} \qquad H' \in E(\mathcal{H}')\}.$$

The chromatic number of the graph is the least k such that the vertices can be coloured in k colours without having monochromatic r^2-tuples. Berge and I [8] estimated the chromatic number of products of graphs (hypergraphs) using Kővári-T. Sós-Turán theorem. The same time a student of Berge, F. Sterboul [164], [165] have proved the same theorem and an *earlier* paper of V. Chvátal [36] used the same technique to prove some assertions roughly equivalent with this part of our paper [8].

1.4 Graph Theory and Probability

Erdős wrote two papers with the above title, one in 1959, [45], and the other in 1961, [46]. These papers were of great importance. In the first one Erdős proved the following theorem.

Theorem 1. *For fixed k and sufficiently large ℓ, $n > \ell^{1+1/(2k)}$, then there exist (many) graphs G_n of girth k and independence number $\alpha(G_n) < \ell$.*

Clearly, the chromatic number of such a graph is at least $\frac{v(G_n)}{\alpha(G_n)}$. So, as Erdős points out, a corollary of his theorem is that

Corollary 2. *For every integer k for $n > n_0(k)$ there exist graphs G_n of girth $\geqslant k$ and chromatic number $\geqslant n^{\frac{1}{2k+1}}$.*

This theorem seems to be a purely Ramsey theoretical result, fairly surprising in those days, but, it has many important consequences in Extremal Graph Theory as well. The same holds for the next theorem, too:

Theorem 3. ([46]). *Assume that $n > n_0$. Then there exist graphs G_n with $K_3 \nsubseteq G_n$ and $\alpha(G_n) = O(\sqrt{n}\log n)$.*

One important corollary of Theorem 1, more precisely, of its proof is that

Theorem 4. *If $\mathcal{L}$ contains no trees, then $\mathrm{ex}(n, \mathcal{L}) > c^*_{\mathcal{L}} n^{1+c_{\mathcal{L}}}$, for some constants $c^*_{\mathcal{L}}, c_{\mathcal{L}} > 0$.*

On the other hand, it is easy to see that if $L \in \mathcal{L}$ is a tree (or a forest), then $\mathrm{ex}(n, \mathcal{L}) = O(n)$.

These theorems use random graph methods. They and some of their generalizations play also important role, in obtaining lower bounds in Turán-Ramsey Theorems. (See also Füredi-Seress, [102].) For general applications of the probabilistic methods in graph theory see, e.g. Erdős-Spencer [89], Bollobás [13], Alon-Spencer [6].

Of course, speaking of Graph Theory and Probability, one should also mention the papers of Erdős and Rényi, perhaps above all, [77].

1.5 The General Theory

In this section, we present the asymptotic solution to the general extremal problem.

General Extremal Problem. Given a family $\mathcal{L}$ of forbidden subgraphs, find those graphs G_n that contain no subgraph from $\mathcal{L}$ and have the maximum number of edges.

The problem is considered to be "completely solved" if all the extremal graphs have been found, at least for $n > n_0(\mathcal{L})$. Quite often this is too difficult, and we must be content with finding $\mathrm{ex}(n, \mathcal{L})$, or at least good bounds for it.

It turns out that a parameter related to the chromatic number plays a decisive role in many extremal graph theorems. The **subchromatic number** $p(\mathcal{L})$ of $\mathcal{L}$ is defined by

$$p(\mathcal{L}) = \min\{\chi(L) : L \in \mathcal{L}\} - 1.$$

The following result is an easy consequence of the Erdős-Storie theorem [90]:

Theorem 1. (The Erdős-Simonovits Theorem [81]). *If $\mathcal{L}$ is a family of graphs with subchromatic number p, then*

$$\mathrm{ex}(n, \mathcal{L}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2).$$

The meaning of this theorem is that $\mathrm{ex}(n, \mathcal{L})$ depends only very loosely on $\mathcal{L}$; up to an error term of order $o(n^2)$, it is already determined by the minimum chromatic number of the graphs in $\mathcal{L}$.

Classification of Extremal Graph Problems

By the above theorem,

$$\mathrm{ex}(n, \mathcal{L}) = o(n^2)$$

if and only if $p(\mathcal{L}) = 1$, i.e. there exist bipartite graphs in $\mathcal{L}$, From the Kövári-T. Sós-Turán Theorem we get that here $\mathrm{ex}(n, \mathcal{L}) = O(n^{2-c})$ for some $c = c(\mathcal{L})$. We shall call these cases **degenerate extremal graph problems** and find them among the most interesting problems in extremal graph theory. One special case is when contains a tree (or a forest). These cases could be called **very degenerate**. Observe, that if a problem is **non-degenerate**, then $T_{n,2}$ contains no excluded subgraphs. Therefore $\mathrm{ex}(n, \mathcal{L}) \geqslant \left[\frac{n^2}{4}\right]$.

Structural Results

The structure of the extremal graphs is also almost determined by $p(\mathcal{L})$, and is very similar to that of $T_{n,p}$ This is expressed by the following results of Erdős and 8imonovits [52], [54], [150]:

Theorem 2. (The Asymptotic Structure Theorem). *Let $\mathcal{L}$ be a family of forbidden graphs with subchromatic number p. If S_n is any graph in* $\mathrm{EX}(n, \mathcal{L})$, *then it can be obtained from $T_{n,p}$ by deleting and adding $o(n^2)$ edges. Furthermore, if $\mathcal{L}$ is finite, then the minimum degree*

$$d_{min}(S_n) = \left(1 - \frac{1}{p}\right)n + o(n).$$

The structure of extremal graphs is fairly stable, in the sense that the almost extremal graphs have almost the same structure as the extremal graphs (for $\mathcal{L}$ or for K_{p+1}). This is expressed in our next result:

Theorem 3. (The First Stability Theorem). *Let $\mathcal{L}$ be a family of forbidden graphs with subchromatic number $p \geqslant 2$. For every $\varepsilon > 0$, there exist a $\delta > 0$ and n_ε such that, if G_n contains no $L \in \mathcal{L}$, and if, for $n > n_\varepsilon$,*

$$e(G_n) > \mathrm{ex}(n, \mathcal{L}) - \delta n^2,$$

then G_n can be obtained from $T_{n,p}$ by changing at most εn^2 edges.

These theorems are interesting on their own and also widely applicable.

In the remainder of this section we formulate a sharper variant of the stability theorem.

One can ask whether further information on the structure of forbidden subgraphs yields better bounds on $\mathrm{ex}(n, \mathcal{L})$ and further information on the structure of extremal graphs. At this point, we need a definition.

Let $\mathcal{L}$ be a family of forbidden subgraphs, and let $p = p(\mathcal{L})$ be its subchromatic number. The **decomposition** $\mathcal{M}$ of $\mathcal{L}$ is the family of graphs M with the property that, for some $L \in \mathcal{L} L$ contains M as an induced subgraph and $L - V(M)$ is $(p-1)$-colorable.

In other words, for $r = v(L), L \subseteq M \times K_{p-1}(r, \ldots, r)$, and M is minimal with this property. The following result is due to Simonovits [150], (see also [54]). In case of $\mathcal{L} = \{K_p\}$ the family $\mathcal{M}$ consists of one graph K_2.

Theorem 4. (The Decomposition Theorem, [150]). *Let $\mathcal{L}$ be a forbidden family of graphs with $p(\mathcal{L}) = p$ and decomposition $\mathcal{M}$. Then every extremal graph $S_n \in \mathrm{EX}(n, \mathcal{L})$ can be obtained from a suitable $K_p(n_1, \ldots, n_p)$ by changing $O(\mathrm{ex}(n, \mathcal{M}) + n)$ edges. Furthermore, $n_j = (n/p) + O(\mathrm{ex}(n, \mathcal{M})/n) + O(1)$, and*

$$d_{\min}(S_n) = \left(1 - \frac{1}{p}\right) n + O(\mathrm{ex}(n, \mathcal{M})/n) + O(1)$$

It follows from this theorem that, with $m = \lceil n/p \rceil$, $\mathrm{ex}(n, \mathcal{L}) = e(T_{n,p}) + O(\mathrm{ex}(m, \mathcal{M}) + n)$. If $\mathrm{ex}(n, \mathcal{M}) > cn$, then $O(\mathrm{ex}(m, \mathcal{M}))$ is sharp: put edges into the first class of a $T_{n,p}$ so that they form a $G_m \in \mathrm{EX}(m, \mathcal{M})$; the resulting graph contains no L, and has $e(T_{n,p}) + \mathrm{ex}(m, \mathcal{M})$ edges.

A second stability theorem can be established using the methods of [150]. To formulate it, we introduce some new terms. Consider a partition $S_1, S_2, \ldots, S_p$ of the vertex-set of G_n, and the complete p-partite graph $H_n = K(S_1, \ldots, S_p)$ corresponding to this partition of $V(G_n)$, where $s_i = |S_i|$. An edge is called an **extra edge** if it is in G_n but not in H_n, and is a **missing edge** if it is in H_n but not in G_n. For given p and G_n, the partition $S_1, S_2, \ldots, S_p$ is called **optimal** if the number of missing edges is minimum. Finally, for a given vertex v, let $a(v)$ and $b(v)$ denote the numbers of missing and extra edges at v.

Theorem 5. (The Second Stability Theorem). *Let $\mathcal{L}$ be a forbidden family of graphs with $p(\mathcal{L}) = p$ and decomposition $\mathcal{M}$, and let $k > 0$. Suppose that G_n contains no $L \in \mathcal{L}$,*

$$e(G_n) \geqslant \mathrm{ex}(n, \mathcal{L}) - k \cdot \mathrm{ex}(n, \mathcal{M}),$$

and let $S_1, \ldots, S_p$ be the optimal partition of $G_n, G_i := G(S_i)$. Then

(i) G_n can be obtain ed from $\times G_i$ by deleting $O(\mathrm{ex}(n, \mathcal{M}) + n)$ edges;

(ii) $e(G_j) = O(\mathrm{ex}(n, \mathcal{M})) + O(n)$, and $v(G_j) = (n/p) + O(\sqrt{\mathrm{ex}(n, \mathcal{M})} + \sqrt{n})$;

(iii) for any constant $c > 0$, the number of vertices v in G_i with $a(v) > cn$ is only $O(1)$, and the number of vertices with $b(v) > cn$ is only $O(\mathrm{ex}(n, \mathcal{M})/n) + O(1)$;

(iv) let $L \in \mathcal{L}$; with $v(L) = r$, and let A_i be the set of vertices v in S_i for which $b(v) < (n/2pr)$; if $M \times K_{p-1}(r_1, \ldots, r) \subseteq L$, then the graph $G(A_i)$ contains no M.

The constant k of the condition cannot be seen in (i)-(iv): it is hidden in the constants of the $O(.)$'s This theorem is useful also in applications. The deepest part is the first part of (iii). This implies (iv), which in turn implies all the other statements. A proof is sketched in [77], where the theorem was needed.

We conclude this section with the theorem characterizing those cases where $T_{n,p}$ is the extremal graph.

Theorem 6. (Simonovits, [153]). *The following statements are equivalent:*

(a) The minimum chromatic number in $\mathcal{L}$ is $p+1$ but there exists (at least one) $L \in \mathcal{L}$ with an edge e such that $\chi(L - e) = p$. (Colour-critical edge.)

(b) There exists an n_0 such that for $n > n_0(\mathcal{L})$, $T_{n,p}$ is extremal.

(c) There exists an n_0 such that for $n > n_0(\mathcal{L})$, $T_{n,p}$ is the **only** *extremal graph.*

The Product Conjecture

When I started working in extremal graph theory, I formulated (and later slightly modified) a conjecture on the structure of extremal graphs in non-degenerate cases. The meaning of this conjecture is that all the non-degenerate extremal graph problems can be reduced to degenerate extremal graph problems.

Conjecture 7. ***(Product structure).*** *Let $\mathcal{L}$ be a family of forbidden graphs and $\mathcal{M}$ be the decomposition family of $\mathcal{L}$, If no trees and forests occur in $\mathcal{M}$, then all the extremal graphs S_n for $\mathcal{L}$ have the following structure: $V(S_n)$ can be partitioned into $p = p(\mathcal{L})$ subsets $V_1, \ldots, V_p$ so that V_i is* ***completely joined*** *to V_j for every $1 \leqslant i < j \leqslant p$.*

This implies that each S_n is the product of p graphs G_i, where each G_i is extremal for some degenerate family $\mathcal{L}_{i,n}$. The meaning of this conjecture is that (almost) all the non-degenerate extremal graph problems can be reduced to degenerate extremal graph problems.

One non-trivial illustration of this conjecture is the Octahedron theorem:

Theorem 8. (Erdős-Simonovits [82]). *Let $O_6 = K_3(2,2,2)$ (i.e. O_6 is the graph defined by the vertices and edges of the octahedron.) If S_n is an extremal graph for O_6 for $n > n_0(O_6)$, then $S_n = H_m \otimes H_{n-m}$ for some $m = \frac{1}{2}n + o(n)$. Further, H_m is an extremal graph for C_4 and H_{n-m} is extremal for P_3.*

Remark 9. *The last sentence of this theorem is an easy consequence of that S_n is the product of two other graphs of approximately the same size.*

Remark 10. *In [82] some generalizations of the above theorem can also be found. Thus e.g., the analogous product result holds for all the forbidden graphs $L = K_{p+1}(2, t_2, \ldots, t_p)$ and $L = K_{p+1}(3, t_2, \ldots, t_p)$.*

Probably the octahedron theorem can be extended to all graphs $L = K_{p+1}(t_1, t_2, \ldots, t_p)$ and even to more general cases. On the other hand, in [155] counterexamples are constructed to the product-conjecture if we allow trees or forests in the decomposition family. In this case, when the decomposition contains trees, both cases can occur: the extremal graphs may be non-products and also they may be products. Turán's theorem itself is a product-case, where the decomposition family contains $K_2 = P_2$.

Szemerédi Lemma on Regular Partitions of Graphs

There are many important tools in Extremal Graph Theory that became quite standard to use over the last 20 years. One of them is the Szemerédi Regularity Lemma.

Let G be an arbitrary graph, X, $Y \subset V(G)$ be two disjoint vertex-sets and let $d(X,Y)$ denote the edge-density between them:

$$d(X,Y) = \frac{e(X,Y)}{|X| \cdot |W|}.$$

Regularity lemma. [168] *For every $\varepsilon > 0$, and every integer κ there exists a $k_0(\varepsilon, \kappa)$ such that for every G_n $V(G_n)$ can be partitioned into sets $V_0, V_1, \ldots, V_k$ - for some $\kappa < k < k_0(\varepsilon, \kappa)$ - so that $|V_0| < \varepsilon n$, each $|V_i| = m$ for $i > 0$ and for all but at most $\varepsilon \cdot \binom{k}{2}$ pairs (i,j), for every $X \subseteq V_i$ and $Y \subseteq V_j$, satisfying $|X|, |Y| > \varepsilon m$, we have*

$$|d(X,Y) - d(V_i, V_j)| < \varepsilon.$$

The applications of Szemerédi's Regularity Lemma are plentiful and are explained in details in [117], so here we shall describe it only very briefly.

One feature of the Regularity Lemma is that - in some sense - it allows us to handle a deterministic graph as if it were a (generalized) random one. One can easily prove for random graphs the existence of various subgraphs and the Regularity Lemma often helps us to ensure the existence of the same subgraphs when otherwise that would be far from trivial.

One example of this is the Erdős-Stone Theorem. Knowing Turán's theorem, the Szemerédi Lemma immediately implies the Erdős-Stene theorem. In the previous section we have mentioned a few improvements of the original Erdős-Stene theorem. The proof of the Chvátal-Szemerédi version also uses the Regularity Lemma as its main tool. Joining the work of Thomason, [170] [171] Fan Chung, Graham, Wilson, [35] [34] and others, V. T. Sós and I used the Regularity Lemma to give a transparent description of the so-called **quasi-random** graph sequences, [158] that was generalized by Fan K. Chung to hypergraphs [33].

The Regularity Lemma can be generalized in various ways. One of these generalizations states that if the edges of G_n are r-coloured for some fixed r, then we can partition the vertices of the graph so that the above Regularity Lemma remains true in all the colours simultaneously. This is what we used among others in proving some Turán-Ramsey type theorems [69], [70], [71] but it has also many other applications.

Generalized Regularity Lemma. *For every $\varepsilon > 0$, and integers r, κ, there exists a $k_0(\varepsilon, \kappa, r)$ such that for every graph G_n the edges of which are r-coloured, the vertex set $V(G_n)$ can be partitioned into sets $V_0, V_1, \ldots, V_k$- for some $\kappa < k < k_0(\varepsilon, \kappa, r)$- so that $|V_0| < \varepsilon n, |V_i| = m$ (is the same) for every $i > 0$, and for all but at most $\varepsilon \begin{pmatrix} k \\ 2 \end{pmatrix}$ pairs (i, j), for every $X \subseteq V_i$ and $Y \subseteq V_j$ satisfying $|X|, |Y| > \varepsilon m$, we have*

$$|d_\nu(X, Y) - d_\nu(V_i, V_j)| < \varepsilon \quad \text{simultaneously for} \quad \nu = 1, \ldots, r,$$

where $d_\nu(X, Y)$ is the edge-density in colour ν.

As I mentioned, we describe the various applications of Szemerédi Lemma in more details in some other place [117]. Here I mention only that it was extended to hypergraphs by Frankl and Rödl [96], see also Chung, [33]. Prömel and Steger also use a hypergraph version of the Regularity lemma in "induced extremal graph problems" [141]. Algorithmic versions were found by Alon, Duke, Leffman and Rödl, Yuster, [4], and in some sense it was extended to sparse graphs by Rödl and independently, by Kohayakawa [116].

I should also mention some new variants due to Komlós, see, for example, [117], [118].

1.6 Turán-Ramsey Problems

V. T. Sós has a survey [162] on

- application of Turán type theorems to distance-distribution, initiated by Erdős and Turán,
- Turán-Ramsey type theorems initiated by her,
- and the connection of these fields.

These fields belong to Extremal Graph Theory and are strongly influenced by Paul Erdős. I will touch on these fields only very briefly.

These problems were partly motivated by applications of graph theory to distance distribution. Turán theorem combined with some geometrical facts can provide us with estimates on the number of short distances in various geometrical situations. Thus they can be applied in some estimates in analysis, probability theory, and so on. It was Erdős who first pointed out this possibility of applying Graph Theory to distance distribution theorems [44] and later Turán in [175] initiated investigating these problems more systematically. This work culminated in 3 joint papers of Erdős, Meir, Sós and Turán [73], [74], [75].

The structure of the extremal graphs in Turán type theorems seems to be too regular. So we arrive at the question: How do the upper bounds in extremal graph theorems improve if we exclude graphs very similar to the Turán graphs? Basically this was what motivated V. T. Sós [160] in initiating a new field of investigation. Erdős joined her and they have proved quite a few nice results, see e.g., [86], [87].

Let $\alpha(G)$ denote the maximum cardinality of vertices in G such that the subgraph spanned by these vertices contains no K_p.

General Problem. Assume that $L_1, \dots, L_r$ are given graphs, and G_n is a graph on n vertices the edges of which are coloured by r colours $\chi_1, \dots, \chi_r$, and

$$\begin{cases} \text{for } \nu = 1, \dots, r \text{ the subgraph of colour } \chi_v \text{ contains no } L_v \\ \text{and } \alpha_p(G_n) \leqslant m. \end{cases}$$

What is the maximum of $e(G_n)$ under these conditions?

Originally the general problem was investigated only for $p = 2$,[8] and one breakthrough was the Szemerédi-Bollobás-Erdős theorem:

Theorem 1. *(**Szemerédi** [169]). If (G_n) is a sequence of graphs not containing K_4 and the stability number $\alpha(G_n) = o(n)$, then*

$$e(G_n) \leqslant \frac{1}{8}n^2 + o(n^2). \tag{1.3}$$

Erdős asks if the $o(n^2)$ error term is necessary: Is it true that in Theorem 1 the stronger

$$e(G_n) \leqslant \frac{n^2}{8}$$

also holds?

[8] In all the papers quoted but in [71] we consider this special case.

Theorem 2. (Bollobás-Erdős [16]). *(1.3) is sharp.*

Many estimates concerning general and various special cases of this field are proved in [69], [70], [71]. Here we mention just one result:

Theorem 3. (Erdős-Hajnal-Simonovits-Sós-Szemerédi [71]). *(a) For any integers $p > 1$ and $q > p$ if $\alpha_p(G_n) = o(n)$ and $K_q \nsubseteq G_n$, then*

$$e(G_n) \leqslant \frac{1}{2}\left(1 - \frac{p}{q-1}\right)\binom{n}{2} + o(n^2).$$

(b) For $q = pk + 1$ this upper bound is sharp.

One of the most intriguing open problems of the field is (among many other very interesting questions)

Problem 4. *Assume that (G_n) is a sequence of graphs not containing the Octahedron graph $K(2,2,2)$. If $\alpha(G_n) = o(n)$, does it follow that $e(G_n) = o(n^2)$?*

I conclude this section with a slightly different result of Ajtai, Erdős, Komlós and Szemerédi. Let t be the average degree of G_n. Turán's theorem guarantees an independent set of size $\frac{n}{t+1}$.

Theorem 5. ([1]). *If the number of $K_3 \subseteq G_n$ is $o(n^3)$, then*

$$\alpha(G_n) > c\frac{n}{t}\log t \qquad \text{for} \qquad t = \frac{2e(G_n)}{n}.$$

The nice feature of the above theorem is that it says: if the number of triangles in G_n is $o(n^3)$, then the size of the maximum independent set jumps by a log-factor. This is sharp: $\frac{n}{t}\log t$ is achieved for random graphs.

Theorem 5 can also be interpreted as follows: excluding the triangles (or assuming that there are only few triangles in our graph) leads to randomlike behaviour.

1.7 Cycles in Graphs

Cycles play central role in graph theory. Many results provide conditions to ensure the existence of some cycles in graphs. Among others, the theory of Hamiltonian cycles (and paths) constitute an important part of graph theory. The Handbook of Combinatorics contains a chapter by A. Bondy [21] giving a lot of information on ensuring cycles via various types of conditions. Also, the book of Walther and Voss [178] and the book of Voss [177] contain many relevant results. Below we shall approach the theory of degenerate extremal graph problems (see Section 1.8) through extremal graph problems with forbidden cycles. Of course, one of the simplest extremal graph problems is when $\mathcal{L}$ is the family of all cycles. If we exclude them, the considered graphs will be all the trees and forests; the extremal graphs are the trees.

Remark 1. *Describing walks and cycles in graphs is perhaps one of those parts of extremal graph theory, where algebraic methods may come in more often than in other extremal problems. So here occasionally, and very superficially, I will speak of Margulis graphs, Ramanujan graphs and Cayley graphs. I feel, these topics are very important, not only because of expander graphs but also because they provide new methods to construct nice graphs in extremal graph theory. Hoping that the Handbook of Combinatorics will sooner or later appear, I warmly recommend Noga Alons' chapter: Tools from Higher Algebra [2], which provides a lot of interesting and useful information - among others - on topics I had to describe very shortly.*

The Long Cycle Problem

One problem posed by Turán was the extremal problem of cycles of length m. If we exclude all the odd cycles, the extremal graph will be the Turán graph $T_{n,2}$. What are the extremal graphs if family $\mathcal{L}_m$ of excluded graphs is the family of cycles of length at least m. The answer is given by the Erdős-Callai theorem:

Theorem 2. (Erdős and Gallai [66]). *Let $\mathcal{L}_m = \{C_k : k \geqslant m\}$. Then*

(i) $\frac{m-1}{2}n - \frac{1}{2}m^2 < \mathrm{ex}(n, \mathcal{L}_m) \leqslant \frac{m-1}{2}n$ *and*

(ii) the connected graphs G_n whose 2-connected blocks are K_{m-1}'s are extremal.

Graphs described in (ii) do not exist for all n, but we get asymptotically extremal graphs for all n, by taking those graphs in which one 2-connected component has size at most $m-1$ and all the other blocks are complete $m-1$-graphs.

The following theorem is the twin of the previous one's.

Theorem 3. (Erdős and Gallai [66]).

$$\mathrm{ex}(n, P_m) \leqslant \frac{m-2}{2}n.$$

The union of $\lfloor \frac{1}{m-1} \rfloor$ vertex disjoint K_{m-1} (and one smaller K_q) shows that this is sharp: $\mathrm{ex}(n, P_m) = \frac{m-2}{2}n + O(m^2)$.

This theorem has a sharper form, proved by Faudree and Schelp [92]. They needed the sharper form to prove some Ramsey theorems on paths.

These theorems can also be used to deduce the existence of Hamilton paths and cycles. Thus, for example, Theorem 2 implies Dirac's famous result:

Theorem 4. (Dirac). *If the minimum degree of G_{2k} is at least k, then G_{2k} is Hamiltonian.*

Erdős and T. Sós observed that the same estimates hold both for the path P_m and the star $K_2(1, m-1)$ and these being two extremes among the trees of m vertices, they conjectured that:

Conjecture 5. ***(Erdős-T. Sós).*** *For any tree T_m,*

$$\mathrm{ex}(n, T_m) = \frac{m-2}{2}n + O(1).$$

Some asymptotical approximations of this conjecture were proved by

Ajtai, Komlós and Szemerédi, (unpublished), also, the conjecture is proved in its sharp form for some special families of trees, like caterpillars.

The Case of Excluded C_{2k}

Since the odd cycles are 3-chromatic colour-critical, one can apply Theorem 6 to them to get

$$\mathrm{ex}(n, C_{2k+1}) = \left[\frac{n^2}{4}\right] \qquad \text{if} \qquad n > n_0(k).$$

The case of even cycles is much more fascinating. The upper bound would become trivial if we assumed that G_n is (almost) regular and contains no cycles of length $\leqslant 2k$. The difficulty comes from that we exclude *only* C_{2k}.

Theorem 6. (Erdős, Bondy-Simonovits [23]).

$$\mathrm{ex}(n, C_{2k}) < ckn^{1+1/k} + o(n^{1+1/k}),$$

Theorem 7. (Bondy-Simonovits [23]). *If $e(G_n) > 100kn^{1+1/k}$, then*

$$C_{2\ell} \subseteq G_n \quad \text{for every integer} \quad \ell \in [k, kn^{1/k}].$$

Erdős stated Theorem 6 in [51] without proof and conjectured Theorem 7, which we proved. The upper bound on the cycle-length is sharp: take a G_n which is the union of complete graphs.

Let us return to Theorem 6. Is it sharp? Finite geometrical (and other) constructions show that for $k = 2, 3, 5$ YES. (Singleton, [159] Benson, [7] Wenger, [179] ...). Unfortunately, nobody knows if this is sharp for C_8, or for other C_{2k}'s.

Faudree and I sharpened Theorem 6 in another direction:

Definition 8. ***(Theta-graph).*** $\Theta(k,p)$ *is the graph consisting of p vertex-independent paths of length k joining two vertices x and y.*

Clearly, $\Theta(k,p)$ is a generalization of C_{2k}. We have proved

Theorem 9. (Faudree-Simonovits [93]). $\mathrm{ex}(n, \Theta(k,p)) < c_{k,p} \cdot n^{1+1/k}$.

The Erdős-Rényi Theorem [77] shows that Theorem 9 is sharp in the sense that

$$\mathrm{ex}(n, \Theta(k,p)) > c^*_{k,p} n^{1+\frac{1}{k}+\frac{1}{kp}} \quad \text{as} \quad n \to \infty.$$

One could ask if there are other global ways to state that if a graph has many edges then it has many cycles of different length. Erdős and Hajnal formulated such a conjecture, which was proved by A. Gyárfás, J. Komlós and E. Szemerédi. Among others, they proved

Theorem 10. (Gyárfás-Komlós-Szemerédi [105]). *If $d_{min}(G) \geqslant \delta$ and $\ell_1, \ldots, \ell_m$ are the cycle-lengths of G, then*

$$\sum \frac{1}{\ell_i} \geqslant c_1 \log \delta.$$

The meaning of this is as follows: If we regard all the graphs with minimum degree δ and try to minimize the sum of the reciprocals of the cycle-lengths, two candidates should first be checked. One is the union of disjoint $K_{\delta+1}$'s, the other is the union of disjoint complete bipartite graphs $K(\delta,\delta)$'s. In the first case we get $\log \delta + O(1)$, in the second one $\frac{1}{2}\log\delta + O(1)$. The above theorem asserts that these cases minimize $\sum \frac{1}{\ell_i}$.

Some graph theorists could be surprised by measuring the density of cycle lengths this way. Yet, whenever we want to express that something is nearly linear, then in number theory we tend to use this measure. Thus, e.g. the famous $3000 problem of Erdős asks for the following sharpening of Szemerédi's theorem on Arithmetic Progressions [167]:

Conjecture 11. *(Erdős).* *Prove that if $A = \{a_1 < a_2 < \cdots\}$ is an infinite sequence of positive integers and*

$$\sum \frac{1}{a_i} = \infty,$$

then for every k, A contains a k-term arithmetic progression.

Very Long Cycles

We know that a graph with minimum degree 3 contains a cycle of length at most $2\log_2 n$. The other extreme is when (instead of short cycles) we wish to ensure very long cycles. We may go much beyond the Erdős-Oallai theorem if we increase the connectivity and put an upper bound on the maximum degree.

Theorem 12. (Bondy-Entringer [22]). *Let $f(n,d)$ be the largest integer k such that every 2-connected G_n with maximum degree contains a cycle of length at least k. Then*

$$4\log_{d-1} n - 4\log_{d-1}\log_{d-1} n - 20 < f(n,d) < 4\log_{d-1} n + 4.$$

9

[9] Here $\log_{d-1} x$ means log base $d-1$ and not the iterated log.

Clearly, the connectivity is needed, otherwise - as we have seen - the Erdős-Gallai theorem is sharp.

Increasing the connectivity to 3 we can ensure longer cycles:

Theorem 13. (Bondy-Simonovits [24]). *If G_n is 3-connected and the minimum degree of G_n is d, the maximum degree is D, then G_n contains a cycle of length at least $e^{c\sqrt{\log n}}$ for some $c = c(d, D)$.*

We conjectured that $e^{c\sqrt{\log n}}$ can be improved to n^c. Bill Jackson, Jackson and Wormald succeded in proving this:

Theorem 14. (B. Jackson [109], [112]). *If G_n is 3-connected and the minimum degree of G_n is d, the maximum degree is D, then G_n contains a cycle of length at least n^c for some $c = c(d, D)$.*

Increasing the connectivity higher does not help in getting longer cycles:

Theorem 15. (Jackson, Parson [110]). *For every $d > 0$ there are infinitely many $d + 2$-regular d-connected graphs without cycles longer than n^γ for some $\gamma = \gamma_d < 1$.*

We close this topic with an open problem:

Conjecture 16. ***(J. A. Bondy).*** *There exists a constant $c > 0$, such that every cyclically 4-connected 3-regular graph G_n contains a cycle of length at least cn.*

Erdős-Pósa Theorem

The following question of Gallai is motivated partly by Menger Theorem. If G is a graph

$$(*)\text{not containing two independent cycles,}$$

how many vertices are needed to represent all the cycles?

K_5 satisfies $(*)$ and we need at least 3 vertices to represent all its cycles. Bollobás [9] proved that in all the graphs satisfying $(*)$ there exist 3 vertices the deletion of which results in a tree (or forest). More generally,

Let $RC(k)$ denote the minimum t such that if a graph G contains no $k + 1$ independent cycles, then one can delete t vertices of G ruining all the cycles of the graph. Determine $RC(k)$!

Erdős and Pósa [76] proved the existence of two positive constants, c_1 and c_2 such that

$$c_1 k \log k \leqslant RC(k) \leqslant c_2 k \log k. \tag{1.4}$$

This theorem is strongly connected to the following extremal graph theoretical question:

Assume that G_n is a graph in which the minimum degree is D. Find an upper bound on the girth of the graph.

Here the usual upper bound is

$$\approx \frac{2\log n}{\log(D-1)}. \tag{1.5}$$

The proof is easy: Assume that the girth is g and let $k = \lfloor \frac{g-1}{2} \rfloor$. Take a vertex x and denote the set of vertices having distance t from x by X_t. Then for $t \leqslant k$ we have $|X_t| \geqslant (D-1)|X_{t-1}|$. Therefore

$$n = v(G_n) \geqslant 1 + D + D(D-1) + D(D-1)^2 + \ldots + D(D-1)^k.$$

This implies (1.5).

These things are connected to many other parts of Graph Theory, in some sense even to the Robertson-Seymour theory. Below I shall try to convince the reader that the Gallai problem is strongly connected to the girth problem.

In [151] I gave a short proof of the upper bound of (1.4). My proof goes as follows (sketch!):

Let G_n be an arbitrary graph not containing $k+1$ independent circuits. Let H_m be a maximal subgraph of G_n all whose degrees are 2,3 or 4. Then one can immediately see that the ramification vertices of H_m, i.e. the vertices of degree 3 or 4 represent all the cycles of G_n.[10] Let μ be the number of these vertices. Replacing the hanging chains[11] by single edges, we get an H_μ each degree of which is 3 or 4. So one can easily find a cycle $C^{(1)}$ of length $\leqslant c_3 \log \mu$ in H_μ. Applying this to $H_\mu - C^{(1)}$ (but first cleaning up the resulting low degrees) we get another short cycle $C^{(2)}$. This cleaning up is where we have to use that the degrees are bounded from above. Iterating this (and using that the degrees are bounded from above) one can find $c_4\mu/\log\mu$ vertex-independent cycles in G_n. Since $k \leqslant c_4\mu/\log\mu$, therefore $\mu \leqslant c_5 k \log k$.

The Erdős-Pósa theorem is strongly connected with the girth problem. If, e.g. we had shorter circuits in graphs with degrees 3 and 4 then the above proof would give better upper bound on $RC(k)$ - but that is ruled out.

The Margulis Graphs and the Lubotzky-Phillips-Sarnak Graphs

Sometimes we insist on finding constructions in certain cases when the randomized methods work easily. Often finding explicit constructions is very difficult. A good example of this is the famous case of the Ramsey 2-colouring, where Erdős offered \$ …for finding a construction of a graph of n vertices not containing complete graphs or independent sets of at least $c\log n$ vertices. (See Frankl, Wilson [97])

Another similar case is the **girth** problem discussed above, with one exception. Namely, in the girth problem Margulis, [131], [132] and Lubotzky-Phillips-Sarnak [126], [127], succeeded in constructing regular graphs G_n of (arbitrary

[10] The vertices of degree 4 are not really needed …

[11] paths all whose inner vertices have degree 2

high) but fixed degree d and girth at least $c_d \log n$. The original random-graph existence proof is due to Erdős and Sachs [79].

These graphs are Cayley graphs. Below (skipping many details)

(a) first we explain, why should one try Cayley graphs of **non-commutative groups**,

(b) then we give a sketch of the description of the first, simpler Margulis graph and

(c) finally we list the main features of the Lubotzky- Phillips-Sarnak graph.

(a) Often cyclic graphs are used in the constructions. Cyclic graphs are the graphs where a set $A_n \subseteq [1, n]$ is given, the vertices of our graphs are the residue classes $Z_i(\mod n)$, and Z_i is joined to Z_j if $|i - j| \in A_n$ (or $|n + i - j| \in A_n$). One such well known graph is Q_p (the Paley graph) obtained by joining Z_i to Z_j if their difference is a quadratic non-residue. The advantage of such graphs is that they have great deal of fuzzy (randomlike) structure. From the point of view of the short cycles they are not the best: they have many short even cycles.

(b) Given an arbitrary group $\mathcal{G}$ and some elements $g_1, \ldots, g_t \in \mathcal{G}$, these elements generate a Cayley graph on $\mathcal{G}$: we join each $a \in \mathcal{G}$ to the elements $ag_1, \ldots, ag_t$. This is a digraph. If we are interested in ordinary graphs, we choose $g_1, \ldots, g_t$ so that whenever g is one of them, then also $g^{-1} \in \{g_1, \ldots, g_t\}$. Thus we get an undirected graph. Still, if $\mathcal{G}$ is commutative, then this Cayley graph will have many even cycles. For example, $a, ag_1, ag_1g_2, ag_1g_2g_1^{-1}, ag_1g_2g_1^{-1}g_2^{-1}$ is (mostly) a C_4 for commutative groups and a P_4 for non-commutative groups. So, if we wanted to obtain Cayley graphs with large girth, we have better to start with non-Abelian groups. This is what Margulis did in [131]:

Let X denote the set of all 2×2 matrices with integer entries and with determinant 1. Pick the following two matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

It is known that they are independent in the sense that there is no non-trivial multiplicative relation between them. So, if we take the 4 matrices A, B, A^{-1} and B^{-1}, they generate an infinite Cayley graph which is a 4-regular tree. If we take everything $\mod p$, then it is easy to see that the tree collapses into a graph of $n \approx p^3$ vertices, in which the shortest cycle has length at least $c \log p$ for some constant $c > 0$. This yields a sequence of 4-regular graphs X_n with girth $\approx c^* \log n$ for some $c^* \approx 0.91\ldots$ Margulis also explains, how the above graphs can be used in constructing certain (explicit) error-correcting codes. Margulis has also generalized this construction (in the same paper) to arbitrary even degrees.

Theorem 17. (Margulis, [131]). *For every $\varepsilon > 0$ we have infinitely many values of r, and for each of them an infinity of regular graphs X_j of degree $2r$ with girth*

$$g(X_j) > \left(\frac{4}{9} - \varepsilon\right) \frac{\log v(X_j)}{\log r}.$$

(c) The next breakthrough was due to Margulis [132] and to Lubotzky, Phillips and Sarnak [126]. The graph of Lubotzky, Phillips and Sarnak was obtained *not* for extremal graph purposes. The authors, investigating the extremal spectral gap of d-regular graphs, constructed graphs where the difference between the first and second eigenvalues is as large as possible. Graphs with large spectral gaps are good expanders, and this was perhaps the primary interest in [126] or in [132]. As the authors of [126] remarked, Noga Alan turned their attention to the fact that their graphs can be "used" also for many other, classical purposes.

Definition 18. *Let X be a connected k-regular graph. Denote by $\lambda(X)$ the second largest eigenvalue (in absolute value) of the adjacency matrix of X.*

Definition 19. *A k-regular graph on n vertices, $X = X_{n,k}$, will be called a **Ramanujan graph**, if $\lambda(X_{n,k}) \leqslant 2\sqrt{k-1}$.*

I do not have the place here to go into details, but the basic idea is that random graphs have roughly the spectral gap[12] required above and vice versa: if the graph has a large spectral gap, then it may be regarded in some sense, as if it were a random graph. So the Ramanujan graphs provide near-extremum in some problems, where random graphs are near-extremal. (See also [3], [35])

Let p, q be distinct primes congruent to 1 mod 4. The Ramanujan graph $X^{p,q}$ of [126] is a $p+1$-regular Cayley graph of $PSL(2, \mathbb{Z}_q)$ if the Legendre symbol $\left(\frac{p}{q}\right) = 1$ and of $PGL(2, \mathbb{Z}_q)$ if $\left(\frac{p}{q}\right) = -1$. (Here $\mathbb{Z}_q$ is the field of integers mod q.)

Theorem 20. (Alon, quoted in [126]). *Let $X_{n,k} = X^{p,q}$ be a non-bipartite Ramanujan graph; $\left(\frac{p}{q}\right) = 1, k = p+1, n = q(q^2-1)/2$. Then the independence number*

$$\alpha(X_{p,q}) \leqslant \frac{2\sqrt{k-1}}{k} n.$$

Corollary 21. ([126]). *If $X_{n,k}$ is a non-bipartite Ramanujan graph, then*

$$\chi(X_{n,k}) \geqslant \frac{k}{2\sqrt{k-1}}.$$

Margulis, Lubotzky, Phillips and Sarnak have constructed Ramanujan graphs which are $p+1$-regular, and

[12]– difference between the largest and second largest eigenvalues

(a) **bipartite** with $n = q(q^2 - 1)$ vertices, satisfying

$$\mathrm{girth}(X_{n,p+1}) \geqslant \frac{4}{3}\frac{\log n}{\log p} - O(1) \quad \text{and} \quad \mathrm{diam}(X_{n,p+1}) \leqslant \frac{2}{3}\frac{\log n}{\log p} + 3.$$

Further, they constructed **non-bipartite Ramanujan graphs** with $n = q(q^2 - 1)/2$ vertices, and with the same diameter estimate and with

$$\mathrm{girth}(X_{n,p+1}) \geqslant \frac{2}{3}\frac{\log n}{\log p} + O(1), \qquad \alpha(X_{n,p+1}) \leqslant \frac{2\sqrt{p}}{p+1} n, \qquad \chi(X_{n,p+1}) \geqslant \frac{p+1}{2\sqrt{p}}.$$

Putting $p = canst$ or $p \approx n^c$ we get constructions of graphs the existence of which were known earlier only via random graph methods. As a matter of fact, they are better than the known "random constructions", showing that

$$\mathrm{ex}(n, C_{2k}) > c_k n^{1+\frac{4}{3k+25}}.$$

1.8 Further Degenerate Extremal Graph Problems

We have already seen the most important degenerate extremal graph problems. Unfortunately we do not have as many results in this field as we would like to. Here we mention just a few of them.

Topological Subgraphs

Given a graph L, we may associate with it all its topologically equivalent forms. Slightly more generally, let $\mathcal{T}(L)$ be the set of graphs obtained by replacing some edges of L by "hanging chains", i.e., paths, all inner vertices of which are of degree 2.

Problem 1. *Find the maximum number of edges a graph G_n can have without containing subgraphs from $\mathcal{T}(L)$.*

Denote the topological complete p-graphs by $< K_p >$. G. Dirac [39] have proved that every G_n of $2n-2$ edges contains a $< K_4 >$. This is sharp: Dirac gave a graph G_n of $2n-3$ edges and not containing $< K_4 >$. Erdős and Hajnal pointed out that there exist graphs G_n of cp^2n edges and not containing $< K_p >$. (This can be seen, e.g. by taking $[n/q]$ vertex-disjoint union $K(q,q)$'s for $q = \binom{p/2}{2}$.) Mader [128] showed that

Theorem 2. *For every integer $p > 0$ there exists a $D = D(p)$ such that if the minimum degree of G is at least $D(p)$, then G contains a $< K_p >$.*

More precisely,

Theorem 3. *There exists a constant $c > 0$ such that if $e(G) > tn$, then G contains a $< K_p >$ for $p = [c\sqrt{\log t}]$.*

Corollary 4. *For every L, $\mathrm{ex}(n, \mathcal{T}(L)) = O(n)$.*

Conjecture 5. (Erdős - Hajnal - Mader [68], [128]). *If $e(G_n) > tn$, then G_n contains a $< K_p >$ with $p \geqslant c\sqrt{t}$.*

Mader's result was improved by Komlós and Szemerédi to almost the best:

Theorem 6. ([119]). *There is a positive c_1 such that if $e(G_n) > tn$, then G_n contains a $< K_p >$ with*

$$p > c_1 \frac{\sqrt{t}}{(\log t)^6}.$$

Very recently, improving some arguments of Alon and Seymour, Bollobás and Thomason completely settled Mader's problem:

Theorem 7. ([20]). *There is a positive c_1 such that if $e(G_n) > tn$, then G_n contains a $< K_p >$ with*

$$p > c_1\sqrt{t}.$$

Their proof-method was completely different from that of Komlós and Szemerédi. Komlós and Szemerédi slightly later also obtained a proof of Theorem 7 along their original lines [[120]].

Recursion Theorems

Recursion theorems could be defined for ordinary graphs and hypergraphs, for ordinary degenerate extremal problems and non-degenerate extremal graph problems, for supersaturated graph problems, ... However, here we shall restrict our considerations to ordinary degenerate extremal graph problems. In this case we have a bipartite L and a procedure assigning an L' to L. Then we wish to deduce upper bounds on $\mathrm{ex}(n, L')$, using upper bounds on $\mathrm{ex}(n, L)$. To illustrate this, we start with two trivial statements.

Claim. Let L be a bipartite graph and L' be a graph obtained from L by attaching a rooted tree T to L at one of its vertices.[13] Then

$$\mathrm{ex}(n, L') = \mathrm{ex}(n, L) + O(n).$$

Claim. Let L be a bipartite graph and L' be a graph obtained by taking two vertex-disjoint copies of L. Then (again)

$$\mathrm{ex}(n, L') = \mathrm{ex}(n, L) + O(n).$$

The proofs are trivial.

[13] This means that we take vertex-disjoint copies of L and T, a vertex $x \in V(T)$ and a vertex $y \in V(L)$ and identify x and y.

One of the problems Turán asked in connection with his graph theorem was to find the extremal numbers for the graphs of the regular (Platonic) polytopes. For the tetrahedron the answer is given by Turán Theorem (applied to K_4). The question of the Octahedron graph is solved by Theorem 8, the problems of the Icosahedron and Dodecahedron can be found in Section 1.12, ([152], [153]). On the cube-graph we have

Theorem 8. (Cube Theorem, Erdős-Simonovits, [83]).

$$\mathrm{ex}(n, Q_8) = O(n^{8/5}).$$

We conjecture that the exponent 8/5 is sharp. Unfortunately we do not have any "reasonable" lower bound.

The above theorem and many others follow from a recursion theorem:

Theorem 9. (Recursion Theorem, [83]). *Let L be a bipartite graph, coloured in BLUE and RED and $K(t,t)$ be also coloured in BLUE and RED. Let L^* be the graph obtained from these two (vertex-disjoint) graphs by joining each vertex of L to all the vertices of $K(t,t)$ of the other colour. If* $\mathrm{ex}(n, L) = O(n^{2-\alpha})$ *and*

$$\frac{1}{\beta} - \frac{1}{\alpha} = t,$$

then $\mathrm{ex}(n, L^*) = O(n^{2-\beta})$.

Applying this recursion theorem with $t = 1$ and $L = C_6$ we obtain the Cube theorem. Another type of recursion theorem was proved by Faudree and me in [93].

Regular subgraphs

Let $\mathcal{L}_{r-reg}$ denote the family of r-regular graphs. Erdős and Sauer posed the following problem [61]:

What is the maximum number of edges in a graph G_n not containing any k-regular subgraph?

Since $K(3,3)$ is 3-regular, one immediately sees that $\mathrm{ex}(n, \mathcal{L}_{3-reg}) = O(n^{5/3})$. Using the Cube Theorem one gets a better upper bound, $\mathrm{ex}(n, \mathcal{L}_{3-reg}) = O(n^{8/5})$. Erdős and Sauer conjectured that for every $\varepsilon > 0$ there exists an $n_0(k, \varepsilon)$ such that for $n > n_0(k, \varepsilon)$ $\mathrm{ex}(n, \mathcal{L}_{k-reg}) \leqslant n^{1+\varepsilon}$. Pyber proved the following stronger theorem.

Theorem 10. (Pyber, [142]). *For every k,* $\mathrm{ex}(n, \mathcal{L}_{k-reg}) = 50k^2 n \log n$.

The proof is based on a somewhat similar but much less general theorem of Alon, Friedland and Kalai [5]. For further information, see e.g. Noga Alon, [2]

One more theorem

We close this section with an old problem of Erdős solved not so long ago by Füredi. Let $F(k,t)$ be the bipartite graph with k vertices $x_1,\dots,x_k$ and $\binom{k}{2} t$ further vertices in groups U_{ij} of size t, where all the vertices of $\cup U_{ij}$ are independent and the t vertices of U_{ij} are joined to x_i and $x_j (1 \leqslant i < j \leqslant k)$. Erdős asked for the determination of $\mathrm{ex}(n, F(k,t))$ for $t = 1$. For $t = 1$ and $k = 2$ this is just C_4, so the extremal number is $O(n^{3/2})$. Erdős also proved (and it follow s from [83] as well) that $\mathrm{ex}(n, F(3,1)) = O(n^{3/2})$.

Theorem 11. (Füredi [100]). $\mathrm{ex}(n, F(k,t)) = O(n^{3/2})$.

1.9 Supersaturated Graphs

Rademacher Type Theorems

Almost immediately after Turán's result, Rademacher proved the following nice theorem (unpublished, see [47]):

Theorem 1. (Rademacher Theorem). *If $e(G_n) > \left[\frac{n^2}{4}\right]$ then G_n contains at least $\left[\frac{n}{2}\right]$ triangles.*

This is sharp: adding an edge to (the smaller class of) $T_{n,2}$ we get $\left[\frac{n}{2}\right] K_3$'s.

Erdős generalized this result by proving the following two basic theorems [47]:

Theorem 2. *There exists a positive constant $c_1 > 0$ sucg that if $e(G_n) > \left[\frac{n^2}{4}\right]$, then G_n contains an edge e with at least $c_1 n$ triangles on it.*

Theorem 3. (Generalized Rademacher Theorem). *There exists a positive constant $c_2 > 0$ such that if $0 < k < c_2 n$ and $e(G_n) > \left[\frac{n^2}{4}\right] + k$, then G_n contains at least $k\left[\frac{n}{2}\right]$ copies of K_3.*

(Lovász and I proved that $c_2 = \frac{1}{2}$ [124]. For further results see Moser and Moon [136], Bollobás, [11], [12], and [124], [125].) Erdős also proved the following theorem, going into the other direction.

Theorem 4. (Erdős [58]). *If $e(G_n) = \left[\frac{n^2}{4}\right] - \ell$ and G_n contains at least one triangle, then it contains at least $\left[\frac{n}{2}\right] - \ell - 1$ triangles.*

(Of course, we may assume that $0 \leqslant \ell \leqslant \left[\frac{n}{2}\right] - 3$.)

The General Case

Working on multigraph and digraph extremal problems, Brown and I needed some generalizations of some theorems of Erdős [50], [56].The results below are direct generalizations of some theorems of Erdős. To avoid proving the theorems in a setting narrower than what might be needed later, Brown and I formulated our results in the "most general, still reasonable" form.

Definition 5. (Directed multi-hypergraphs [31]). *A directed (r,q)-multi-hypergraph has a set V of vertices, a set $\mathcal{H}$ of directed hyperedges, i.e. ordered r-tuples, and a multiplicity function $\mu(\boldsymbol{H}) \leqslant q$ (the multiplicity of the ordered hyperedge) $\boldsymbol{H} \in \mathcal{H}$.*

We shall return to the multigraph and digraph problems later, here I formulate only some simpler facts. The extremal graph problems directly generalize to directed multi-hypergraphs with bounded hyper-edge-multiplicity:

Given a family $\mathcal{L}$ of excluded directed (r,q)-multi-hypergraphs, we may ask the maximum number of directed hyperedges (counted with multiplicity) a directed (r,q)-multi-hypergraph $\mathbf{G}_n$ can have without containing forbidden sub-multi-hypergraphs from $\mathcal{L}$. The maximum is again denoted by $\text{ex}(n,\mathcal{L})$.

Let $\mathbf{L}$ be a directed (r,q)-multi-hypergraph and $\mathbf{L}[t]$ be obtained from $\mathbf{L}$ by replacing each vertex v_i of $\mathbf{L}$ by a set X_i of t independent vertices, and forming a directed multihyperedge $(y_1,\dots,y_r)$ of multiplicity μ if $y_1 \in X_{i_1},\dots,y_r \in X_{i_r}$ and the corresponding $(v_{i_1},\dots,v_{i_r})$ is a directed hyperedge of multiplicity μ in $\mathbf{L}$.

Theorem 6. (Brown-Simonovits, [31]).

$$\text{ex}(n,\mathbf{L}[t]) - \text{ex}(n,\mathbf{L}) = o(n^r).$$

Again, the influence of Erdős is very direct: the above theorem is a direct generalization of his result in [56].

Theorem 7. (Brown-Simonovits, [31]). *Let $\mathcal{L}$ be an arbitrary family of (r,q)-hypergraphs, and $\gamma = \lim \frac{\text{ex}(n,\mathcal{L})}{n^r}$, as $n \to \infty$. There exists a constant $c_2 = c_2(\mathcal{L},\varepsilon)$ such that, if*

$$e(\mathbf{G}_n) \geqslant (\gamma+\varepsilon)n^r$$

and n is sufficiently large, then there exists some $\mathbf{L} \in \mathcal{L}$ for which $\mathbf{G}_n$ contains at least $c_2 n^{v(\mathbf{L})}$ copies of this $\mathbf{L}$.

1.10 Typical K_p-Free Graphs: The Erdős-Kleitman-Rothschild Theory

Erdős, Kleitman and Rothschild [72] started investigating the following problem:

How many labelled graphs not containing L exist on n vertices?

Denote this number by $M(n, L)$. We have a trivial lower bound on $M(n, L)$: take any fixed extremal graph S_n and take all the $2^{\mathrm{ex}(n,L)}$ subgraphs of it:

$$M(n, L) \geqslant 2^{\mathrm{ex}(n,L)}.$$

In some sense it is irrelevant if we count *labelled* or *unlabelled* graphs. The number of labelled graphs is at most $n!$ times the number of unlabelled graphs and $\mathrm{ex}(n, L) \geqslant \left[\frac{n^2}{4}\right]$ for all non-degenerate cases, (and $\mathrm{ex}(n, L) \geqslant cn^{1+\alpha}$ for all the *non-tree-non-forest* cases). So, if we are satisfied with rough estimates, we may say: counting only labelled graphs is not a real restriction here.

Strictly speaking, this problem is not an extremal graph problem, neither a supersaturated graph problem. However, the answer to the question shows that this problem is in surprisingly strong connection with the corresponding extremal graph problem.

Theorem 1. (Erdős-Kleitman-Rothschild [72]). *The number of K_p-free graphs on n vertices and the number of $p-1$-chromatic graphs on n vertices are in logarithm asymptotically equal: For every $\varepsilon(n) \to 0$ there exists an $\eta(n) \to 0$ such that if $M(n, K_p, \varepsilon)$ denotes the number of graphs of n vertices and with at most εn^p subgraphs K_p, then*

$$\mathrm{ex}(n, K_p) \leqslant M(n, K_p, \varepsilon) \leqslant \mathrm{ex}(n, K_p) + \eta n^2.$$

In other word, we get "almost all of them" by simply taking all the $(p-1)$-chromatic graphs.

More generally, Erdős, Frankl and Rödl [65] proved that if $\chi(L) > 2$, then

$$M(n, L) = 2^{\mathrm{ex}(n,L)+o(\mathrm{ex}(n,L))}.$$

The corresponding question for bipartite graphs is unsolved. Even for the simplest non trivial case, i.e. for C_4 the results are not satisfactory. This is not so surprising. All these problems are connected with random graphs, where for low edge-density the problems often become much more difficult. Kleitman and Winston [115] showed that

$$M(n, C_4) \leqslant 2^{cn\sqrt{n}},$$

but the best value of the constant c is unknown. Erdős conjectured that

$$M(n, L) = 2^{(1+o(1))\mathrm{ex}(n,L)}.$$

Then the truth should be, of course

$$M(n, C_4) = 2^{((1/2)+o(1))n\sqrt{n}}.$$

I finish with a recent problem of Erdős.

Problem 2. ***(Erdős)***. *Determine or estimate the number of **maximal** triangle-free graphs on n vertices.*

Some explanation. In the Erdős-Kleitman-Rothschild case the number of bipartite graphs was large enough to give a logarithmically sharp estimate. Here $K(a, n-a)$ are the maximal *bipartite* graphs, their number is negligible. This is why the situation becomes less transparent.

1.11 Induced Subgraphs

One could ask, why do we always speak of *not necessarily induced* subgraphs. What if we try to exclude *induced* copies of L? If we are careless, we immediately run into a complete nonsense. If L is not a complete graph and we ask:

> *What is the maximum number of edges a G_n can have without having an induced copy of L?*

the answer is the trivial $\binom{n}{2}$ and the only extremal graph is K_n. So let us give up this question for a short while and try to at tack the corresponding *counting problem* which turned out in the previous section to be in a strong connection with the extremal problem.

> *How many labelled graphs not containing induced copies of L are on n vertices'?*

Denote this number by $M^*(n, L)$. Prömel and Steger succeeded in describing $M^*(n, L)$. They started with the case of C_4 and proved that almost all G_n not containing an induced C_4 have the following very specific structure. They are **split graphs**, which means that they are obtained by taking a K_p and $(n-p)$ further independent points and joining them to K_p arbitrarily. (Trivially, these graphs contain no induced C_4's.)

Theorem 1. (Prömel-Steger, [139]). *If S_n^* is the family of split graphs, then*

$$\frac{M^*(n, C_4)}{|S_n^*|} \to 1 \quad \text{as} \quad n \to \infty.$$

This implies - by a result of Prörnel [138] that

Corollary 2. *There exist two constants, $c_{even} > 0$ and $c_{odd} > 0$, such that*

$$\frac{M^*(n, C_4)}{2^{n^2/4+n-(1/2)n\log n}} \to \begin{cases} c_{even} & for\ even\ n, \\ c_{odd} & for\ odd\ n. \end{cases}$$

Can one generalize this theorem to arbitrary excluded induced subgraphs? To answer this question, first Prömel and Steger generalized the notion of chromatic number.

Definition 3. *Let $\tau(L)$ be the largest integer k for which there exists an integer $j \in [0, k-1]$ such that no $k-1$-chromatic graph in which j colour-classes are replaced by cliques contains L as an induced subgraph.*

Clearly, if $\sigma(L)$ denotes the clique covering number, (= the minimum number of complete subgraphs of L to cover all the vertices of L) then

Lemma 4. (Prömel-Steger). $\chi(L), \sigma(L) \leqslant \tau(L) \leqslant \chi(L) + \sigma(L)$.

Now, taking a $T_{n,p}$ for $p = \tau(L) - 1$ and replacing j appropriate classes of it (in the above definition) by complete graphs and then deleting arbitrary edges of the $T_{n,p}$ we get graphs not having induced L's:

Theorem 5. (Prömel-Steger). *Let H be a fixed nonempty subgraph with $\tau \geqslant 3$. Then*

$$M^*(n, H) = 2^{\left(1-\frac{1}{\tau-1}\right)\binom{n}{2}+o(n^2)}.$$

Definition 6. *Given a sample graph L, call G_n "good" if there exists a* **fixed** *subgraph $U_n \subseteq \overline{G_n}$ (= the complementary graph of G_n) such that whichever way we add some edges of G_n to U_n, the resulting U' contains no induced copies of L.* $\mathrm{ex}^*(n, H)$ *denotes the maximum number of edges such a G_n can have.*

Example 1. *In case of C_4, any bipartite graph $G(A, B)$ is "good", since taking all the edges in A, no edges in B and some edges from $G(A, B)$ we get a U_n not containing C_4 as an induced subgraph.*

Theorem 7. (Prömel-Steger [140]).

$$\mathrm{ex}^*(n, L) = \left(1 - \frac{1}{\tau - 1}\right)\binom{n}{2} + o(n^2).$$

Thus Prömel and Steger convincingly showed that there is a possibility to generalize ordinary extremal problems and the corresponding counting problems to induced subgraph problems. For further information, see [140] [141].

1.12 The Number of Disjoint Complete Graphs

There are many problems where instead of ensuring many K_{p+1}'s (see Section 1.9) we would like to ensure many **edge-disjoint** or **vertex-disjoint** copies of K_{p+1}. Let us start with the case of **vertex-disjoint** copies.

If G_n is a graph from which one can delete $s-1$ vertices so that the resulting graph is p-chromatic, then G_n cannot contain s vertex-disjoint copies of K_{p+1}. This is sharp: let $H_{n,p,s} = T_{n-s+1,p} \otimes K_{s-1}$. Then $H_{n,p,s}$ has the most edges among the graphs from which one can delete $s - 1$ vertices to get a graph of chromatic number at most p. Further:

Theorem 1. (Moon [135]). *Among all the graph not containing s vertex independent K_{p+1}'s $H_{n,p,s}$ has the most edges, assumed that $n > n_0(p, s)$.*

This theorem was first proved by Erdős and Gallai for $p = 1$, then for K_3 by Erdős [48], and then it was generalized for arbitrary p by J. W. Moon, and finally, a more general theorem was proved by me [153]. This more general theorem contained the answer to Turán's two "Platonic" problem: it guaranteed that $H_{n,2,6}$ is the only extremal graph for the dodecahedron graph and $H_{n,3,3}$ for the icosahedron, if n is sufficiently large.

We get a slightly different result, if we look for edge-independent complete graphs. Clearly, if one puts k edges into the first class of $T_{n,p}$, then one gets

k edge-independent K_p's as long as $k < cn$. One would conjecture that this is sharp. As long as k is fixed, the general theorems of [153] provide the correct answer. If we wish to find the maximum number of edge-independent copies of K_{p+1} for

$$e(G_n) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + k,$$

for $k = k(n) \to \infty$, the problem changes in character, see e.g. recent papers of Győri [106], [108]. We mention just one theorem here:

Theorem 2. (Győri [107]). *Let $e(G_n) = e(T_{n,p}) + k, (p \geqslant 3)$, where $k \leqslant 3\lfloor \frac{n+1}{p} \rfloor - 5$. The G_n contains k edge-independent K_{p+1}'s, assumed that $n > n_0(p)$.*

For $p = 2$ (for triangles) the result is different in style, see [106].

1.13 Extremal Graph Problems Connected to Pentagonlike Graphs

A lemma of Erdős asserts that each graph G_n can be turned into a bipartite graph by deleting at most half of its edges. (Above: Theorem 6) The proof of this triviality is as follows. Take a bipartite $H_n \subseteq G_n$ of maximum number of edges. By the maximality, each $x \in V(G_n)$ having degree $d(x)$ in G_n must have degree $\leqslant \frac{1}{2}d(x)$ in H_n. Summing the degrees in both graphs we get $e(H_n) \geqslant \frac{1}{2}e(G_n)$. This estimate is sharp for random graphs of edge probability $p > 0$, in asymptotical sense. Now, our first question is if this estimate can be improved in cases when we know some extra information on the structure of the graph, say, excluding triangles in G_n. The next theorem asserts that this is not so. Let $D(G_n)$ denote the minimum number of edges one has to delete from G_n to turn it into a bipartite graph.

Theorem 1. (Erdős, [53]). *For every $\varepsilon > 0$ there exists a constant $c = c_\varepsilon > 0$ such that for infinitely many n, there exists a G_n for which $K_3 \nsubseteq G_n$, $e(G_n) > c_\varepsilon n^2$, and*

$$D(G_n) > \left(\frac{1}{2} - \varepsilon\right) e(G_n).$$

Conjecture 2. *If $K_3 \nsubseteq G_n$, then one can delete (at most) $n^2/25$ edges so that the remaining graph is bipartite.*

Let us call a graph G_n **pentagonlike** if its vertex-set V can be partitioned into $V_1, \ldots, V_5$ so that $x \in V_i$ and $y \in V_j$ are joined iff $i - j \equiv \pm 1 \mod 5$. The pentagonlike graph $Q_n := C_5[n/5]$ shows that, if true, this conjecture is sharp. The conjecture is still open, in spite of the fact that good approximations of its solutions were obtained by Erdős, Faudree, Pach and Spencer. This conjecture is proven for $e(G_n) \geqslant \frac{n^2}{5}$ (see below) and the following (other) weakening is also known, [64]:

Theorem 3. *If $K_3 \nsubseteq G_n$ then*

$$D(G_n) \leqslant \frac{n^2}{18+\delta}.$$

for some (explicite) constant $\delta > 0$.

In fact, Erdős, Faudree, Pach, and Spencer [64] proved that

Theorem 4. *For every triangle-free graph G with n vertices and m edges*

$$D(G_n) \leqslant \max\left\{\frac{1}{2}m - \frac{2m(2m^2-n^3)}{n^2(n^2-2m)}, m - \frac{4m^2}{n^2}\right\} \tag{1.6}$$

Since the second term of (1.6) decreases in $\left[\frac{1}{8}n^2, \frac{1}{2}n^2\right]$, and its value is exactly $\frac{1}{25}n^2$ for $m = \frac{1}{5}n^2$, therefore (1.6) implies that if $e(G_n) > \frac{1}{5}n^2$, and $K_3 \nsubseteq G_n$, then $D(G_n) \leqslant \frac{1}{25}n^2$. It is trivial that if $e(G_n) \leqslant \frac{2}{25}n^2$, then $D(G_n) < \frac{1}{25}n^2$. However, the general conjecture is still open: it is unsettled in the middle interval $\frac{2n^2}{25} < e(G_n) < \frac{n^2}{5}$.

The next theorem of Erdős, Győri and myself [67] states that if $e(G_n) > \frac{1}{2}n^2$, then the pentagon-like graphs need the most edges to be deleted to become bipartite. (This is sharper than the earlier results, since it provides also information on the near-extremal structure.)

Theorem 5. *If $K_3 \nsubseteq G_n$ and $e(G_n) \geqslant \frac{n^2}{5}$, then there is a pentagonlike graph H_n^* with at least the same number of edges: $e(G_n) \leqslant e(H_n^*)$, for which $D(G_n) \leqslant D(H_n^*)$.*

1.14 Problems on the Booksize of a Graph

We have already seen a theorem of Erdős, stating that if a graph has many edges, then it has an edge e with cn triangles on it. Such configurations are usually called books. The existence of such edges is one of the crucial tools Erdős used in many of his graph theorems. Still, it was a longstanding open problem, what is the proper value of this constant c above. Without going into details we just mention three results:

Theorem 1. (Edwards [41, 42]). *If $e(G_n) > \frac{n^2}{5}$, then G_n has an edge with $[n/6]+1$ triangles containing this edge.*

This is sharp. The theorem would follow if we knew that there exists a $K_3 = (x, y, z)$ for which the sum of the degrees, $d(x) + d(y) + d(z) > \frac{3n}{2}$. Indeed, at least $\frac{n}{6}$ vertices would be joined to the same pair, say, to xy. An other paper of Edwards contains results of this type, but only for $e(G_n) > \frac{1}{3}n^2$. Let $\Delta_r = \Delta_r(G_n)$ denote the maximum of the sums of the degrees in a $K_r \in G_n$. (For instance, in a random graph $R_n \Delta_r(R_n) \approx r \cdot \frac{2e(R_n)}{n}$.)

Theorem 2. (Edwards [41]). *If* $\frac{1}{r}\Delta_r > \left(1 - \frac{1}{r+1}\right)n$, $n \geqslant 1$ *then*

$$\frac{1}{r+1}\Delta_{r+1} \geqslant \frac{2e(G_n)}{n}.$$

This theorem says that if G_n has enough edges to ensure a K_{r+1}, then it also contains a K_{r+1} whose vertex-degree-sum is as large as it should be by averaging. Erdős, Faudree and Győri have improved Theorem 1 if we replace the edge- density condition by the corresponding degree-condition. Among others, they have shown that

Theorem 3. (Erdős-Faudree-Győri [63]). *There exists a* $c > 0$ *such that if the minimum degree of* G_n *is at least* $[n/2] + 1$, *then* G_n *contains an edge with* $[n/6] + cn$ *triangles containing this edge.*

1.15 DigraphjMultigraph Extremal Graph Problems

We have already seen supersaturated extremal graph theorems on multi-digraphs. Here we are interested in simple asymptotically extremal sequences for digraph extremal problems.

Multigraph or digraph extremal problems are closely related and in some sense the digraph problems are the slightly more general ones. So we shall restrict ourselves to digraph extremal problems. A digraph extremal problem means that some q is given and we consider the class of digraphs where loops are excluded and any two vertices may be joined by at most q arcs in one direction and by at most q arcs of the opposite direction. This applies to both the excluded graphs and to the graphs on n vertices the edges of which should be maximized. So our problem is:

Fix the multiplicity bound q described above. A family $\mathcal{L}$ of digraphs is given and $\mathrm{ex}(n, \mathcal{L})$ denotes the maximum number of arcs a digraph $\mathbf{D}_n$ can have under the condition that it contains no $\mathbf{L} \in \mathcal{L}$ and satisfies the multiplicity condition. Determine or estimate $\mathrm{ex}(n, \mathcal{L})$.

The Digraph and Multigraph Extremal graph problems first occur in a paper of Brown and Harary [30]. The authors described fairly systematically all the cases of small forbidden multigraphs or digraphs. Next Erdős and Brown extended the investigation to the general case, finally I joined the “project”. Our papers [26], [28] and [29] describes fairly well the situation $q = 1$ for digraphs (which is roughly equivalent with $q = 2$ for multigraphs). We thought that our results can be extended to all q but Sidorenko [147] and then Rödl and Sidorenko [144] ruined all our hopes. One of our main results was in a somewhat simplified form:

Theorem 1. *Let* $q = 1$ *and* $\mathcal{L}$ *be a given family of excluded digraphs. Then there exists a matrix* $A = (a_{ij})$ *of* r *rows and columns, depending only on* $\mathcal{L}$, *such that there exists a sequence* $(\boldsymbol{S}_n)$ *of asymptotically extremal graphs for* $\mathcal{L}$

whose vertex-set V can be partitioned into $V_1, \dots, V_r$ so that for $1 \leqslant i < j \leqslant r$, $av \in V_i$ is joined to $av' \in V_j$ by an arc of this direction, iff the corresponding matrix-element $a_{ij} = 2$; further, the subdigraphs spanned by the V_i's are either independent sets or tournaments, depending on whether $a_{ii} = 0$ or 1.

One crucial tool in our research was a density notion for matrices. We associated with every matrix A a quadratic form and maximized it over the standard simplex:

$$g(A) = \max \left\{ uAu^T : \sum u_i = 1, u_i \geqslant 0 \right\}.$$

The matrices are used to characterize some generalizations of graph sequences like $(T_{n,p})_{n>n_0}$ of the general theory for ordinary graphs, and $g(A)$ measures the edge-density of these structures: replaces $\left(1 - \frac{1}{p}\right)$ of the Erdős-Stone-Simonovits theorem.

Definition 2. *A matrix A is called* **dense** *if for every submatrix B' of symmetric position $g(B') < g(B)$. In other words, B is minimal for $g(B) = \lambda$.*

We conjectured that - as described below - the numbers $g(B)$ are of **finite multiplicity** and **well ordered** if the matrices are dense:

Conjecture 3. *If q is fixed, then for each λ there are only finitely many dense matrices B with $g(B) = \lambda$. Further, if (B_n) is a sequence of matrices of bounded integer entries then $(g(B_n))$ cannot be strictly monotone decreasing.*

One could wonder how one arrives at such conjectures, but we do not have the space to explain that here. Similar matrices (actually, multigraph extremal problems) occur when one attacks Turán-Ramsey problems, see [69], [70], [71].

Our conjecture was disproved by Sidorenko and Rödl [144]. As a consequence, while we feel that the case $q = 1$ (i.e. the case of digraphs where any two points can be joined only by one arc of each direction) is sufficiently well described, for $q > 1$ the problem today seems to be fairly hopeless. Multidigraphs have also been considered by Katona in [113], where he was primarily interested in continuous versions of Turán-type extremal problems.

1.16 Erdős and Nassredin

Let me finish this paper with an anecdote. Nassredin, the hero of many middle-east jokes, stories (at least this is how we know it in Budapest), once met his friends who were eager to listen to his speech. 'Do you know what I wish to speak about" Nassredin asked them. "No, we don't" they answered. "Then why should I speak about it" said Nassredin and left.[14] Next time the friends really wanted to listen to the clever and entertaining Nassredin. So, when Nassredin asked the audience "Do you know what I want to speak about", they answered:

[14] I am not saying I follow his logic, but this is how the story goes.

"YES, we do". "Then why should I speak about it" said Nassredin again and went home. The third time the audience decided to be more clever. When Nassredin asked them "Do you know what I will speak about", half of the people said "YES" the other half said "NO". Nassredin probably was lasy to speak: "Those who know what I wanted to tell you should tell it to the others" he said and left again.

I am in some sense in Nassredin's shoes. How could I explain on 30 or 50 pages the influence of Erdős on Extremal Graph Theory to people who do not know it. And why should I explain to those who know it. Yet I think, Nassredin did not behave in the most appropriate way. So I tried - as I promised - to illustrate on some examples this enourmous influence of Paul. I do not think it covered half the topics and I have not tried to be too systematic.

Long Live Paul Erdős!

Bibliography

[1] M. Ajtai, P. Erdős, J. Komlós and E. Szemerédi: On Turán's theorem for sparse graphs, Combinatorica, **1**(4) (1981) 313–317.

[2] N. Alon: Tools from Higher Algebra, Chapter 32 of Handbook of Combinatorics (ed. Graham, Lovász, Grötschel), 1995.

[3] N. Alon: Eigenvalues and expanders, Combinatorica **6**, (1986) 83–96.

[4] N. Alon, R. Duke, H. Leffman, V. Rödl and R. Yuster: Algorithmic aspects of the regularity lemma, FOCS, 33 (1993) 479–481, Journal of Algorithms, **16**(1) (1994) 80–109.

[5] N. Alon, S. Friedland and A. Kalai: Regular subgraphs of almost regular graphs, J. Combinatorial Theory, Series B **37** (1984) 79–91.

[6] N. Alon and J. Spencer: The Probabilistic Method, Wiley Interscience, 1992.

[7] C. Benson: Minimal regular graphs of girth eight and twelve, Canad. J. Math., **18** (1966) 1091–1094.

[8] C. Berge and M. Simonovits: The colouring numbers of the direct product of two hypergraphs, Lecture Notes in Math., **411**, Hypergraph Seminar, Columbus, Ohio 1972 (1974) 21–33.

[9] B. Bolloás: Graphs without two independent cycles (in Hungarian), Mat. Lapok, **14** (1963) 311–321.

[10] B. Bollobás: Extremal Graph Theory, Academic Press, London, (1978).

[11] B. Bollobás: Relations between sets of complete subgraphs, Proc. Fifth British Combinatorial Conf. Aberdeen, (1975) 79–84.

[12] B. Bollobás: On complete subgraphs of different orders, Math. Proc. Cambridge Philos. Soc., **79** (1976) 19–24.

[13] B. Bollobás: Random Graphs, Academic Press, (1985).

[14] B. Bollobás: Extremal graph theory with emphasis on probabilistic methods, Conference Board of Mathematical Sciences, Regional Conference Series in Math, N062, AMS (1986).

[15] B. Bollobás and P. Erdős: On the structure of edge graphs, Bull. London Math. Soc., 5 (1973) 317–321.

[16] B. Bollobás and P. Erdős: On a Ramsey-Turán type problem, Journal of Combinatorial Theory, (B) **21** (1976) 166–168.

[17] B. Bollobás, P. Erdős and M. Simonovits: On the structure of edge graphs II, Journal of London Math. Soc. (2), **12** (1976) 219–224.

[18] B. Bollobás and Y. Kohayakawa: An extension of the Erdős-Storie theorem, Combinatorica, **14**(3) (1994) 279–286.

[19] B. Bollobás and A. Thomason: Large dense neighbourhoods in Tunin's theorem, Journal of Combinatorial Theory, (B) **31** (1981) 111–114.

[20] B. Bollobás and A. Thomason: Topological subgraphs, European Journal of Combinatorics.

[21] J. A. Bondy, Basic Graph Theory: Paths and Circuits, Manuscript (1992), to appear as a chapter in Handbook of Combinatorics, (eds Graham, Grötschel, Lovász)

[22] J. A. Bondy and R. C. Entringer: Longest cycles in 2-connected graphs with prescribed maximum degree, Can. J. Math, **32** (1980) 987–992.

[23] J. A. Bondy and M. Simonovits: Cycles of even length in graphs, Journal of Combinatorial Theory, **16**B (2) April (1974) 97–105.

[24] J. A. Bondy and M. Simonovits: Longest cycles in 3-connected 3-regular graphs, Canadian Journal Math. XXXII (4) (1980) 987–992.

[25] W. G. Brown: On graphs that do not contain a Thomsen graph, Canad. Math. Bull.,**9** (1966) 281–285.

[26] W. G. Brown, P. Erdős and M. Simonovits: Extremal problems for directed graphs, Journal of Combinatorial Theory, B **15**(1) (1973) 77–93.

[27] W. G. Brown, P. Erdős and M. Simonovits: On multigraph extremal problems, Problèmes Combin. et Theorie des graphes, (ed. J. Bermond et al.), Proc. Conf. Orsay 1976 (1978) 63–66.

[28] W. G. Brown, P. Erdős, and M. Simonovits: Inverse extremal digraph problems, Proc. Colloq. Math. Soc. János Bolyai **37** Finite and Infinite Sets, Eger (Hungary) 1981 Akad. Kiadó, Budapest (1985) 119–156.

[29] W. G. Brown, P. Erdős and M. Simonovits: Algorithmic Solution of Extremal digraph Problems, Transactions of the American Math Soc., **292/2** (1985) 421–449.

[30] W. G. Brown and F. Harary: Extremal digraphs, Combinatorial theory and its applications, Colloq. Math. Soc. J. Bolyai, **4** (1970) 1. 135–198; MR 45 #8576.

[31] W. G. Brown and M. Simonovits: Digraph extremal problems, hypergraph extremal problems, and densities of graph structures. Discrete Mathematics, **48** (1984) 147–162.

[32] De Caen and L. Székely: The maximum edge number of 4-and 6-cycle free bipartite graphs, Sets, Graphs, and Numbers, Proc. Colloq. Math. Soc. János Bolyai **60** (1992) 135–142.

[33] F. R. K. Chung: Regularity lemmas for hypergraphs and quasi-randomness, Random Structures and Algorithms, Vol. **2**(2) (1991) 241–252.

[34] F. R. K. Chung and R. L. Graham: Quasi-random hypergraphs, Random Structures and Algorithms, **1** (1990) 105–124.

[35] F. K. Chung, R. L. Graham and R. M. Wilson: Quasi-random graphs, Combinatorica, **9**(4) (1989) 345–362.

[36] V. Chvátal: On finite polarized partition relations, Canad. Math. Bull., **12** (1969) 321–326.

[37] V. Chvátal and E. Szemerédi: On the Erdős-Stene theorem, Journal of the London Mathematics Society, Ser. 2, **23** (1981) 207–214.

[38] K. Clarkson, H. Edelsbrunner, L. Guibas, M. Sharir and E. Welzl: Combinatorial complexity bounds for arrangements of curves and spheres, Discrete Computational Geometry, **55** (1990) 99–160.

[39] G. Dirac: Some theorems on abstract graphs, Proc. London Math. Soc. (3), 2 (1952),69–81.

[40] G. Dirac: Extensions of Turán's theorem on graphs, Acta Math., **14** (1963) 417–422.

[41] C. S. Edwards: Complete subgraphs with degree-sum of vertex-degrees, Combinatorics, Proc. Colloq. Math. Soc. Janos Bolyai **18** (1976), 293–306.

[42] C. S. Edwards: The largest degree-sum for a triangle in a graph, Bull. London Math. Soc., **9** (1977) 203–208.

[43] P. Erdős: On sequences of integers no one of which divides the product of two others and related problems, Mitt. Forsch. Institut Mat. und Mech. Tomsk **2** (1938) 74–82.

[44] P. Erdős: Neue Aufgaben 250. Elemente der Math., **10** (1955) p114.

[45] P. Erdős: Graph Theory and Probability, Canad. Journal of Math., **11** (1959) 34–38.

[46] P. Erdős: Graph Theory and Probability, II. Canad. Journal of Math., **13** (1961) 346–352.

[47] P. Erdős: On a theorem of Rademacher-Turán, Illinois J. Math., **6** (1962) 122–127. (Reprinted in [57].)

[48] P. Erdős: Über ein Extremalproblem in Graphentheorie, Arch. Math. (Basel), **13** (1962) 222–227.

[49] P. Erdős: On a problem in graph theory, Math. Gazette, **47** (1963) 220–223.

[50] P. Erdős: On extremal problems of graphs and generalized graphs, Israel J. Math, **2**(3) (1964) 183–190.

[51] P. Erdős: Extremal problems in graph theory, Theory of Graphs and its Appl., (M. Fiedler ed.) Proc. Symp. Smolenice, 1963), Acad. Press, NY (1965) 29–36.

[52] Erdős: Some recent results on extremal problems in graph theory (Results), Theory of Graphs (International Symposium, Rome, 1966), Gordon and Breach, New York and Dunod, Paris, (1967), 117–130, MR 37, #2634.

[53] P. Erdős: On bipartite subgraphs of graphs (in Hungarian), Matematikai Lapok, (1967) pp283–288.

[54] P. Erdős: On some new inequalities concerning extremal properties of graphs, Theory of Graphs, Proc. ColI. Tihany, Hungary (eds. P. Erdős and G. Katona) Acad. Press. N. Y. (1968) 77–81.

[55] P. Erdős: On some applications of graph theory to number theoretic problems, *Publ. Ramanujan Inst.*, **1** (1969) 131–136. (Sharpness of [43].)

[56] P. Erdős: On some extremal problems on r-graphs, Discrete Mathematics **1**(1), (1971) 1–6.

[57] P. Erdős: The Art of Counting, Selected Writings (in Combinatorics and Graph Theory, ed. J. Spencer), The MIT Press, Cambridge, Mass., (1973) MR58#27144

[58] P. Erdős: On the number of triangles contained in certain graphs, Canad. Math. Bull., **7**(1) January, (1974) 53–56.

[59] P. Erdős: Problems and results in combinatorial analysis, Theorie Cornbinatorie , Proc. Conf. held at Rome, 1973, Roma, Acad. Nazionale dei Lincei (1976) 3–17.

[60] P. Erdős: Paul Turán 1910–1976: His work in graph theory, J. Graph Theory, **1** (1977) 96–101.

[61] P. Erdős: On the combinatorial problems which I would most like to see solved, Combinatorica **1**, (1981), 25–42.

[62] P. Erdős: On some of my favourite problems in various branches of cornbinatorics, Fourth Czechoslovakian Symposium on Combinatorics, Graphs and Complexity, J. Nesetril, M Fiedler (editors) (1992), Elsevier Science Publisher B. V.

[63] P. Erdős, R. Faudree, and E. Györi: On the booksize of graphs with large minimum degree, Studia. Sci. Math. Hungar., **30** (1995) 1–2.

[64] P. Erdős, R. Faudree, J. Pach, J. Spencer: How to make a graph bipartite, Journal of Combinatorial Theory, (B) **45** (1988) 86–98.

[65] P. Erdős, P. Frankl and V. Rödl: The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent, Graphs and Combinatorics, **2** (1986) 113–121.

[66] P. Erdős and T. Gallai: On maximal paths and circuits of graphs, Acta Math. Acad. Sci. Hungar., **10** (1959) 337–356.

[67] P. Erdős, E. Gyor! and M. Simonovits: How many edges should be deleted to make a triangle-free graph bipartite, Sets, Graphs, and Numbers, Proc. Colloq. Math. Soc. János Bolyai 60 239–263.

[68] P. Erdős, A. Hajnal: On complete topological subgraphs of certain graphs, Annales Univ. Sci. Budapest, **7** (1964) 143-149. (Reprinted in [57].)

[69] P. Erdős, A. Hajnal, V. T. Sós, E. Szemerédi: More results on Ramsey-Turán type problems, Combinatorica, **3**(1) (1983) 69-82.

[70] P. Erdős, A. Hajnal, M. Simonovits, V. T. Sós, and E. Szemerédi: Turán- Ramsey theorems and simple asymptotically extremal structures, Cornbina- torica, **13** (1993) 31-56.

[71] P. Erdős, A. Hajnal, M. Simonovits, V. T. Sos and E. Szemerédi: Thran-Ramsey theorems for K_p-stability numbers, Combinatorics, Probability and Computing, **3** (1994) 297-325. (Proc. Cambridge Conf on the occasion of 80th birthday of P. Erdős, 1994.)

[72] P. Erdős, D. J. Kleitman and B. L. Rothschild: Asymptotic enumeration of K_n-free graphs, Theorie Combinatorie, Proc. Conf. held at Rome, 1973, Roma, Acad. Nazionale dei Lincei 1976, vol II, 19–27.

[73] P. Erdős, A. Meir, V. T. Sós and P. Turán: On some applications of graph theory I. Discrete Math., **2** (1972) (3) 207–228.

[74] P. Erdős, A. Meir, V. T. Sós and P. Turán: On some applications of graph the- ory II. Studies in Pure Mathematics (presented to R. Rado) 89–99, Academic Press, London, 1971.

[75] P. Erdős, A. Meir, V. T. Sós and P. Turán: On some applications of graph theory III. Canadian Math. Bulletin, **15** (1972) 27–32.

[76] P. Erdős and L. Pósa: On independent circuits contained in a graph, Canadian J. Math., **17** (1965) 347–352. (Reprinted in [57].)

[77] Erdős and A. Rényi: On the evolution of random graphs, Magyar Tud. Akad. Mat. Kut. Int. Közl., **5** (1960) 17–65. (Reprinted in [89] and in [57].)

[78] P. Erdős, A. Renyi and V. T. Sós: On a problem of graph theory, Studia Sci. Math. Hung., **1** (1966) 215–235.

[79] P. Erdős and H. Sachs: Reguläre Graphen gegebener taillenweite mit minimalen Knotenzahl, Wiss. Z. Univ. Halle-Wittenberg, Math-Nat. R. 12(1963) 251–258.

[80] P. Erdős, A. Sárközy and V. T. Sós: On product representation of powers, I, Reprint of the Mathematical Inst. of Hung. Acad. Sci, 12/1993., submitted to European Journal of Combinatorics.

[81] P. Erdős and M. Simonovits: A limit theorem in graph theory, Studia Sci. Math. Hungar., **1** (1966) 51–57. (Reprinted in [57].)

[82] P. Erdős and M. Simonovits: An extremal graph problem, Acta Math. Acad. Sci. Hung. , **22**(3–4) (1971) 275–282.

[83] P. Erdős and M. Simonovits: Some extremal problems in graph theory, Combinatorial Theory and its Appl., Proc. Colloq. Math. Soc. János Bolyai **4** (1969) 377–384. (Reprinted in [57].)

[84] P. Erdős and M. Simonovits: Compactness results in extremal graph theory, Combinatorica, **2**(3) (1982) 275–288.

[85] P. Erdős and M. Simonovits: Supersaturated graphs and hypergraphs, Combinatorica, **3**(2) (1983) 181–192.

[86] P. Erdős, V. T. Sós : Some remarks on Ramsey's and Turán's theorems, Combin. Theory and Appl. (P. Erdős et al eds) Proc. Colloq. Math. Soc. János Bolyai **4** Balatonfiired (1969), 395–404.

[87] P. Erdős and V. T. Sós: On Ramsey-Turán type theorems for hypergraphs Combinatorica **2**, (3) (1982) 289–295.

[88] P. Erdős, V. T. Sós : Problems and results on Rarnsey-Turán type theorems Proc. Conf. on Comb., Graph Theory, and Computing, Congr. Num., **26** 17–23.

[89] P. Erdős and J. Spencer: Probabilistic Methods in Combinatorics, Acad. Press, NY , 1974; MR52 #2895.

[90] P. Erdős and A. H. Stone: On the structure of linear graphs, Bull. Amer. Math. Soc., **52** (1946) 1089–1091.

[91] P. Erdős and G. Szekeres: A combinatorial problem in geometry, Compositio Math., **2** (1935) 463–470.

[92] R. J. Faudree and R. H. Schelp: Ramsey type results, Proc. Colloq. Math. Soc. János Bolyai **10** Infinite and Finite Sets, Keszthely, 1973, 657–665.

[93] R. J. Faudree and M. Simonovits: On a class of degenerate extremal graph problems, Combinatorica, **3**(1) (1983) 83–93.

[94] R. J. Faudree and M. Simonovits: Ramsey problems and their connection to Turán type extremal problems, Journal of Graph Theory, Vol **16**(1) (1992) 25–50.

[95] P. Frankl and V. Rödl: Hypergraphs do not jump, Combinatorica, **4**(4) (1984).

[96] P. Frankl and V. Rödl: The Uniformity lemma for hypergraphs, Graphs and Combinatorics, **8**(4) (1992) 309–312.

[97] P. Frankl and R. M. Wilson: Intersection theorems with geometric consequences, Combinatorica **1**, (4) (1981) 357–368.

[98] Z. Füredi: Graphs without quadrilaterals, Journal of Combinatorial Theory, (B) **34** (1983) 187–190.

[99] Z. Füredi: Turán type problems, Surveys in Combinatorics, (A. D. Keedwell, ed.) Cambridge Univ. Press, London Math. Soc. Lecture Note Series, **166** (1991) 253–300.

[100] Z. Füredi: On a Turán type problem of Erdős, Combinatorica, **11**(1) (1991) 75–79.

[101] Z. Füredi: New asymptotics for bipartite Turán numbers, submitted to Journal of Combinatorial Theory, (B) (see also the abstracts of invited lectures at ICM Zürich, 1994)

[102] Z. Fiiredi and Á. Seress: Maximal triangle-free graphs with restrictions on the degrees, Journal of Graph Theory, **18**(1) 11–24.

[103] R. K. Guy: A problem of Zarankiewicz, Proc. Coll. Theory of Graphs, (Tihany, 1966), (eds: Erdős, Katona) Akad. Kiadó, Budapest, 1968, 119–150.

[104] R. K. Guy and S. Znám: A problem of Zarankiewicz, Recent Progress in Combinatorics, (eds: J. A. Bondy, R. Murty) Academic Press, New York, 1969, 237–243.

[105] A. Gyárfás, J. Komlós and E. Szemerédi: On the distribution of cycle length in graphs, Journal of Graph Theory, **8**(4) (1984) 441–462.

[106] E. Győri: On the number of edge-disjoint triangles in graphs of given size, Proc. Colloq. Math. Soc. János Bolyai **52** 7th Hungarian Combinatorial ColI. (Eger) North Holland (1987) 267–276.

[107] E. Győri: On the number of edge-disjoint cliques in graphs, Combinatorica **11**, (1991) 231–243.

[108] E. Győri: Edge-disjoint cliques in graphs, Proc. Colloq. Math. Soc. János Bolyai **60** (Proc. ColI. dedicated to the 60th birthday of A. Hajnal and V. T. Sós, Budapest, 1991), 357–363.

[109] B. Jackson: Longest cycles in 3-connected cubic graphs, Journal of Combinatorial Theory, (B) **41** (1986) 17–26.

[110] B. Jackson and T. D. Parson: On r-regular, r-connected non-hamiltonian graphs, Bull. Australian Math. Soc., **24** (1981) 205–220.

[111] B. Jackson and T. D. Parson: Longest cycles in r-regular r-connected graphs, Journal of Combinatorial Theory, (B) **32** (3) (1982) 231–245.

[112] B. Jackson, N. C. Wormald: Longest cycles in 3-connected graphs of bounded maximum degree, *Graphs, Matrices, Designs*, Lecture Notes in Pure and Applied Math, Marcel Dekker Inc., (1993) 237–254.

[113] Gy. Katona: Continuous versions of some extremal hypergraph problems, Proc. Colloq. Math. Soc. János Bolyai 18 Combinatorics, (Keszthely, 1976) II. 653–678, MR 80e#05071.

[114] G. Katona, T. Nemetz and M. Simonovits: A new proof of a theorem of P. Turán and some remarks on a generalization of it, (In Hungarian), Mat. Lapok, XV. 1–3 (1964) 228–238.

[115] D. J. Kleitman and K. J. Winston: On the number of graphs without 4-cycles, Discrete Mathematics **41**, (1982),167–172.

[116] Y. Kohayakawa: The Regularity Lemma of Szemerédi for Sparse Graphs, (Manuscript, August, 1993)

[117] J. Komlos and M. Simonovits: Szemeredi regularity lemma and its application in graph theory, to appear in Paul Erdős is **80**, Proc. Colloq. Math. Soc. János Bolyai (1995)

[118] J. Komlós, V. T. Sos: Regular subgraphs in graphs, manuscript, (1992)

[119] J. Komlós and E. Szemerédi: Topological cliques in graphs, Combinatorics, Probability and Computing **3** (1994), 247–256.

[120] J. Komlós and E. Szerneredi: Topological cliques in graphs II, Combinatorics, Probability and Computing, to appear.

[121] T. Kővári, V. T. Sós, P. Turán: On a problem of Zarankiewicz, Colloq. Math., **3** (1954), 50–57.

[122] F. Lazebnik and V. A. Ustimenko: New examples of graphs without small cycles and of large size, European Journal of Combinatorics, **14**(5) (1993) 445–460.

[123] F. Lazebnik, V. A. Ustimenko, and A. J. Woldar: New constructions of bipartite graphs on m, n vertices with many edges and without small cycles, to appear in JCT(B).

[124] L. Lovász and M. Simonovits: On the number of complete subgraphs of a graph I. Proc. Fifth British Combin. Conf. Aberdeen (1975) 431–442.

[125] L. Lovász and M. Simonovits: On the number of complete subgraphs of a graph II. Studies in Pure Math (dedicated to the memory of P. Turán). Akadémiai Kiadó+Birkhäuser Verlag, (1983) 459–495.

[126] A. Lubotzky, R. Phillips, and P. Sarnak: Ramanujan Conjecture and explicite construction of expanders, (Extended Abstract), Proc. STOC 1986, 240–246

[127] A. Lubotzky, R. Phillips, and P. Sarnak: Ramanujan graphs, Combinatorica, **8**(3) 1988,261–277.

[128] W. Mader: Hinreichende Bedingungen für die Existenz von Teilgraphen die zu einem vollständigen Graphen homöomorph sind, Math. Nachr. **53** (1972), 145–150.

[129] W. Mader: Homomorphieeigenschaften und mittlere Kantendichte von Graphen, Math. Annalen **174** (1967), 265–268.

[130] W. Mantel: Problem 28, Wiskundige Opgaven, **10** (1907) 60–61.

[131] G. A. Margulis: Explicit constructions of graphs without short cycles and low density codes, Combinatorica, **2**(1) (1982) 71–78.

[132] G. A. Margulis: Arithmetic groups and graphs without short cycles, 6th Internat. Symp. on Information Theory, Tashkent 1984, Abstracts, Vol. **1**, 123–125 (in Russian).

[133] G. A. Margulis: Some new constructions of low-density parity-check codes, convolution codes and multi-user communication, 3rd Internat. Seminar on Information Theory, Sochi (1987), 275–279 (in Russian)

[134] G. A. Margulis: Explicit group theoretic construction of group theoretic schemes and their applications for the construction of expanders and concentrators, Journal of Problems ofInformation Transmission, 1988 pp 39–46 (translation from Problemy Peredachi Inforrnatsii, **24**(1) 51–60 (January-March 1988)

[135] J. W. Moon: On independent complete subgraphs in a graph, Canad. J. Math., **20** (1968) 95–102. also in: International Congress of Math. Moscow, (1966), vol 13.

[136] J. W. Moon and Leo Moser: On a problem of Turán, MTA, Mat. Kutató Int. Közl., **7** (1962) 283–286.

[137] J. Pach and P. Agarwal: Combinatorial Geometry, Courant Institute Lecture Notes, New York University, 1991 (200 pages) to appear in ...

[138] H. J. Prömel: Asymptotic enumeration of ℓ-colorable graphs, TR Forshungsinstitute für Diskrete Mathematik, (1988) Bonn, Report No 88???-OR

[139] H. J. Prömel and A. Steger: Excluding induced subgraphs: Quadrilaterals, Random Structures and Algorithms, **2**(1) (1991) 55–71.

[140] H. J. Prömel and A. Steger: Excluding induced subgraphs II TR Forshungsinstitute für Diskrete Mathematik, (1990) Bonn, Report No 90642–0R to appear in Discrete Applied Math.

[141] H. J. Prömel and A. Steger: Excluding induced subgraphs III A general asymptoties, Random Structures and Algorithms, **3**(1) (1992) 19–31.

[142] L. Pyber, Regular subgraphs of dense graphs, Combinatorica, **5**(4) (1985) 347–349.

[143] F. P. Ramsey: On a problem of formal logic, Proc. London Math. Soc. 2nd Series, **30** (1930) 264–286.

[144] V. Rodl and A. Sidorenko: On jumping constant conjecture for multigraphs, (Manuscript, 1993)

[145] I. Z. Ruzsa and E. Szemeredi: Triple systems with no six points carrying three triangles, *Combinatorics* (Keszthely, 1976), (1978), **18**, Vol. II., 939–945. North-Holland, Amsterdam-New York.

[146] G. N. Sárközy: Cycles in bipartite graphs and an application in number theory, (1993), TR DIMACS forthcoming in Journal of Graph Theory, ( )

[147] A. F. Sidorenko: Boundedness of optimal matrices in extremal multigraph and digraph problems, Combinatorica, **13**(1) (1993) 109–120.

[148] A. F. Sidorenko: Extremal estimates of probability measures and their combinatorial nature Math. USSR - Izv 20 (1983) N3 503–533 MR 84d: 60031. (=Translation) Original: Izvest. Acad. Nauk SSSR. ser. matem. 46(1982) N3 535–568.

[149] A. F. Sidorenko: What do we know and what we do not know about Turán Numbers, Manuscript, (submitted to Graphs and Combinatorics, 1992, Febr.)

[150] M. Simonovits: A method for solving extremal problems in graph theory, Theory of graphs, Proc. ColI. Tihany, (1966), (Ed. P. Erdős and G. Katona) Acad. Press, N.Y., (1968) 279–319.

[151] M. Simonovits: A new proof and generalizations of a theorem of Erdős and Pósa on graphs without $k+1$ independent circuits, Acta Math. Acad. Sci. Hungar., **18**(1–2) (1967) 191–206.

[152] M. Simonovits: The extremal graph problem of the icosahedron, Journal of Combinatorial Theory, **17**B (1974) 69–79.

[153] M. Simonovits: Extremal graph problems with symmetrical extremal graphs, additional chromatic conditions, Discrete Math., **7** (1974) 349–376.

[154] M. Simonovits: On Paul Turán's influence on graph theory, J. Graph Theory, **1** (1977) 102–116.

[155] M. Simonovits: Extremal graph problems and graph products, Studies in Pure Math. (dedicated to the memory of P. Turán) Akadémiai Kiadó+Birkhauser Verlag (1982)

[156] M. Simonovits: Extremal Graph Theory, Selected Topics in Graph Theory, (ed. by Beineke and Wilson) Academic Press, London, New York, San Francisco, 161–200. (1983)

[157] M. Simonovits: Extremal graph problems, Degenerate extremal problems and Supersaturated graphs, Progress in Graph Theory (Acad Press, ed. Bondy and Murty) (1984) 419–437.

[158] M. Simonovits and V. T. Sós: Szemerédi's partition and quasi-randomness, Random Structures and Algorithms, Vol 2, No.1 (1991) 1–10.

[159] R. Singleton: On minimal graphs of maximum even girth, Journal of Combinatorial Theory 1 (1966), 306–332.

[160] V. T. Sós: On extremal problems in graph theory, Proc. Calgary International Conf. on Combinatorial Structures and their Application, (1969) 407–410.

[161] V. T. Sós: Some remarks on the connection between graph-theory, finite geometry and block designs Theorie Combinatorie, Ace. Naz.dei Lincei (1976) 223–233

[162] V. T. Sos: Survey on Turán-Ramsey Problems, manuscript, to be published.

[163] J. Spencer, E. Szemerédi and W. T. Trotter: Unit distances in the Euclidean plane, Graph Theory and Cominatorics, Proc. Cambridge Combin. Conf. (ed B. Bollobás) Academic Press (1983) 293–304.

[164] F. Sterboul: A class of extremal problems, Recent Advances in Graph Theory, Proc. Conf. Praga, 1974, 493–499.

[165] F. Sterboul: On the chromatic number of the direct product of hypergraphs, Lecture Notes in Math., **411**, Hypergraph Seminar, Columbus, Ohio 1972 (1974),

[166] T. Szele: Combinatorial investigations, Matematikai és Physikai Lapok, **50** (1943) 223–256.

[167] E. Szemerédi: On a set containing no k elements in an arithmetic progression, Acta Arithmetica, **27** (1975) 199–245.

[168] E. Szemerédi: On regular partitions of graphs, Problemes Combinatoires et Théorie des Graphes (ed. J. Bermond et al.), CNRS Paris, 1978,399–401.

[169] E. Szemerédi: On graphs containing no complete subgraphs with 4 vertices (in Hungarian) Mat. Lapok, **23** (1972) 111–116.

[170] A. Thomason: Random graphs, strongly regular graphs and pseudo-random graphs, in Surveys in Combinatorics, 1987 (Whitehead, ed.) LMS Lecture Notes Series 123, Cambridge Univ. Press, Cambridge, 1987, 173–196

[171] A. Thomason: Pseudo-random graphs, in Proceedings of Random graphs, Poznan, 1985, (M. Karonski, ed.), Annals of Discrete Math., **33** (1987) 307–331.

[172] Collected papers of Paul Turán: Akadémiai Kiadó, Budapest, 1989. Vol 1–3, (with comments of Simonovits on Turán's graph theorem).

[173] P. Turán: On an extremal problem in graph theory, Matematikai Lapok, **48** (1941) 436–452 (in Hungarian), (see also [174], [172]).

[174] P. Turán: On the theory of graphs, Colloq. Math., **3** (1954) 19–30, (see also [172]).

[175] P. Turán, Applications of graph theory to geometry and potential theory, Proc. Calgary International Conf. on Combinatorial Structures and their Application, (1969) 423–434 (see also [172]).

[176] P. Turán: A Note of Welcome, Journal of Graph Theory, **1** (1977) 7–9.

[177] H.-J, Voss: Cycles and bridges in graphs, Deutscher Verlag der Wissenschaften, Berlin, Kluwer Academic Publisher, (1991)

[178] H. Walter and H.-J, Voss: Über Kreise in Graphen, (Cycles in graphs, book, in German) VEB Deutscher Verlag der Wissenschaften, Berlin, 1974.

[179] R. Wenger: Extremal graphs with no C^4, C^6 and C^{10}, Journal of Combinatorial Theory, (B) **52** (1991) p.113–116.

[180] K. Zarankiewicz: Problem P101, Colloq. Math, **2** (1951) 301.

[181] A. A. Zykov: On some properties of linear complexes, Mat Sbornik, **24** (1949) 163–188, Amer. Math. Soc. Translations, **79** 1952.