 

# Paul Turán's influence in combinatorics

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**Abstract.** This paper is a survey on the topic in extremal graph theory influenced directly or indirectly by Paul Turán. While trying to cover a fairly wide area, I will try to avoid most of the technical details. Areas covered by detailed fairly recent surveys will also be treated only briefly. The last part of the survey deals with random $\pm 1$ matrices, connected to some early results of Szekeres and Turán.

**Keywords.** Extremal graph theory, hypergraphs, regularity lemma, quasi-randomness, applications, random matrices.

**AMS classification.** Primary 05C35, Secondary 05C65.

## 1. Preface

Paul Turán was one of my professors who had the greatest influence, – not only on me, on my way of thinking of Mathematics, of doing Mathematics, but – on my whole mathematical surrounding.

Once I read that Hilbert was the last polyhistor in Mathematics. This meant that after him not too many people had an overview over the whole Mathematics. I do not really know if this is true or not: I know only that "most" of the mathematicians I know concentrate basically on one or two fields, while some of my professors, like Erdős, Turán, and Rényi were covering several parts of Mathematics. I think of Turán as a polyhistor in Mathematics.

YES: Today only the best can excel in more than one branch. Turán was one of them. His main work, his most important results concern primarily number theory, interpolation and approximation theory, the theory of polynomials and algebraic equations, complex analysis, and Fourier analysis. He invented a new method in analysis, called the power sum method [369], giving interesting results in themselves and applicable in several distinct branches of Mathematics. His results in combinatorics and graph theory were definitely *not his most important results*, still they were very important in graph theory. He found theorems that later became the roots of whole theories. Definitely this is the case with his – today already classical – graph theorem. Paul Erdős wrote [121] that

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> Turán had the remarkable ability to write perhaps only one paper or to state one problem in various fields distant from his own; later others would pursue his idea and a new subject would be born.
>
> In this way Turán initiated the field of extremal graph theory. He started this subject in 1941 (see [358] and [359])...

I should also mention here that – though the big breakthrough in the application of probabilistic methods in combinatorics is due to Erdős – Turán's new proof of the Hardy–Ramanujan theorem [356] (later becoming the root of statistical number theory) and the Szekeres–Turán proof of the existence of "almost Hadamard matrices" [347] were important contributions.

I have just written that Paul Turán greatly influenced our way of thinking. Both Erdős and Turán quite often set out from some particular problem and then built up a whole theory around it. However, for Turán the motivation seemed to be much more important. When he spoke about Mathematics, he went a long way to explain why that problem he was speaking of was interesting for him. My impression was that he preferred building theories, at the same time was cautious not to build too general theories that might seem to be already vacuous.

I shall explain this through some "stories".[1]

(a) I started working in extremal graph theory, basically at the end of my first year – as a student – at the Eötvös Loránd University. This happened as follows: Vera Sós (the wife of Turán) was our lecturer in "Mathematical Analysis" and in "Combinatorics and Graph Theory". (Our group of 26 first-year honours students in Mathematics had nine 50 minute lectures with her weekly. A year earlier she had also taught combinatorics to the group of Bollobás.) After our first year she was definitely our most popular lecturer. The second semester Vera decided to start a so called "special lecture" on Graph Theory, as a continuation of her "introductory course". Most probably most of the dedicated students in Mathematics attended this course. Here she spoke – among others – about Turán's hypergraph conjecture. Next week three of us, (independently?) Katona, Nemetz, and myself told her that we have proved some theorems in connection with Turán's hypergraph conjecture. Vera suggested to write them up, in Hungarian, in the Matematikai Lapok, in a joint paper. First Katona and I wrote up the paper, but that was not good enough for Vera, so Katona and Nemetz rewrote it, and finally the paper [215] appeared and became one of our most cited papers.[2] Having finished the paper, I continued working on these types of questions, while Katona and Nemetz went into other directions. So I proved several theorems which today would be called Turán type results. I wrote them up in a "student paper" and submitted it to the "Students Research Society" (Matematikai Diákkör) whose "professor" leaders were András Hajnal and

[1] Telling stories is a very dangerous thing: the reader may think that I promised to write of Paul Turán and instead I am speaking of Vera Sós, or even worse, of myself. No, No, No: I am speaking of our excellent professors, Turán, Erdős, Vera Sós, András Hajnal, Rényi, Gallai ....

[2] This was my first joint paper with Nemetz, and the second one of Katona, who finished his fourth year at the university at that time.

Vera Sós in those days. Most probably I won some prize, and the question was if to publish my new results in some mathematical journal, say in Acta Math. Acad. Hungarica. However, a little later Vera Sós informed me that "unfortunately" Gábor Dirac had just published a paper on related topics [100]. So my second paper was "killed".

Anyway, slightly later I met Turán, and tried to inform him of my results, starting in a "very abstract way". Basically I defined a monotone property $\mathcal{P}$ and maximized the number of edges in the family of $n$-vertex graphs of property $\mathcal{P}$. Turán suggested to take the simpler but equivalent formulation that "We have a finite or infinite family of excluded subgraphs…". Even today I stick to this "more transparent" formulation.

(b) Actually, the first time I met Turán – as a mathematics professor – was slightly earlier. In the first semester Vera Sós taught us Analysis, however, one day she got flu, had fever and had to stay home. So her husband, Turán came in to give the lecture, on the Lagrange Mean Value Theorem. Despite the fact that in those days Vera was our favourite lecturer, I was shocked by the spellbinding style of Turán, while speaking of this relatively simple theorem.

Actually, I heard some opinions, according to which Turán was excellent for the best students but sometimes difficult to follow for the less gifted ones.[3] The reason for this was that he not only proved the theorems but (a) explained the background very carefully and (b) explained what would fail if we tried to prove it in some other ways.

(c) When I became a third year student, I started learning Function Theory (Theory of Complex Analytic Functions), from Kató Rényi, the wife of Alfréd Rényi. I enjoyed her lectures very much and having finished this two-semester course, for some reason I dropped into the Mathematical Institute.[4] There I met Gábor Halász and asked what he was doing there. He answered that in 10 minutes there would be a seminar of Turán in Number Theory and Complex Analysis, and he would give a lecture there. I happened to be free, so I decided to attend Gábor's lecture. I enjoyed that whole atmosphere and the Mathematics there so much that I became a regular participant of the "Turán seminar" for many, many years. And that was partly due to Halász, but primarily to Turán. The seminar was interactive, very friendly, anyone could ask any (relevant mathematical) questions, to help one to understand the details, and the background ….

(d) Several years later, as an assistant professor, once I entered Turán's office. He was reading a letter, which informed him about some new results (about the convergence properties of power series on the unit complex disk). He started explaining it to me. I asked him why that result was interesting and the answer was very convincing. Actually, I was "slightly frightened": I felt that Turán could convince me of any mathematical result being interesting, if he felt it interesting.

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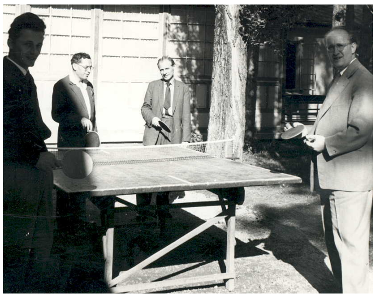
[3] Unfortunately Turán have not given regular Number Theory courses those years. Here the "gifted" would mean the best 10 students in our group.

[4] I was a student and later an assistant professor … at the Eötvös University while this was a Research Institute, part of the Academy, headed by Alfréd Rényi. Fortunately in those days the walking distance between the two places was roughly five minutes.

Kató Rényi, Turán, Vera Sós, Erdős (and somebody covered by Vera?)



Knapowski, Erdős, Szekeres, and Turán

We are often asked: what is the secret of Hungarian Mathematics that it is so good? Of course, we have standard answers to this, despite the fact that the question itself may be slightly dangerous.

It is nice to hear that our Mathematics is outstanding, but at the same time one should keep checking in which areas can one be satisfied and where we have to do something to improve "Hungarian Mathematics".

I myself have at least three answers to this question. The first one is that in Hungary there is a very strong tradition to support talented young students in Mathematics and Physics (and most probably, in many other fields as well). We had our KöMaL: the High School Mathematics Journal. Most of those who are today math professors in Hungary still remember how much we owe to it and have gained from participating in the contests organized in this surrounding.[5] Also, there were organized math lectures and meetings while we were still high school students. This is where I first met Bollobás, Komlós, Halász, and many others when I was a second year high school student.

Yet, definitely, one of the most important factors was that we had excellent professors at the University. Excellent in Mathematics and excellent in conveying their Mathematics to us. I myself, selecting those who really influenced my Mathematics, (following the timeline) would list first Vera Sós, Paul Erdős and Paul Turán.[6]

## 1.1. Apologizing?

In this survey I will try to cover several areas, but not in too much detail. Often I will start some topic, give a few theorems, and then refer the reader to other surveys or papers.

[5] Actually, Erdős and Turán learnt of each other from this journal.

[6] If I wanted to extend this list, of course, I would add my mother, perhaps Hajós, definitely Hajnal, Gallai, and Rényi. We met Rényi relatively late, when we became third year students, however, when he started giving special lectures about Random Methods in Analysis, Random Methods in Combinatorics, Introduction to Information Theory, again, all the best students were sitting there and eagerly listening to him. He – similarly to Turán – also gave long explanations on the background of the theorems he was speaking of.

While writing this survey, I looked at several other surveys, of excellent authors, and many of them started with apologizing sentences that there was no way to try to be complete, and the author had to leave out several interesting and important results. The same applies to this survey as well. In several cases – selecting a paper – I had to restrict myself to including its first, or most characteristic results, and leave the other, at least for me very important, results to the reader. One reason for this was that I tried to write a readable survey. And the same is the reason why I was not afraid to repeat some parts: be occasionally "redundant".

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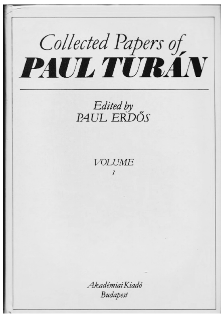


When Turán died in 1976, his collected papers were published in a three-volume book [368], which is an annotated edition of his works in the sense that the grateful mathematical surrounding added mathematical notes to his papers. I myself was responsible for Graph Theory and Combinatorics. I wrote three mini-surveys for [368]: one on "pure extremal graph theorems", another one on applications of extremal graph theorems in Analysis, Geometry (and Potential Theory), and the third one on "random matrices". *This survey* includes a large part of those surveys, however, it goes much further: the new developments in the field showing Turán's influence in Discrete Mathematics greatly surpass what I could write in those days. Here I include many results showing these new developments (and leave out certain parts covered by other surveys of this volume, see Katona e.g., [214]. I also cut short describing areas that are covered by the very recent survey papers of the Erdős Centennial volume, e.g., Gowers [189], Rödl and Schacht [303] or Füredi and myself [180], ....

Of course, the most important subject covered here (where Turán's influence can be seen) is Extremal Graph Theory. One basic source to provide a lot of information is the book of Bollobás, Extremal Graph Theory [55]. There are many surveys covering distinct parts of this very large area. Among them are mine, e.g., [327], [328], [330], [332] and there is a survey by Bollobás in the Handbook of Combinatorics [51]. Of course, the Handbook contains several further chapters basic to this field, just to mention the chapters by Bondy [64] and by Alon [9]. I should also mention many excellent, more detailed further surveys related to this one, e.g., of Füredi [167], Keevash [218], Kühn and Osthus [255].

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Since the very recent survey of Füredi and myself [180] covers a huge and important area of extremal graph theory, namely the so-called Degenerate Extremal Graph Problems, here we shall concentrate on the non-degenerate cases, where the extremal structures have positive density. In this non-degenerate case I will select five topics:

(a) New results attained with the help of the Szemerédi Regularity Lemma [349] (for the older ones see, e.g., [249]). There are very many new developments in this area, which will be touched on only very briefly, in Section 6.2. Here I mention only its connection to Property Testing [16] [14], ... and to graph limits, where I refer the reader to some papers of Christian Borgs, Jennifer Chayes, László Lovász, Vera Sós, Kati Vesztergombi, e.g., [68, 69, 70], to the homepage of Lovász, where many of these can easily be found, and to the very new book of Lovász [263];

(b) Ramsey–Turán type results, where for the older results see the survey of Vera Sós and myself [335], and for the many new interesting developments, see among others Balogh and Lenz [39].

(c) and also the Andrásfai–Erdős–Sós type theorems [24], Erdős–Simonovits [139], Łuczak [268], Thomassen [355], ...

(d) Applications in multicolor Ramsey problems, e.g., results of Łuczak [269], Gyárfás, Ruszinkó, Sárközy, and Szemerédi [194], Kohayakawa, Simonovits, Skokan [231], and many others.

(e) Typical Structures: Erdős–Kleitman–Rothschild type theorems, [131], Erdős, Frankl and Rödl [125], and Balogh, Bollobás, and Simonovits, e.g., [34], ...

Again, there is no way to be complete here. Rather I chose to indicate the main lines of some of these theories .... It is also very useful and informative to read the corresponding problem-posing papers of P. Erdős [113] [119] [120], [123]. I should also mention the book of Chung and Graham on Erdős problems [93].

In Section 16 I will discuss the theory of Random Matrices, but only shortly: a relatively new and excellent survey of Van Vu [370] describes this area in detail. There is also another reason: Subsection 16.2 on determinants is connected to Turán the most, while in the next two parts on the probability of being singular and on the distribution of eigenvalues of random matrices is where many new interesting results were proved after Turán's death. Yet, they are connected to Turán in a slightly weaker way.[7]

Overlapping with my older surveys is inevitable. Yet I will try to "overemphasize" those parts that had to be left out from [180] and [331]. Some further related surveys and pseudo-survey papers are Füredi [167], Sidorenko [318] Simonovits [330], Simonovits and Sós [335], Kohayakawa and Rödl [229], Rödl and Schacht [303], and many others.

## 2. Introduction

Today one of the most developed and fastest developing areas of Graph Theory is Extremal Graph Theory and the parts of Graph Theory connected to it. There are several reasons for this. One of them is that this is a real theory with many important, highly non-trivial subfields and many related larger fields of combinatorics. I have already mentioned some some of them. Further ones are

[7] Yet I decided to include a short part on them, too.

(a) Although Extremal Hypergraph theory is still an extremely hard field to achieve new results in, several very interesting new theorems were proved for hypergraphs in the last decade.
(b) New tools were created, above all, Hypergraph Regularity Lemmas, and, connected to them, Removal Lemmas and Counting Lemmas, and Graph Limit Theory.
(c) Computers were used to solve several extremal graph and hypergraph problems, mostly using a new theory, the Razborov Flag Algebras [293, 296].
(d) Some parts of Theoretical Computer Science are connected to the above fields. I mention here four such topics:
  (i) Graph Property Testing, very strongly connected to applying Szemerédi Regularity Lemma, (see e.g. papers of Alon and Shapira) [21], [16].
  (ii) Applications of graph results, e.g., Degenerate Extremal Graph Theorems in Computer Science.
  (iii) Theory of quasi-random graphs (initiated in some sense by Thomason, [353], then by Chung, Graham and Wilson [94] ...
  (iv) Application of random graph methods and expanders – that are strongly connected to extremal graph theory – in Computer Science,[8]
(e) As to the tools used in Extremal Graph Theory, they are connected to the theory of Random Graphs:
  (i) it uses random graphs to get lower bounds,
  (ii) it investigates extremal subgraphs of random graphs,
  (iii) and it motivates the description of typical structures,
(f) It is connected among others, to Finite Geometry (also used for constructions providing lower bounds in our problems), to Commutative Algebra, also used to get lower bounds, ... (Vera Sós wrote one of the first surveys on the connections to Finite Geometries [339]).

Reading this “list” the reader immediately sees that describing the new developments in this area is much more than what such a survey paper can cover, even if in many cases it only refers to other papers or surveys. So we shall try to provide a “random tour” in this huge area.

Also, I plan to post on my homepage a slightly longer version of this survey, providing more details.

## 2.1. Structure of the paper

(a) We shall start with the Theory of Extremal Graphs. We shall describe the huge development of the Theory of Extremal Graphs, primarily areas neglected in [332] and [180].

[8] For two “mini-surveys” see e.g. Spencer [341] and Alon [10].

(b) Section 5 describes the theory of supersaturated graphs.

(c) In Section 13 I shall describe those *applications of extremal graph results* which were initiated by Paul Turán, in the last years of his life. Also we shall describe other applications of Turán's theorem.

(d) These applications led also to the Ramsey–Turán Theory, described in more detail in the survey paper of Vera Sós and myself [335]. There are quite a few new developments in this field. I shall describe some of them in Section 10.

(e) There are several connections between Ramsey Theory and the theory of Turán type problems. Section 12 contains some results on this.

(f) There is one more, very important area not to be forgotten: Erdős and Turán greatly influenced our day's mathematics just by asking about the density version of Van der Waerden's theorem. This is well described, at least its early period, in the book of Graham, Rothschild and Spencer [190]. Many important details can be learned from the paper of Vera Sós [340], papers of Gowers, Green, Tao, . . . I also will include a very short section on this topic.

(g) Section 16 discusses a paper of Szekeres and Turán on the average of the square of the determinants of random $\pm 1$ matrices.

## 3. Turán type graph problems

Paul Turán's graph theoretical and combinatorial results can roughly be classified as follows:

(a) His classical extremal graph theorem [358, 359] and the analogous results of Kővári, V. T. Sós and Turán [252] on the extremal number of $K_2(a, b)$.

(b) His results on applications of his graph theorem, see [363, 364, 365, 366], and also the papers of Erdős, Meir, V.T. Sós and Turán [132, 133, 134][9].

(c) Results on random $\pm 1$ matrices, estimating the average of the $k^{th}$ power of their determinants [347, 357, 360, 362].

(d) Beside this, it was Turán who asked the first general question in connection with the crossing numbers (see e.g., one of his last papers [367], or Beineke and Wilson [46]).

### 3.1. Turán's graph theorem

In 1935 Erdős and Szekeres proved [149] that

**Theorem 3.1.** *For every $k$ there exists an $n_k$ such that if we fix $n_k$ points in the plane arbitrarily (but in general position), then there are always $k$ of them spanning a convex $k$-gon.*

[9] and a corrigendum to [134] (misprints).

To prove this, they applied Ramsey's theorem. Actually they did not know it, but rediscovered it. Motivated by the Ramsey Theorem, Turán proved his famous theorem. Before formulating it we introduce some notations.

**Notation.** Given a graph, hypergraph, the first subscript will almost always denote the number of vertices: $G_n$, $S_n$, $H_n$ will mostly denote graphs (digraphs, hypergraphs) of $n$ vertices.[10] Mostly we shall restrict our considerations to ordinary graphs (without loops and multiple edges). Given a graph (digraph, hypergraph) $G$, $v(G)$ and $e(G)$ denote the number of vertices and edges respectively, and $\chi(G)$ is $G$'s chromatic number. $K_p$ denotes the complete graph on $p$ vertices, $C_\ell$ and $P_\ell$, are the cycle and path of $\ell$ vertices, respectively. $K_p(n_1,\dots,n_p)$ is the complete $p$-partite graph with $n_i$ vertices in its $i^{\text{th}}$ class, and $T_{n,p}$ is the Turán graph of $n$ vertices and $p$ classes, that is, $T_{n,p} = K_p(n_1,\dots,n_p)$ where $\sum n_i = n$ and $|n_i - \frac{n}{p}| < 1$.

Given two graphs $G$ and $H$, denote by $G \otimes H$ the graph obtained from vertex-disjoint copies of $G$ and $H$ by joining each vertex of $G$ to each one of $H$. (Occasionally we denote their disjoint union by $G + H$, and the disjoint union of $k$ copies of $H$ by $kH$.)

**Turán's problem.** *Given $p$ and $n$, how large can $e(G_n)$ be if $G_n$ does not contain a $K_{p+1}$ ?*

Clearly, $T_{n,p}$ does not contain $K_{p+1}$. Turán's theorem asserts that $T_{n,p}$ is extremal in the following sense:

**Turán's Theorem ([358] (1940)).** *For given $n$ and $p$ any graph having more edges than $T_{n,p}$ or having exactly as many edges as $T_{n,p}$ but being different from it must contain a $K_{p+1}$, as a subgraph.*

As Turán remarks, from this form one can easily verify that the maximum number of edges a graph $G_n$ can have without containing a $K_{p+1}$ is

$$\frac{1}{2}\left(1-\frac{1}{p}\right)(n^2-r^2)+\binom{r}{2}, \quad \text{if } n \equiv r \ (\mathrm{mod}\ p) \text{ and } 0 \le r < p. \tag{3.1}$$

In this sense Turán's theorem yields a complete solution of the posed question.[11]

**How did Turán arrive at this theorem?** In Ramsey's theorem we ask (in some sense): Assume we know that $G_n$ contains no $k$ independent vertices. For how large $p$ can we ensure the existence of a $K_{p+1}$ in $G_n$? Turán replaced the condition that $G_n$ had no $k$ independent vertices by a simpler condition that the graph had many edges. He asked:

[10] Very rarely we shall consider some "excluded" graphs and the subscript will just enumerate them.

[11] Letters: Mostly we shall exclude $p + 1$-chromatic graphs but there will be cases when we shift the indices and exclude $p$-chromatic graphs.

> Given a graph $G_n$ of $e$ edges, how large must $K_{p+1}$ occur in $G_n$? Or, in other words, given $n$ and $p$, how large an $e$ ensures the occurrence of a $K_{p+1}$ in $G_n$?

**The "complementary" form.** A lesser known but equally useful form of Turán's theorem can be obtained by switching to the complementary graph $\overline{G_n} = H_n$. If $H_n$ has no $p+1$ independent vertices, then $e(H_n) \geq e(\overline{T_{n,p}})$ and the equality implies that $H_n = \overline{T_{n,p}}$. (This is Theorem III in his original paper [358].)

**On the history of Turán's theorem.** As Turán remarks in the "Added in Proof" of [358], he has learnt from J. Kraus that W. Mantel has already proved his theorem in the special case $p = 3$, [273]. It is interesting to realize that this theorem could have been found by Mantel back in 1907, but he missed it. It is even more surprising that P. Erdős missed finding this theorem in 1938. As a matter of fact, Erdős and E. Klein have proved an analog result in [106]. Here Erdős investigated a number theoretical question and arrived at the following graph theoretical result:

**Theorem 3.2.** *If $G_n$ contains no $C_4$, then $e(G_n) = O(n^{3/2})$.*

At the same time, E. Klein gave a "finite geometric" construction showing that there exist graphs $G_n$ with $e(G_n) > cn^{3/2}$ edges and without containing 4-cycles. Turán, proving his theorem, immediately posed several other analog problems (such as the problem of excluded path $P_k$, excluded loops, the problem when $L$ is the graph determined by the vertices and edges of a regular polyhedron). This started a new line of investigation. Erdős (as he stated many times), felt it was a kind of blindness on his side not to notice these nice problems.

In 1949 Zykov [375] rediscovered Turán's theorem, giving a completely different proof. He used an operation which could be called *symmetrization* and which was later successfully used to prove many analog results. Since that many further proofs of Turán's theorem have been found. Some of them are similar to each other, some are completely different. Thus e.g. proofs of Andrásfai [23] G. Dirac [100] and the proofs of Katona, Nemetz and Simonovits [215] are somewhat similar, the proof of Motzkin and Straus [277] seems to be completely new, though it is actually strongly related to Zykov's proof [375]. Most of these proofs led to interesting new generalizations. In other cases the generalizations were formulated first and only then were they proved. This is the case of the proof of Erdős, and also with the proofs of Erdős and T. Sós, Bollobás and Thomason, and Bondy, see [146], [60], [63]. Before turning to the general case I state three of these results.

**Dirac's theorem.** *Assume that $n > p$ and $e(G_n) > e(T_{n,p})$. Then, for every $j \leq p$, $G_n$ contains not only a $K_{p+1}$ but a $K_{p+2}$ with an edge missing, ..., a $K_{p+j+1}$ with $j$ edges missing, assuming that $n > p + j + 1$.*

Observe that for each $j$ this immediately implies Turán's theorem, since a $K_{p+j+1} - (j$ edges) contains a $K_{p+1}$.

**Erdős theorem ([118]).** *If $G_n$ contains no $K_{p+1}$ then there exists a $p$-chromatic graph $H_n$ such that if $d_1 \le d_2 \le d_3 \le \cdots \le d_n$ and $d_1^* \le d_2^* \le d_3^* \le \cdots \le d_n^*$ are the degree sequences of $G_n$ and $H_n$ respectively, then $d_i^* \ge d_i$, $(i = 1, 2, \dots, n)$.*

This again immediately implies Turán's theorem, by

$$2e(G_n) = \sum d_i \le \sum d_i^* = 2e(H_n) \le 2e(T_{n,p}).$$

Denote by $N(x)$ the neighborhood of $x$.

**Erdős–T. Sós–Bollobás–Thomason theorem [60, 146].** *If $G_n$ is a graph with $e(G_n) > e(T_{n,p})$, then $G_n$ has a vertex $x$ of, say, degree $d$, for which for $G_{n-d} := G_n - N(x)$, we have $e(G_{n-d}) > e(T_{n-d,p-1})$*

This theorem was slightly improved by Bondy [63]. This result implies Turán's theorem if we apply induction on $p$: $G_{n-d}$ contains a $K_p$ yielding together with $x$ a $K_{p+1}$ in $G_n$. (Above I deliberately forgot the case $e(G_n) = e(T_{n,p})$, for the sake of simplicity.)

## 3.2. General problem

Since 1941 a wide theory has developed around Turán's theorem.

> Let $\mathcal{L}$ be a finite or infinite family of graphs and let $\mathbf{ex}(n, \mathcal{L})$ denote the maximum number of edges a graph $G_n$ (without loops and multiple edges) can have without containing any $L \in \mathcal{L}$ as a subgraph. Further, let $\mathbf{EX}(n, \mathcal{L})$ denote the family of graphs attaining this maximum. Given a family $\mathcal{L}$, determine $\mathbf{ex}(n, \mathcal{L})$ and $\mathbf{EX}(n, \mathcal{L})$.

When $\mathcal{L} = \{L\}$, we shall replace $\mathbf{ex}(n, \{L\})$ by $\mathbf{ex}(n, L)$. The general asymptotics on $\mathbf{ex}(n, \mathcal{L})$ was given by

**Theorem 3.3** (Erdős and Simonovits [136], Erdős [114], [115] and Simonovits [321]). *For any family $\mathcal{L}$ of excluded graphs, if*

$$p(\mathcal{L}) = \min_{L \in \mathcal{L}} \chi(L) - 1, \tag{3.2}$$

*then*

$$\mathbf{ex}(n, \mathcal{L}) = \left(1 - \frac{1}{p(\mathcal{L})}\right) \binom{n}{2} + o(n^2) \qquad \text{as} \qquad n \to \infty. \tag{3.3}$$

*Further, if $S_n$ is any extremal graph for $\mathcal{L}$, then it can be obtained from $T_{n,p}$ by changing $o(n^2)$ edges.*

(The weaker result of Erdős and Simonovits, namely (3.3), is an easy consequence of the Erdős–Stone theorem [148]. The most important conclusion of these theorems is that the maximum number of edges and the structure of the extremal graphs depend only very weakly on the actual family $\mathcal{L}$, it is asymptotically determined by the minimum chromatic number. A further interesting conclusion is that for any $\mathcal{L}$ we can find a single $L \in \mathcal{L}$ such that $\mathbf{ex}(n,\mathcal{L}) - \mathbf{ex}(n,\{L\}) = o(n^2)$. This is a *compactness* type phenomenon asserting that there is not much difference between excluding many graphs or just one appropriate member of the family.)

**Remark 3.4.** Several authors call the result according to which (3.2) implies (3.3) the Erdős–Stone theorem, in my opinion, incorrectly. This "theorem" did not exist before our first joint paper with Erdős [136]. It changed the whole approach to this field. Finally, Erdős always considered it as an Erdős–Simonovits result.

### 3.3. Degenerate extremal graph problems

If $\mathcal{L}$ contains at least one bipartite $L$, then $\mathbf{ex}(n,\mathcal{L}) = o(n^2)$, otherwise

$$\mathbf{ex}(n,\mathcal{L}) \geq e(T_{n,2}) = \left[\frac{n^2}{4}\right].$$

This is why we shall call the case $p(\mathcal{L}) = 1$ *degenerate*.

Here we arrive at the second – and again very important – graph paper of Turán. In 1954 Kővári, V. T. Sós and Turán proved the following result.

**Kővári–T. Sós–Turán theorem [252].**

$$\mathbf{ex}(n, K_2(p,q)) \leq \frac{1}{2}\sqrt[p]{q-1}\, n^{2-(1/p)} + O(n). \tag{3.4}$$

We should remark that an important footnote on the first page of [252] states:

> "As we learned, after giving the manuscript to the Redaction, from a letter of P. Erdős, he has found independently most of the results of this paper."

This theorem can be regarded as a sharpening of the Erdős–Stone theorem [148] asserting that

$$\mathbf{ex}(n, K_d(m,\dots,m)) = \left(1 - \frac{1}{d-1}\right)\binom{n}{2} + o(n^2)$$

and yielding that $\mathbf{ex}(n, K_2(m,m)) = o(n^2)$. Both these theorems were motivated by some topological problems. (3.4) is probably sharp for every $p \leq q$, apart from the value of the multiplicative constant, however this is not known in general. As a construction of Erdős, Rényi and T. Sós [135] and of W. G. Brown [76] shows, (3.4) is sharp for $p = 1, 2$, and 3. For $p = q = 2$ even the value of the multiplicative constant is sharp. A construction of $H$ by Hylten-Cavallius [204] shows that it is also sharp for $p = 2$, $q = 3$. Further, the Mörs construction [278] on the analog matrix problem, and the Füredi construction [171] show that (3.4) is sharp for $p = 2$ and all $q \geq 2$. We shall return to this question (that is, to the corresponding matrix problem) below.

**Remark 3.5.** It was a great surprise when it turned out that $\mathbf{ex}(n, K(3,3)) \approx \frac{1}{2}n^{5/3}$: by the lower bound given by Brown [76] we knew that the exponent $5/3$ in (3.4) is sharp, however, when Füredi [169] improved the upper bound, that showed that the multiplicative constant $\frac{1}{2}$ of the Brown construction is the right one.

Another interesting degenerate problem is the problem when a path $P_k$ is excluded. As I learnt from Gallai, this was one of those problems asked by Turán (in a letter written to Erdős) which started the new development in this field. The answer was given much later by the

**Erdős–Gallai theorem [126].** $\mathbf{ex}(n, P_k) \leq \frac{k-2}{2}n.$

Clearly, if $n$ is divisible by $k-1$, the disjoint union of $n/(k-1)$ $K_{k-1}$'s shows that the theorem is sharp. If $n$ is not divisible, this construction yields only $\mathbf{ex}(n, P_k) \geq \frac{k-2}{2}n - O(k^2)$. The exact value of $\mathbf{ex}(n, P_k)$ was found by Faudree and Schelp, who used it to prove some generalized Ramsey theorems [153]. Erdős and Gallai also proved [126] that if $\mathcal{L}_k$ is the family of all the cycles of at least $k$ vertices, then $\mathbf{ex}(n, \mathcal{L}_k) = \frac{1}{2}(k-1)n + O(k^2)$, and in some cases the extremal graphs are exactly those graphs whose doubly connected components (blocks) are $K_{k-1}$'s. Kopylov [250] considered the problem of connected graphs without $P_k$, and his results implied the earlier ones. Balister, Győri, Lehel and Schelp [31] also have results sharpening Kopylov's theorems. The reader can find further information in [180].

It is worth mentioning that Erdős and T. Sós conjectured [113] that for every tree $T_k$, $\mathbf{ex}(n, T_k) \leq \frac{1}{2}(k-2)n$. Ajtai, Komlós, Simonovits and Szemerédi proved (under publication) this for all sufficiently large $k$:

**Theorem 3.6** (Ajtai, Komlós, Simonovits and Szemerédi [2], [3],[4]). *There exists a $k_0$ such that for $k > k_0$ and $n \geq k$*

$$\mathbf{ex}(n, T_k) \leq \frac{1}{2}(k-2)n.$$

We close this part with the following

**Theorem 3.7** (G. Dirac, [98]). *If $P_\ell \subseteq G$, and $G$ is (at least) 2-connected, then $G$ also contains a $C_m$ with $m \geq \sqrt{2\ell}$.*

### 3.4. Even cycles

An unpublished result of Erdős states that

$$\mathbf{ex}(n, C_{2t}) = O(n^{1+(1/t)}). \tag{3.5}$$

Two different generalizations of this result were given by Bondy and Simonovits [66], and by Faudree and Simonovits [155]. I skip this area since it is fairly well described in [180]. Let me discuss the Cube theorem. Turán asked that if $L$ denotes the graph defined by the vertices and edges of a regular polyhedron, how large is $\mathbf{ex}(n, L)$? Erdős and Simonovits [138] proved that if $Q_8$ denotes the cube graph, then

**Theorem 3.8** (Cube theorem). $\mathbf{ex}(n, Q_8) \le C_Q \cdot n^{8/5}$ .

Actually if $\tilde{Q}_8$ is obtained from $Q_8$ by joining two opposite vertices, then $\mathbf{ex}(n, \tilde{Q}_8) = O(n^{8/5})$, too. One intriguing open question is whether there exists a $c > 0$ such that $\mathbf{ex}(n, Q_8) > c \cdot n^{8/5}$, or at least, $\mathbf{ex}(n, \tilde{Q}_8) > c \cdot n^{8/5}$.

**Remark 3.9.** As I mentioned above, this topic is also discussed in much more details in the recent survey of Füredi and Simonovits [180]. The same applies to large part of the next subsection.

### 3.5. Finite geometric constructions

If the extremal graph problem for $\mathcal{L}$ in consideration is non-degenerate, and $p$ is defined by (3.2) then $T_{n,p}$ yields an asymptotically extremal sequence in the sense that $T_{n,p}$ contains no $L \in \mathcal{L}$ and has asymptotically maximum number of edges. The extremal graph is often (but not always, see [329], [325]) obtained from $T_{n,p}$ by

(a) first slightly changing the sizes of the classes, that is, replacing $T_{n,p}$ by a $K_p(n_1, \dots, n_p)$, where $n_i = \frac{n}{p} + o(n)$;

(b) then adding $o(n^2)$ edges to this $K_p(n_1, \dots, n_p)$.

(c) The assertion that this is not always the case means that sometimes we need a third step too, namely, to delete $o(n^2)$ edges in a suitable way, see [329].

In this sense the non-degenerate case is relatively easy: $(T_{n,p})$ is an asymptotically extremal sequence of graphs. The extremal structures in the degenerate cases seem to be much more complicated in the sense that in most cases we do not have lower and upper bounds differing only in a constant multiplicative factor. Thus for example we do not know whether the upper bound in the cube theorem is sharp, or that the upper bound given by the Kővári–T. Sós–Turán theorem is sharp for any $p, q \ge 4$. We do not even know the existence of a positive constant $c$ such that

$$\frac{\mathbf{ex}(n, K_2(4,4))}{n^{2-(1/3)+c}} \to \infty.$$

Still, whenever we know that our upper bound for a bipartite $L$ is sharp, we always use either explicitly or in an equivalent form some finite geometric construction, or some algebraic construction very near to it. I have already mentioned some of these constructions, namely that of E. Klein in [106], of Erdős, Rényi and T. Sós [135] for graphs without $C_4$, and that of Hylten-Cavallius for graphs not containing $K_2(2,3)$. Two further very important constructions are the Brown construction [76] for graphs not containing $K_2(3,3)$ and the Benson [48] construction (see also the Singleton construction [336]) of graphs not containing $C_3$, $C_4$, $C_5$, $C_6$ and $C_7$, and of graphs not containing $C_3, \ldots, C_{11}$. These constructions of Benson show that (3.5) is sharp for $t = 3$ and $t = 5$, while W. G. Brown's construction shows that the Kővári–T. Sós–Turán theorem is sharp for $p = q = 3$ (and therefore for all $p = 3, q \geq 3$), apart from the value of the multiplicative constants.

**Remark 3.10.** Since [180] is a much more detailed survey, however mostly restricted on the Degenerate Extremal Graph Problems, and since these finite geometric problems mostly refer to degenerate cases, we suggest to the interested reader to read the corresponding parts from [180]. Here we mention only that several constructions using finite geometries or related methods were found since Turán died. Perhaps Mörs [278], Füredi [171], Ball and Peppe [32], and Wenger [371], should be mentioned here, and several slightly different constructions of Lazebnik, Ustimenko, and their school (see e.g., [256, 257, 258]) and also the breakthrough results of Kollár, Rónyai, and Tibor Szabó, [235] and Alon, Rónyai and Szabó [18] (see also [9] and [180]).

### 3.6. A digression: the extremal matrix problems

If $G_n$ is a graph, the condition that $G_n$ does not contain any $L \in \mathcal{L}$ implies that if we consider the adjacency matrix $A$ of $G_n$ and a $v(L) \times v(L)$ symmetrical submatrix of $A$,[12] then this submatrix cannot be the adjacency matrix of $L$. If for every $L \in \mathcal{L}$ we add to $\mathcal{L}$ all those graphs which are obtained from $L$ by addition of edges, and denote by $\widehat{\mathcal{L}}$ the resulting family of forbidden graphs, then the extremal graph problems for $\mathcal{L}$ and $\widehat{\mathcal{L}}$ are the same, further the exclusion of every $L \in \widehat{\mathcal{L}}$ is equivalent to the exclusion of their adjacency matrices as symmetrical submatrices of $A$.

The number of edges of $G_n$ is half of the 1's in the adjacency matrix, thus each extremal graph problem generates an equivalent problem for 0-1 matrices, where the number of 1's is to be maximized. Sometimes this approach is very useful, e.g., enables us to find continuous versions of graph theorems. However, in our case there is an even better matrix theoretical approach. Assume that $G_n$ is a bipartite graph with $n$ vertices in its first class and $m$ vertices in the second one. Then we often represent $G$ by an $n \times m$ 0-1 matrix, and e.g. the exclusion of $K_2(p,q)$ in $G$ is equivalent to the condition that taking arbitrary $p$ rows and $q$ columns of $A$, at least one of the corresponding $p \times q$ entries of the matrix will be 0, further, taking arbitrary $q$ rows and $p$ columns the same holds.

[12] where symmetric submatrix means that if we take some $j^{th}$ row of $A$ then we also take the corresponding $j^{th}$ column and vice versa.

Now, as one can read on the first page of the Kővári, T. Sós and Turán paper, K. Zarankiewicz raised the following interesting question: given a 0-1 matrix $A$, of $n$ rows and $n$ columns, and an integer $j$, how large should the number of 1's be to guarantee that $A$ contains a minor of order $j$ consisting merely of 1's? If the solution of this problem is denoted by $k_j(n)$, then one main result of the Kővári, T. Sós, and Turán paper asserts in a somewhat more complicated but sharper form that

$$k_j(n) = O(n^{2-(1/j)}). \tag{3.6}$$

Further, they show that $\lim_{n\to\infty} k_2(n)/n^{3/2} = 1$. Then they point out that their matrix results imply

$$\mathbf{ex}(n, K_2(p,p)) \le \frac{1}{2}\sqrt[p]{p-1}\cdot n^{2-\frac{1}{p}} + O(n). \tag{3.7}$$

**Note 1:** Should this be "mention the general problem"?

**Some historical remarks.** (a) The authors of [252] mention the general of excluding a $p\times q$ submatrix of 1's and that they restrict the discussion to the Zarankiewicz problem, where $a = b$.

(b) Kővári, T. Sós and Turán used a finite geometric construction to prove that $k_2(n) \ge n^{3/2} - o(n^{3/2})$. However, they did not use finite geometric language. Neither did Erdős, describing E. Klein's construction [106].

(c) Here again we should make a historical remark. According to [252]

> "S. Hartman, J. Mycielski and C. Ryll-Nardzewski have proved that
>
> $$c_1 n^{4/3} \le k_2(n) \le c_2 n^{3/2} \tag{1.2}$$
>
> with numerical $c_1$ and $c_2$".

Of course the Erdős–Klein result from 1938 was sharper, though it was formulated for graphs, and therefore formally it did not imply the Hartman–Mycielski–Ryll-Nardzewski result.

Two more historical notes should be made. Above we made a sharp distinction between degenerate and non-degenerate extremal graph problems. The germ of this distinction can be found in [252]. In Section 3 the authors write: "Let us call attention to a rather surprising fact". And this fact is that $\mathbf{ex}(n, K_2(p,p)) = O(n^{2-(1/p)})$, while to ensure a fairly similar graph, namely $K_{p+1}$, we need $\approx \frac{1}{2}\left(1-\frac{1}{p}\right)n^2$ edges, which is much more. Further, in Section 6 the authors formulate the conjecture that $k_j(n) \ge c_j n^{2-(1/j)}$, which is equivalent with the conjecture that (3.4) is sharp.

The reader more interested in this topic is referred to the survey of R. K. Guy [193] and to the paper of Mörs [278] completely solving the case of the Zarankiewicz problem when a $2\times p$ submatrix of an $n\times m$ 0-1 matrix is excluded.

# 4. Some non-degenerate extremal problems

Let $R_k$ denote the graph determined by the vertices and edges of a regular polyhedron.[13] Clearly, $R_4 = K_4$ is the tetrahedron graph, $R_6 = K_3(2,2,2)$ is the octahedron graph, $R_8 = Q_8$ is the cube graph and $R_{12}$, $D_{20} = R_{20}$ are the icosahedron graphs and the dodecahedron graphs. As we have mentioned, Turán raised the question: how many edges can $G_n$ have without containing $R_k$ as a subgraph? For $K_4$ Turán's theorem yields the answer. For the cube $Q_8$ Theorem 3.8 describes the situation. For the dodecahedron and the icosahedron Simonovits [325, 324] gave a sharp answer. (It is strange that the simplest polyhedron, namely the cube, creates the most trouble.) To formulate some results, we need a definition.

**Definition 4.1.** $H(n,p,s) := T_{n-s+1,p} \otimes K_{s-1}$: we join each vertex of $K_{s-1}$ to each vertex of $T_{n-s+1,p}$.

It turns out that in very many cases this graph is the (only?) extremal graph. Below first I will give some examples, and then, in Section 4.1 a very general theorem on the symmetric extremal graph sequences, and finally, in Section 4.2, a few further examples.

**Why is $H(n,d,s)$ a good candidate to be extremal?** The simpler, shorter answer is that $H(n,p,s)$ is a simple generalization of $T_{n,p}$. But then comes the question: why is $(T_{n,p})$ a good candidate to be the extremal graph sequence for various extremal problems? The answer is

**Theorem 4.2** (Simonovits, critical edge, [321]). *If $p(\mathcal{L})$ is defined by (3.2), and some $L_0 \in \mathcal{L}$ has an edge $e$ for which*

$$\chi(L_0 - e) = p, \tag{4.1}$$

*then there exists an $n_0$, such that for $n > n_0$ $T_{n,p}$ is extremal for $\mathcal{L}$, moreover, it is the only extremal graph (for each fixed $n > n_0$).*

*On the other hand, if (3.2) holds and for infinitely many $n$ $T_{n,p}$ is extremal for $\mathcal{L}$, then there is an $L \in \mathcal{L}$ and an edge $e$ in $L$ for which $\chi(L - e) = p$.*

**Remarks 4.3.** (a) Erdős had some results from which he could have easily deduced the above result for $p = 2$.

(b) The above theorem has the corollary that if $T_{n,p} \in \mathbf{EX}(n,\mathcal{L})$ for infinitely many $n$, then for $n > n_0$ there are no other extremal graphs.

(c) In those days I formulated the meta-theorem:

**"Meta-Theorem" 4.4.** If we can prove some results for $L = K_{p+1}$, then most probably we can extend them to any $L$ with critical edges.

[13] Here $k = 4, 6, 8, 12, 20$ is the number of vertices.

This can be seen in the Kolaitis, Prömel and Rothschild paper [234], which extends the main results of Erdős, Kleitman and Rothschild [131], and in many, many other cases of which we list only Mubayi [279], Babai, Simonovits and Spencer [28], Prömel and Steger, [291], Balogh and Butterfield [37] . . . .

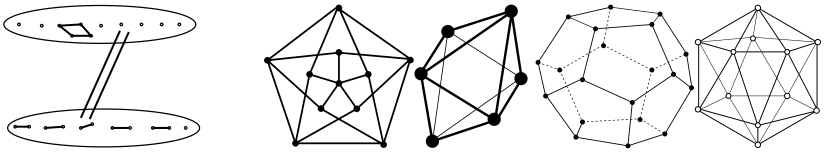
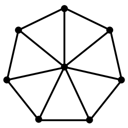


Figure 1. $O_6$-extremal, Grötzsch, octahedron, dodecahedron, icosahedron.

One interesting immediate corollary of Theorem 4.2 is the following.

**Theorem 4.5.** *$T_{n,2}$ is (the only) extremal graph for $L = C_{2k+1}$ for $n > n_0(k)$.*

The value of $\mathbf{ex}(n, C_{2k+1})$ can be read out from the works of Bondy [62], Woodall [373], and Bollobás [55] (pp. 147–156) concerning (weakly) pancyclic graphs for all $n$ and $k$. It implies that the bound for $n_0(k)$ is $4k$ in Theorem 4.5. Füredi and Gunderson [172] gave a new streamlined proof based on works of Kopylov [250] and Brandt [71] and completely described the extremal graphs. They are unique for $n \notin \{3k-1, 3k, 4k-2, 4k-1\}$ (for $2k+1 \geq 5$).



Another related result is that of Tomasz Dzido [103]. According to this, if we consider the even wheel $W_{2k} := K_1 \otimes C_{2k-1}$ – where we know by Theorem 4.2 that for sufficiently large $n$ $T_{n,3}$ is the only extremal graph, Dzido also proves that

**Theorem 4.6** (Dzido, even wheels [103]). *For all $n > 6k - 10$, $\mathbf{ex}(n, W_{2k}) = \mathbf{ex}(n, K_4)$.*

Theorem 4.2 immediately yields the extremal number for the 4-color-critical graphs, among others for the Grötzsch graph seen on Figure 1.

**Theorem 4.7** (Grötzsch extremal [321, 325, 330]). *Let $\Gamma_{11}$ be the Grötzsch graph on Figure 1. For $n > n_0$, $T_{n,3}$ is the only extremal graph.*

**Theorem 4.8** (Dodecahedron theorem [325]). *For $n > n_0$, $H(n, 2, 6)$ is the only extremal graph for the dodecahedron graph $D_{20} = R_{20}$.*

**Theorem 4.9** (Icosahedron theorem [324]). *For $n > n_0$ $H(n, 3, 3)$ is the only extremal graph for the icosahedron graph $R_{12}$.*

Let us return to the questions:
($\alpha$) "When is $H(n,p,s)$ extremal for $\mathcal{L}$?", and
($\beta$) "When is $H(n,p,s)$ the only extremal graph for $\mathcal{L}$, for $n > n_{\mathcal{L}}$?"
In [330] I asked if there are cases when $H(n,p,s)$ is an extremal graph but there are infinitely many other extremal graphs as well. Now I know that YES, there are. (We skip the details). The next question is: why is $H(n,p,s)$ an extremal graph in many cases? In particular, why is $H(n,2,6)$ extremal for $D_{20}$? Of course, for such questions there are no clear cut answers, yet I try to answer this later, see Remark 4.22.

The octahedron graph problem was solved (or, at least reduced to the sufficiently well-described problem of $\mathbf{ex}(n,C_4)$) by Erdős and Simonovits.

**Theorem 4.10** (Octahedron theorem [137]). *If $S_n$ is extremal for $R_6$, then one can find an extremal graph $A_m$ for $C_4$ and an extremal graph $B_{n-m}$ for $P_3$ of $\frac{1}{2}n + O(\sqrt{n})$ vertices each, such that $S_n = A_m \otimes B_{n-m}$.*

Clearly, $B_{n-m}$ is either a set of $(n-m)/2$ independent edges or a set of $\frac{1}{2}(n-m-1)$ independent edges and an isolated vertex.

Some very similar theorems can be found in Griggs, Simonovits and Thomas [192], see Section 15.1, and some general results on $L = K_p(a,b,c,\dots,c)$ in [137].

In the late 1960s and early 1970s some basic techniques were found, mainly by Erdős and Simonovits, to prove non-degenerate extremal graph theorems. Often sharp solutions are given in terms of the solution of some degenerate problems. This is the case in the Octahedron theorem (which is the simplest case of some more general theorems [137]). The reason of this phenomenon is discussed in detail in [326], [327] and [329]. Further, many particular extremal graph results can mechanically be deduced from a fairly general theorem of Simonovits [325]. This is the case e.g. with Moon's theorem, [275] or with the dodecahedron theorem. In some other cases, e.g, in the case of the icosahedron, this deduction is possible but not too easy.

Questions related to this will be discussed in the next subsection.

## 4.1. How to solve non-degenerate extremal problems?

Given a family $\mathcal{L}$ of forbidden subgraphs, beside the subchromatic number $p(\mathcal{L})$ defined in (3.2) the so called "Decomposition family" of $\mathcal{L}$ is the second most important factor influencing $\mathbf{ex}(n,\mathcal{L})$ and $\mathbf{EX}(n,\mathcal{L})$. So first we define it, then give a few examples and show how it influences the extremal structures.

**Definition 4.11** (Decomposition $\mathbb{M}$ of $\mathcal{L}$). Given a family $\mathcal{L}$ of forbidden subgraphs, with a $p$ defined by (3.2), we collect in $\mathbb{M}$ those graphs $M$ for which there exists an $L \in \mathcal{L}$, such that $M \otimes K_{p-1}(v(L),\dots,v(L))$ contains $L$. [14]

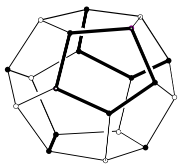
[14] To get finite families $\mathbb{M}$ when $\mathcal{L}$ is finite, we may also assume that $M$ is minimal for the considered property, or at least $M \subseteq L$.

In other words, $M \in \mathbb{M}$ if putting[15] it into a class $A_i$ of a large $T_{n,p}$, the resulting graph contains some $L \in \mathcal{L}$. The extremal graph problem of $\mathbb{M}$ is always degenerate, since $p+1$-coloring some $L_0 \in \mathcal{L}$ and taking subgraphs spanned by any two color-classes of $L_0$ we get (several) bipartite $M \in \mathbb{M}$.

In the general results of Erdős [114, 115] and myself [321] we proved that comparing an extremal graph for $\mathcal{L}$ and $T_{n,p}$, the error terms are determined up to some multiplicative constants, by $\mathbf{ex}(n, \mathbb{M}(\mathcal{L}))$.

EXAMPLES

(a) If $\mathcal{L} = \{K_{p+1}\}$, then $\mathbb{M}(\mathcal{L}) = \{K_2\}$. More generally, if there is an $L \in \mathcal{L}$ of minimum chromatic number: $\chi(L) = p(\mathcal{L}) + 1$, and there is a critical edge $e \in E(L)$, i.e., $\chi(L-e) = p$, then $\mathbb{M} = \{K_2\}$.



(b) If $\mathcal{L} = \{D_{20}\}$, the Dodecahedron graph, then $6K_2 \in \mathbb{M}(\mathcal{L})$ where $6K_2$ is the graph consisting of 6 independent edges. However, $\mathbb{M}(D_{20})$ contains also $C_5 + P_4 + K_2$, see the figure.

(c) If $\mathcal{L} = \{R_{12}\}$, the Icosahedron graph, then $P_6, 2K_3 \in \mathbb{M}(\mathcal{L})$.

(d) The decomposition class of $\mathcal{L} = \{K_3(a,b,c)$ consists of $K(a,b)$, if $a \le b \le c$.

**Remark 4.12.** The Decomposition family does not (always) determine the extremal graphs. Thus e.g., $K(2,2,2)$ and $K(2,2,3)$ have the same decomposition, however, by [137], their extremal numbers are different.

## 4.2. Some further examples

If the decomposition $\mathbb{M}(\mathcal{L})$ contains a tree (or forest), then the remainder terms in the general theorems become linear. A subcase of this, when $\mathbb{M}(\mathcal{L})$ contains a path (or a subgraph of a path) is described in my paper [325].

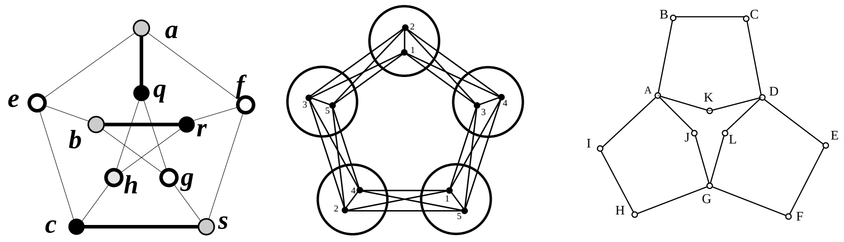
Giving a lecture in Štiřin (1997) I wanted to illustrate the general power of these results to solve extremal graph problems. So I selected one excluded graph from Łuczak's lecture, another one from Nešetřil's lecture, seen in Figure 2. I called in [330] these graphs shown in Figure 2 accordingly Łuczak and Nešetřil graphs.

**Theorem 4.13** (Łuczak-extremal). *For $n > n_0$, $H(n,4,2)$ is the only extremal graph for the Łuczak graph $L_{10}$.*

**Theorem 4.14** (Nešetřil-extremal). *For $n > n_0$, $H(n,2,2)$ is the only extremal graph for the Nešetřil-graph $N_{12}$.*

**Theorem 4.15** ($H_{n,p,k}$-theorem). *(i) Let $L_1, \dots, L_\lambda$ be given graphs with* $\min \chi(L_i) = p+1$. *Assume that omitting any $k-1$ vertices of any $L_i$ we obtain*

[15] "putting" means selecting $v(M)$ vertices in this class and joining them so that the resulting subgraph is isomorphic to $M$.




(a) Petersen graph (b) Łuczak graph (b) Nesetril graph

Figure 2. Some excluded subgraphs.

*a graph of chromatic number $\geq p+1$, but $L_1$ can be colored in $p+1$ colors so that the subgraph of $L_1$ spanned by the first two colors is the union of $k$ independent edges and (perhaps) of some isolated vertices. Then, for $n > n_0(L_1, \dots, L_\lambda)$, $H_{n,p,k}$ is the (only) extremal graph.*

*(ii) Further, there exists a constant $C > 0$ such that if $G_n$ contains no $L_i \in \mathcal{L}$ and*

$$e(G_n) > e(H_{n,p,k}) - \frac{n}{p} + C,$$

*then one can delete $k-1$ vertices of $G_n$ so that the remaining $G_{n-k+1}$ is $p$-colorable.*

This theorem is strongly connected with Theorem 4.2. [325] and [330] contain much more general theorems than the above ones, these are just illustrations of the general results. Without going too much into detail, I define a sequence of symmetric graphs and provide a fairly general theorem.

**Definition 4.16.** $\mathcal{G}(n,p,r)$ is the family of graphs $G_n$, where $V(G_n)$ can be partitioned into $p+1$ classes $U_1, \dots, U_p$ and $W$ with

$$\left| |U_i| - \frac{n}{p} \right| < r, \qquad |W| < r$$

where $G[U_i]$ is the vertex-disjoint union of the connected, pairwise isomorphic subgraphs of $G_n$, the "blocks" $B_{i,j}$. Further, each $x \in W$ is joined – for each $i = 1, \dots, p$ – to each block $B_{i,j}$ in the same way: the isomorphisms $\psi_{i,j} : B_{i,1} \to B_{i,j}$ are fixed and $x \in W$ is joined to a $y \in B_{i,1}$ iff it is joined to each $\psi_{i,j}(y)$.

**Theorem 4.17.** *If $\mathbb{M}(\mathcal{L})$ contains a path $P_\tau$ then there exists an $r$ such that for every sufficiently large $n$, $\mathcal{G}(n,p,r)$ contains an extremal graph $S_n \in \mathbf{EX}(n,\mathcal{L})$.*

This theorem helps to prove many extremal graph results. Some other results of [325] ensure the uniqueness of the extremal graphs, too. One reason why these results are easily applicable in several cases is that they apply not only to ordinary extremal graph problems but to extremal graph problems with "chromatic conditions".

Assume that instead of only excluding subgraphs from $\mathcal{L}$ we also have some additional conditions on $G_n$:

> Consider a graph property $\mathcal{P}$ and assume that $G_n \in \mathcal{P}$. Does this change the maximum in a Turán type problem?

Denote by $\mathbf{ex}(n, \mathcal{L}, \mathcal{P})$ the maximum of $e(G_n)$ under the condition that $G_n$ has no subgraphs from $\mathcal{L}$ and satisfies $\mathcal{P}$. Mostly we think of "chromatic properties" (see Definition 4.18).

Clearly, if no $\mathcal{L}$-extremal graph has property $\mathcal{P}$, then $\mathbf{ex}(n, \mathcal{L}, P) < \mathbf{ex}(n, \mathcal{L})$. If the condition is that $\chi(G_n) > t$, for some $t > p$, that will only slightly diminish the maximum: we can take a fixed graph $H_v$ of high chromatic number and high girth and then consider $H_v + T_{n-v,p}$.[16]

**Definition 4.18** (Chromatic conditions). The chromatic property $\mathcal{C}_{s,t}$ is the family of graphs from which one cannot delete $s$ vertices of $L$ to get a $t$-chromatic graph.

**Theorem 4.19.** *Assume that $\mathcal{L}$, $s, t$ are given, and $\mathbf{ex}(n, \mathcal{L}, \mathcal{C}_{s,t})$ is the maximum number of edges an $\mathcal{L}$-free $G_n \in \mathcal{C}_{s,t}$ can have. If $\mathbb{M}(\mathcal{L})$ contains a path $P_\tau$ then there exists an $r$ such that for every sufficiently large $n$, $\mathcal{G}(n, p, r)$ contains an extremal graph $S_n \in \mathbf{EX}(n, \mathcal{L}, \mathcal{C}_{s,t})$.*

Theorem 4.17 can be used to solve the extremal graph problem "algorithmically", since $W$ and $B_{i,\ell}$ have bounded sizes. The details are omitted.

Below we describe an algorithm to solve extremal graph problems: This algorithm works if we know the appropriate information on $\mathcal{L}$.

**Algorithm 4.20** (The stability method). (a) We look for a property $\mathcal{P}$ which we feel is an important feature of the conjectured extremal graphs $S_n$.

(b) Show that if $G_n$ does not contain some $L \in \mathcal{L}$ and does not have the property $\mathcal{P}$, then $e(G_n)$ is significantly smaller than the conjectured extremal number.

(c) This shows that all the extremal graphs have property $\mathcal{P}$. Using this extra information we prove the conjectured structure of the extremal graphs.

**Example 4.21.** If the decomposition class $\mathbb{M}$ contains an $M$ consisting of $r$ independent edges, then we can immediately see that if any $B_{i,\ell}$ has at least two vertices (and therefore, being connected, has an edge), then the symmetric graph sequences contain some $L$, a contradiction. Hence the blocks $B_{i,\ell}$ reduce to vertices. Therefore any $x \in W$ is either joined to each vertex of $U_i$ or to none of them. Now it is not too difficult to see that the extremal graphs must be (almost) the $H(n, p, k)$ graphs: The only difference which can occur is that the vertices of degree $n - O(1)$ do not necessarily form a complete subgraph.

[16] There is an exception when $\mathcal{L}$ contains some trees.

**Remark 4.22.** So we have seen that if the decomposition class $\mathbb{M}(\mathcal{L})$ contains an $M$ consisting of independent edges, then we have can apply the theorems from [325] and have a good chance to have $H(n,p,s)$ as the extremal graph.

Following this line, one can easily deduce Theorem 4.15 from Theorem 4.19. The next few results follow from these theorems.

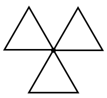
**Theorem 4.23** (Petersen-extremal graphs). *For $n > n_0$, $H_{n,2,3}$ is the (only) extremal graph for the Petersen graph $\mathbb{P}_{10}$.*

(An alternative proof of this can be derived from Theorem 4.30 of the next section.)

I close this part with two cases when Theorem 4.17 is applicable but the extremal graph is not a $H(n,p,s)$. Both results follow from Theorem 4.15.[17] Let $\mathcal{L}_{k,\ell}$ denote the graphs with $k$ vertices and $\ell$ edges.

**Theorem 4.24** (Simonovits [323]). *Let $k$ be fixed and $\ell := e(T_{k,p}) + b$, for $1 \le b \le k/(2p)$. If $n$ is sufficiently large, then*

$$\mathbf{ex}(n, \mathcal{L}_{k,\ell}) = e(T_{n,p}) + b - 1.$$



A theorem of Erdős, Füredi, Gould, and Gunderson determines $\mathbf{ex}(n, F_{2k+1})$, where $F_{2k+1} := (kK_2) \otimes K_1$: $k$ triangles with one common vertex. Clearly, here the Decomposition class contains a $kK_2$, hence Theorem 4.17 is applicable. Yet the extremal graph is not a $H(n,2,s)$, since even one vertex completely joined to a $T_{2k,2}$ creates an $F_{2k+1}$. (For even $k$, the extremal graph is obtained from a $T_{n,2}$ by putting two $K_k$'s into its first class.)

## 4.3. Andrásfai–Erdős–Sós type theorems

We have seen that $\mathbf{ex}(n, \mathcal{L}) - \mathbf{ex}(n, \mathcal{L}) = O(n)$ if $\mathcal{P}$ is that $\chi(G_n)$ is high. The situation completely changes if we try to maximize $d_{\min}(G_n)$, instead of $e(G_n)$.

**Theorem 4.25** (Andrásfai–Erdős–Sós [24]). *If $G_n$ does not contain $K_p$, and $\chi(G_n) \ge p$, then*

$$d_{\min}(G_n) \le \left(1 - \frac{1}{p - \frac{4}{3}}\right) n + O(1).$$

Comparing this with Turán's theorem, where $d_{\min}(T_{n,p-1}) \approx (1 - \frac{1}{p-1})n$, we see that because of the extra condition $\chi(G_n) \ge p$, the maximum of $d_{\min}(G_n)$ dropped by $c_p n$, for some $c_p \approx \frac{1}{3p^2} > 0$. Below we shall need

[17] They can be obtained directly, by much simpler arguments, as well.

**Definition 4.26** (Blowing up a graph). Given a graph $M_v$, its blown-up version $M[a_1, \dots, a_v]$ is a graph where each vertex $x_i \in V(M_v)$ is replaced by a set $X_i$ of $a_i$ independent vertices (and these $X_i$'s are disjoint) and we join a $u \in X_i$ and a $w \in X_j$ if the original vertices $x_i$ and $x_j$ were joined in $M_v$. If $a_1 = \cdots = a_v = a$, then we use the simpler notation $M[a]$.

To generalize Theorem 4.25, Erdős and Simonovits [139] defined

$$\psi(n, L, t) := \max\{e(G_n) \; : \; L \not\subseteq G_n \text{ and } \chi(G_n) \geq t\},$$

where $L$ is a fixed excluded graph, $t$ is fixed, and $n \to \infty$. Using this language and including some further results of [24], we can say that

**Theorem 4.27** (Andrásfai–Erdős–Sós [24]).

$$\psi(n, K_p, p) = \left(1 - \frac{1}{p - \frac{4}{3}}\right) n + O(1). \tag{4.2}$$

*For $n > n_0$, the extremal graph $S_n$ for this problem is a product: $S_n = T_{m,p-3} \otimes C_5[a_1, a_2, \dots, a_5]$, where the parameters $m$ and $a_i$ should be chosen to maximize $e(S_n)$ among these structures.*

The above description of $S_n$ almost completely determines its structure: if $T_{m,p-3} = K_{p-3}(m_1, \dots, m_{p-3})$, then

$$a_i = \frac{n}{3n - 4} + O(1) \qquad \text{and} \qquad m_i = \frac{3n}{3n - 4} + O(1).$$

To formulate a more general and sharper result, assume that

$$L \text{ has a critical edge: an } e \text{ for which } \chi(L - e) < \chi(L). \tag{4.3}$$

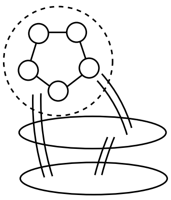


**Fig. 3:** Extremal structure.

**Note 2:** Please insert a reference to Figure 3 in the text

**Theorem 4.28** (Erdős–Simonovits [139]). *If $\chi(L) = p$ and $L$ has a critical edge, then, for $n > n_0(L)$,*

$$\psi(n, L, p) \leq \psi(n, K_p, p).$$

Actually, equality may hold only for $L = K_p$.

**Theorem 4.29** (Erdős–Simonovits [139]). *Let $\chi(L) = p$ and $L \neq K_p$ satisfy (4.3). Then, for $n > n_0(L)$,*

$$\psi(n, L, p) \leq \left(1 - \frac{1}{p - \frac{3}{2}}\right) n + O(1). \tag{4.4}$$

Of course, this theorem does not cover the case of the Petersen graph: it has no critical edge. Figure 2 shows that one can delete 3 independent edges from $\mathbb{P}_{10}$ to get a bipartite graph. Moreover, if $T(v,p,s)$ is the graph obtained from $T_{n,p}$ by putting $s$ independent edges into the first class of $T_{n,p}$, then Figure 2 shows that $\mathbb{P}_{10} \subseteq T_{12,2,3}$. So the "stability" of $\mathbb{P}_{10}$-extremal graphs is covered by

**Theorem 4.30** (Simonovits [330]). *For every $v$ (and $t \le v/2$) there exists a $K = K(v)$ such that if*

$$d_{\min}(G_n) > \frac{2}{5}n + K$$

*and $T_{v,2,t} \not\subset G_n$, then one can delete $K$ vertices of $G_n$ to get a bipartite graph.*

**Remarks 4.31.** (a) Theorem 4.30 is sharp, as shown by $C_5[\frac{1}{5}n]$. Clearly, $\delta(C_5[\frac{1}{5}n]) \ge \frac{2}{5}n - 2$ and $T_{v,2,t} \not\subset C_5[\frac{1}{5}n]$. Further, replacing $T_{v,2,t}$ by any graph $L \subseteq T_{v,2,t}$ we get the same sharpness if $K_3 \subseteq L$, since $C_5[\frac{1}{5}n]$ contains no $K_3$.

(b) Moreover, Theorem 4.30 is sharp also for $\mathbb{P}_{10}$: one can relatively easily show that $\mathbb{P}_{10}$ cannot be embedded into $C_5[\frac{1}{5}n]$.

(c) The theorem is *not sharp* if $\chi(L) = 3$ and $L \subseteq C_5[\mu]$ for some $\mu$.[18]

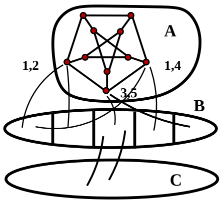



**Fig. 4: Hajnal construction.**

The real question was if $\psi(n, K_3, t) \le c_t n + o(n)$ for some constants $c_t \to 0$ as $t \to \infty$. In other words, is it true that if the chromatic number tends to $\infty$, we can push down the degree density arbitrarily?

In [24] it was conjectured that YES, however, it turned out in the Erdős and Simonovits paper [139] that NO. This follows from Construction 4.33 of A. Hajnal below.[19] For this we shall need the definition of the Kneser graph $\mathbf{KN}(2k+\ell, k)$. Its vertices are the $k$-subsets of a $(2k+\ell)$-element set $U$ and we join $X, Y \subseteq U$ if $X \cap Y = \emptyset$. It is easy to color $\mathbf{KN}(2k+\ell, k)$ with $\ell + 2$ colors. The Petersen graph $\mathbb{P}_{10} = \mathbf{KN}(5,2)$ is the simplest non-trivial Kneser graph.

**Theorem 4.32** (Kneser conjecture, Lovász theorem [262]).

$$\chi(\mathbf{KN}(2k+\ell, k)) = \ell + 2. \qquad (4.5)$$

**Construction 4.33** (A. Hajnal, in [139]). Let $k, \ell, h \to \infty$, $\ell = o(k)$, $k = o(n)$. Our graph $H_n$ has $n \approx 3h$ vertices partitioned into three groups $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$, where

$$H[\mathbb{A}] = \mathbf{KN}(2k+\ell, k), \qquad |\mathbb{B}| \approx 2h, \qquad |\mathbb{C}| \approx h.$$

(Case $k = 2$, $\ell = 1$ can be seen in Figure 4).

[18] $C_{2\mu+1} \subseteq C_5[\mu]$ for $\mu > 1$.

[19] I think that this construction was found by Hajnal, but now that I reread our paper, I cannot exclude that it was found by Erdős and Hajnal.

(a) Each vertex $v$ of $\mathbf{KN}(2k+\ell,k)$ is a subset of $\{1,\dots,2k+\ell\}$: call its elements the “names” of $v$. The vertices of $\mathbb{B}$ are partitioned into $2k+\ell$ subclasses $B_j$. $j = 1,2,\dots,2k+\ell$ of approximately equal sizes. We join the vertices of $B_j$ to those vertices of $\mathbb{A}$ whose nameset contains $j$. Finally, join each vertex from $\mathbb{C}$ to each one of $\mathbb{B}$.

Let us verify the implicitly or explicitly stated properties of $H_n$. $\chi(H_n) \ge \ell+2$, by (4.5). $H_n$ contains no $K_3$, because there are no edges between $\mathbb{C}$ and $\mathbb{A}$, so all the triangles have to be in $\mathbb{A}\cup\mathbb{B}$. However, $\mathbb{A}$ does not contain $K_3$'s, and by the “name rule”, if $x,y\in\mathbb{A}$ are connected, then they have no common neighbors in $\mathbb{B}$. Finally, if $k,\ell,n\to\infty$, $k=o(n)$, $\ell=o(k)$, then $d_{\min}(H_n)\ge n/3-o(n)$, since the vertices $x\in\mathbb{A}$ have

$$d(x) \approx \frac{k}{2k+\ell}\frac{2n}{3}, \tag{4.6}$$

because of the name rule, while for the vertices of $\mathbb{B}$ (4.6) is trivial; for an $x\in\mathbb{C}$, $d(x)=\frac{2}{3}n-o(n)$.

**Remark 4.34.** When we described this construction originally, the Kneser conjecture was still unproved: we used a much weaker assertion (an unpublished argument of Szemerédi, based on a theorem of Kleitman) that $\chi(\mathbf{KN}(2k+\ell,k))\to\infty$. Soon the Kneser conjecture was proved by Lovász [262], then an alternative proof was given by Bárány [44] and then many nice results were proved, of which we mention here just one, due to Schrijver [315], describing the color-critical subgraphs of $\mathbf{KN}(m,k)$.

There are many interesting related results in this area. We mention here only a few of them:

**Theorem 4.35** (Häggkvist [197], Guoping Jin [207]).

$$\psi(n,K_3,4)=\frac{11}{29}n+O(1).$$

The sharpness of this result follows from an “optimally” blown-up version of the Grötzsch graph, where “optimally” means that $n$ vertices are partitioned into 11 classes $U_1,\dots U_{11}$ and the classes are joined as in the Grötzsch graph, however the proportions are chosen so that the number of edges be maximized, which happens when each degree is approximately the same. Improving earlier an result of Thomassen [355], Łuczak proved

**Theorem 4.36** (Łuczak [268]). *For every $\varepsilon>0$ there exists an $L=L(\varepsilon)$ such that if $G_n$ is triangle-free and $d_{\min}(G_n)>(\frac{1}{3}+\varepsilon)n$, then $G_n$ is contained in some blown-up version of a triangle free $H_m$ for some $m\le L(\varepsilon)$.*

As Erdős and myself, using the construction of Hajnal, pointed out, such a result does not hold below $n/3$, more precisely, with an $\varepsilon < 0$. The results above leave open the case $\varepsilon = 0$ which was very recently answered by Brandt and Thomassé [74], who also completely described the structure of triangle free graphs $G_n$ with $d_{\min}(G_n) > n/3$. Their results imply

**Theorem 4.37.** *All graphs $G_n$ with $d_{\min}(G_n) > \frac{1}{3}n$ are 4-colorable.*

## 4.4. The structure of dense $L$-free graphs

Below we shall write $G \to H$ if $H$ contains a homomorphic image of $G$, or, in other words, a blown-up version $H(t)$ of $H$ contains $G$. To avoid too technical arguments, we restrict ourselves to the 3-chromatic case. For a graph $L$ we define

$$\begin{aligned} \xi(L) &= \max\left\{m : \ m \text{ is odd and } L \to C_m\right\} \\ &= \max\left\{m : \ m \text{ is odd and } L \subseteq C_m[v(L)]\right\}. \end{aligned}$$

Note that if $\chi(L) = 3$, then $\xi(L)$ cannot be larger than $girth_{\text{odd}}(L)$, the length of the shortest odd cycle contained in $L$. Finally, by $\beta(G)$ we denote the minimum number of edges that must be deleted from $G$ to make it bipartite.

In this section we study the structure of $L$-free graphs of large minimum degree for a general 3-chromatic graph $L$. Our main result can be stated as follows.

**Theorem 4.38** (Łuczak and Simonovits [271]). *Let $L$ be a 3-chromatic graph. Then for every $\alpha, \eta > 0$, there exists an $n_0$ such that for every $L$-free graph $G$ with $v(G) = n \geq n_0$ and*

$$d_{\min}(G) > \left\lceil \frac{2n}{\xi(L)+2} \right\rceil + \eta n\,, \tag{4.7}$$

*we have $\beta(G) \leq \alpha n^2$.*

*Furthermore, for every $\alpha > 0$ there exist an $\bar{\eta} > 0$ and an $\bar{n}_0$ such that each $L$-free graph $G$ with $v(G) = n \geq \bar{n}_0$ and*

$$d_{\min}(G) > \left\lceil \frac{2n}{\xi(L)+2} \right\rceil - \bar{\eta} n\,, \tag{4.8}$$

*contains a subgraph $G'$ with at least $e(G) - \alpha n^2$ edges such that $G' \to C_{\xi(L)+2}$.*

Similar but sharper results were proved by Győri, Nikiforov and Schelp for the special case when $L$ is an odd cycle.

**Theorem 4.39** (Győri, Nikiforov and Schelp [196]). *If a non-bipartite graph $G_n$ has minimum degree $d_{\min}(G_n) \geq n/(4k+2) + c_{k,m}$, where $c_{k,m}$ does not depend on $n$ and $n$ is sufficiently large, and if $C_{2s+1} \subset G_n$ for some $k \leq s \leq 4k+1$ then $C_{2s+2j+1} \subset G_n$ for every $j = 1, \dots, m$.*

They describe the structure of all graphs on $n$ vertices with $d_{\min}(G_n) \geq n/(4k+2)$ not containing odd cycles longer than $2k+1$. In particular they prove that these graphs can be made bipartite by deletion of a fixed number of edges or vertices.

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**Further sources to read:** Alon and Sudakov [22].

# 5. Problem of supersaturated graphs

## 5.1. Counting complete subgraphs

For the sake of simplicity we restrict ourselves to the case when $\mathcal{L}$ has only one member $L$. By definition, if $e(G_n) = \mathbf{ex}(n, L) + 1$, then $G_n$ contains an $L$. It is rather surprising that generally $e(G_n) > \mathbf{ex}(n, L)$ ensures much more than just one $L$. The first result in this direction is an unpublished theorem of Rademacher (1941) according to which a graph $G_n$ with $\left[\frac{n^2}{4}\right] + 1$ edges contains at least $\lfloor \frac{n}{2} \rfloor$ copies of $K_3$. This was immediately generalized by

**Theorem 5.1** (Erdős [109]). *There exists a constant $c > 0$ such that if $e(G_n) = \left[\frac{n^2}{4}\right] + k$, $1 \leq k \leq cn$, then $G_n$ contains at least $k\lfloor \frac{n}{2} \rfloor$ copies of $K_3$.*

$T_{n,2,k}$ shows that this result is sharp, apart from the value of $c$. Indeed, $e(T_{n,2,k}) = \left[\frac{n^2}{4}\right] + k$ and it has only $k\lfloor \frac{n}{2} \rfloor$ triangles. Later Erdős extended this result to $K_{p+1}$ and graphs $G_n$ with $e(T_{n,p}) + k$ edges [117]. Many similar results were proved by Erdős [117, 112], Moon and Moser [276], Bollobás [53, 54], Lovász and Simonovits, [264, 265].

For complete graphs, Lovász and Simonovits proved a conjecture of Erdős and formulated a general conjecture in [264, 265] which they could prove only for special values of $k = e(G_n) - \mathbf{ex}(n, K_{p+1})$, namely, when $k \in [1, \varepsilon n^2]$.[20] Later, in several steps it was solved by Fisher [158, 159], Razborov [295], Nikiforov [286] and finally, “completely”, by Reiher [297].

We have already mentioned the “meta-theorem” that if one can prove a result for $K_p$, then one can also prove it for graphs with critical edges. One example of this is

**Theorem 5.2** (D. Mubayi, [279]: critical edges). *Let $L$ be $p+1$-chromatic with a critical edge. Let $c(n, L)$ be the minimum number of copies of $L$ produced by the addition of an edge to $T_{n,p}$. There exist $n_0(L)$ and $\delta(L)$ such that every graph $G_n$ of order $n > n_0$ with $e(G_n) = \mathbf{ex}(n, K_{p+1}) + k$ edges contains at least $kc(n, L)$ copies of L, provided $k \leq \delta n$.*

The proof uses the graph removal lemma and the Erdős–Simonovits stability theorem.

[20] More precisely, when for some $q \geq p$, $e(T_{n,q}) < e(G_n) < e(T_{n,q}) + \varepsilon_q n^2$.

### 5.2. General sample graphs

Turning to the general case we fix an arbitrary $L$ and call a graph $G_n$ supersaturated if $e(G_n) > \mathbf{ex}(n, L)$. The problem is, at least how many copies of $L$ must occur in a $G_n$ with $\mathbf{ex}(n, L) + k$ edges. Erdős and Simonovits [140] proved that

**Theorem.** *For every $c > 0$ there exists a $c^* > 0$ such that if $e(G_n) > \mathbf{ex}(n, L) + cn^2$ and $v = v(L)$, then $G_n$ contains at fewest $c^* n^v$ copies of $L$.*

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**Further sources to read:** The reader interested in further information is suggested to read the papers of Lovász and Simonovits on structural stability [265], Erdős and Simonovits, [140], or Brown and Simonovits [85], or my survey [328].

### 5.3. Razborov's method, flag algebras

Given a graph $G_n$, we may count the occurrences of several possible subgraphs in it. Denote by $c(L, G_n)$ the number of occurrences of $L$ in $G_n$. Inequalities for such "counting functions" were the basic tools in several cases, see e.g. [252], [276] [265]. The connection between Supersaturated Graph theorems and proofs of ordinary extremal graph problems was discussed e.g. in [328]. In the last few years Razborov has developed a new method which enables the researcher to apply computers to prove inequalities between counting functions on a graph. This method turned out to be very successful and popular. To describe it and its applications would go far beyond our scope. I just mention one of the first papers of A. Razborov [293] and his very recent survey [296] on this topic, or Keevash [218].

### 5.4. The general case, bipartite graphs

As we have mentioned, the theory of supersaturated graphs started with Rademacher's theorem, and the first few papers in the field counted complete subgraphs of supersaturated graphs, [117], [100] .... (Perhaps one exception should be mentioned here: counting walks in graphs, e.g. Blakley and Roy [49], that was found independently also by [282], [260]. Counting walks is important e.g., if we wish to get information on the eigenvalues of a graph.)

The theory of supersaturated graphs is completely different for (a) the case when the excluded graph, $L$ is bipartite, and (b) when it is not. The case when it is bipartite is described in detail in [180], and from other viewpoints, in my survey, [328], so I will describe the situation here only very shortly.

For $e(G_n) \le \mathbf{ex}(n, L)$, of course, it may happen that $G_n$ contains no copies of $L$. As soon as we go above $\mathbf{ex}(n, L)$, we immediately have very many copies. Yet, to give a precise description is hopeless, even for one of the the simplest cases, for $C_4$:

we do not know enough of the finite geometries to tell how many $C_4$ must occur in $G_n$ if $e(G_n) = \mathbf{ex}(n, C_4) + 1$.

Erdős and I conjectured (see [328]) that if $\chi(L) = 2$ then for every $\varepsilon > 0$ there exists an $\eta(\varepsilon) > 0$ such that if $e(G_n) > (1+\varepsilon)\mathbf{ex}(n, L)$, then $G_n$ contains at least $\eta n^{v(L)}$ copies of $L$. We also formulated a weaker conjecture, asserting that – for any fixed $L$ – there exist a (small) $\eta > 0$ and a $C > 0$ such that if $e(G_n) > C\mathbf{ex}(n, L)$, then $G_n$ contains at least $\eta n^{v(L)}$ copies of $L$. It is also mentioned (implicitly?) in [328] that these conjectures mean that the random graph has the fewest copies of $L$.[21] Sidorenko [319], [320] considered dense graph sequences, turned the corresponding inequalities into integrals, the error terms disappeared, and he formulated more explicitly that for a given number of edges the Random Graph has the least copies of $L$.

Today this has become one of the most important conjectures in this area. The simplest case when the conjecture is unknown is when $L$ is obtained from a $K(5,5)$ by deleting edges of a $C_{10}$. We could mention here several results, however basically we refer the reader to [180] and mention only Simonovits, [328], Conlon, Fox and Sudakov [95].

**Remark 5.3.** Earlier we always first proved an extremal graph theorem and then the corresponding supersaturated graph theorem. Today this is not quite so: For $k \ge 4$ we do not really know any reasonable upper bound on $\mathbf{ex}(n, Q_{2^k})$ (for the $k$-dimensional cube), while the corresponding Erdős–Simonovits–Sidorenko conjecture is proved by Hatami [199]. This may seem to be surprising, however, the Sidorenko conjecture is about *dense* graphs.

### 5.5. Ramsey-supersaturated?

The general question would be (though not the most general one) that if we have a sample graph $L$ and $n > n_0$, and we $r$-color $K_n$, at least how many monochromatic subgraphs must occur.[22] The simplest case is to determine

$$\min\left(c(K_p, G_n) + c(K_p, \overline{G_n})\right).$$

For $K_3$ the answer is relatively easy, see Goodman [184]. Erdős conjectured [110] that the minimum is achieved by the Random Graph. This was disproved by Thomason [354]. (See also [205].)

## 6. Regularity lemma

When the Szemerédi Regularity Lemma [349] “arrived”, first it seemed somewhat too complicated. The reason for this was that in those days most graph theorists felt uneasy

[21] In those days quasi-random graphs were “non-existent”, today we know that from this point of view the random and the quasi-random graphs are indistinguishable.

[22] A related question is, how many monochromatic forbidden subgraphs appear near the Ramsey bound, see e.g., Rosta and Surányi, [307], Károlyi and Rosta [212], ....

about having this "approximation type statements".[23]

Today we know that (a) it is not that complicated and that (b) it is one of the *most important* tools in Extremal Graph Theory. This is not the place to explain it. Surveys like Komlós and Simonovits [249], [248] describe sufficiently well the usage of the Regularity Lemma in our setting, for "dense graph sequences",[24] several excellent newer surveys are also available, like Kohayakawa and Rödl [229], Rödl and Schacht [302], Gerke and Steger [183], and many others. Yet, for the sake of completeness we formulate it.

## 6.1. The original regularity lemma

**Definition 6.1** ($\varepsilon$-regular pairs). The pair of two disjoint vertex sets, $A, B \subseteq V(G)$ is $\varepsilon$-regular in $G$, if for every $X \subseteq A$ and $Y \subseteq B$ satisfying $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$, we have

$$\left|\frac{e(X,Y)}{|X||Y|} - \frac{e(A,B)}{|A||B|}\right| < \varepsilon. \tag{6.1}$$

**Theorem 6.2** (Szemerédi Regularity Lemma). *For every $\kappa > 0$ and $\varepsilon > 0$ there exists a $k_0 = k_0(\varepsilon, \kappa)$ such that for each graph $G_n$, $V(G_n)$ can be partitioned into $k \in (\kappa, k_0)$ vertex sets $(U_1, \dots, U_k)$, of $\le \lceil n/k \rceil$ vertices (each), so that for all but $\varepsilon\binom{k}{2}$ pairs $(U_i, U_j)$ ($1 \le i < j \le k$) the subgraph $G[U_i, U_j]$ induced by $U_i, U_j$ is $\varepsilon$-regular.*

The meaning of this "lemma" is that any graph can be approximated by a "generalized random graph". Its applicability comes from the fact that embedding certain structures into random-like graphs is much easier than into arbitrary graphs. This approximation helps us to prove (instead of statements on "embedding into arbitrary graphs") the simpler assertions on "embedding into generalized random graphs".

The Regularity Lemma completely changed that part of graph theory we are considering here. There are many excellent introductions to its applications. One of the first ones was that of Komlós and myself [249], or its extension [248].

**Remarks 6.3.** (a) The Regularity Lemma can be applied primarily when a graph sequence $(G_n)$ is given with positive edge density: $e(G_n) > cn^2$, for some fixed $c > 0$.

(b) For ordinary graphs it has several weaker or stronger versions, and one could assert that if one knows the statement, the proofs are not that difficult: the breakthroughs came from finding the right Regularity Lemma versions.

(c) For hypergraphs the situation completely changes: the regularity lemmas are much more complicated to formulate and often their proofs are also very painful (?). For a related survey see the PNAS paper of Rödl, Nagle Skokan, Schacht and Kohayakawa [298] and the "attached" Solymosi paper [337], and Gowers, [188], and Tao [351].

[23] Harary, e.g., did not like assertions containing statements like "for $n > n_0$" ....

[24] where $e(G_n) > cn^2$ for some constant $c > 0$ as $n \to \infty$.

(d) Regularity Lemmas are connected with “removal lemmas”, and “counting lemmas”. However, for ordinary graphs they are easy, while for hypergraphs they are much deeper.

(e) Regularity Lemmas can be applied to sparse graph sequences $(G_n)$ as well, [225, 228] assuming that the graphs $G_n$ satisfy some technical assumptions, according to which they do not have too dense subgraphs. Subgraphs of random graphs satisfy this condition, therefore Sparse Regularity Lemmas were applicable in several cases for non-random subgraphs of sparse random graphs.

(f) Regularity Lemmas were “invented” to ensure small subgraphs of given properties of a graph $G_n$. Later Komlós, G. N. Sárközy, and Szemerédi started using it to ensure spanning subgraphs. This is what the “Blow-Up Lemmas” were invented for, see Komlós, [245], Komlós, Sárközy and Szemerédi, [240]. Later they worked out algorithmic versions of the blow-up lemma too [242] (see also Rödl and Ruciński [300]) and hypergraph versions (Keevash, [217]) were established. We return to this topic in Subsection 6.6.

(g) There are many cases where Regularity Lemmas are used to give a first proof for some theorems, but later it turns out that the “regularity lemma” can be eliminated.

(h) Regularity Lemmas play a crucial role in the theory of quasi-randomness, in “property testing”, and in the theory of graph limits.

## 6.2. Some newer regularity lemmas

In [249] we tried to give an easy introduction to the applications of the Regularity Lemma. We have described the earliest applications, the Alon, Duke, Lefmann, Rödl and Yuster paper [13] about the algorithmic aspects of the Regularity Lemma, which helps to turn existence theorems using the Regularity Lemma into algorithms, the Frieze and Kannan version [164] which helps to make algorithms faster, since it uses a weaker Regularity Lemma, however, with much fewer classes. Beside [164], see also [?]. The weak Regularity Lemma in my opinion also connects the combinatorial approach to Mathematical Statistics, above all, to Principal Component Analysis.

Note 3: The reference {MatrixMultip} is missing from the Bibliography

There are also continuous versions of Regularity Lemmas. Here we refer the reader to the paper of Lovász and B. Szegedy [266] and to the book of Lovász [263]. Many further remarks and references could be added here but we have to cut it short.

## 6.3. Regularity Lemma for sparse graphs

The Kohayakawa–Rödl version of the Szemerédi Regularity Lemma uses a “technical” assumption that the considered $G_n$ does not contain subgraphs $G_m$ of much higher density than $G_n$. Very recently Alex Scott proved a new version of the Regularity Lemma, for sparse graphs [316]. Yet this has not solved all the problems. As Scott points out, it may happen in the applications of the Scott Lemma that most of the edges are in the “wrong place”. We skip the details. On the connection of random graph models and Regularity Lemmas, we mention Bollobás and Riordan [59].

## 6.4. Regularity lemma and quasi-randomness

Quasi-randomness informally means that

(Q) We consider graph sequences $(G_n)$ and look for "properties" $\mathcal{P}_i$ that are obvious for the usual random graphs (say, from the binomial distribution $\mathcal{R}_{n,p}$) and equivalent to each other.

Here there are two notions relatively near to each other: the pseudo-random and the quasi-random graphs. The investigations in this area were initiated by Andrew Thomason (see e.g. his survey [353]) and were motivated (partly?) by Ramsey problems. Chung, Graham and Wilson [94] showed that if we weaken the error terms, then there are six properties satisfying (Q). Vera Sós and I proved that there is another property $\mathcal{P}_R$ equivalent to quasi-randomness:

**Theorem 6.4** (Simonovits–Sós [333]). *A graph sequence $(G_n)$ is $p$-quasi-random in the Chung–Graham–Wilson sense iff for every $\kappa$ and $\varepsilon > 0$ there exist two integers $k(\varepsilon, \kappa)$ and $n_0(\varepsilon, \kappa)$ such that for $n > n_0$ $V(G_n)$ has a (Szemerédi) partition into $k$ classes $U_1, \dots, U_k$ (where $|U_i - n/k| \leq 1$, $\kappa < k < k(\varepsilon, \kappa)$) where all but at most $\varepsilon k^2$ pairs $1 \leq i < j \leq k$ are $\varepsilon$–regular with densities $d(U_i, U_j)$ satisfying*

$$|d(U_i, U_j) - p| < \varepsilon.$$

Several extensions exist for sparse graph sequences and hypergraph sequences, however, we do not discuss them in detail. For the sparse case see, e.g., Kohayakawa and Rödl [229]. For hypergraph extensions (which are much more technical) see, e.g., Keevash [217].

## 6.5. Regularity lemma and property testing

Property testing is among the important "Computer Science motivated" areas. It is perhaps two steps away from Turán's results, yet I write very shortly about it. Assume that we have a graph property $\mathcal{P}$. We would like to decide if a graph $G_n \in \mathcal{P}$ or not. However, we may ask only a few questions about pairs $xy$ if they are edges of $G_n$ or not? For example, we would like to decide if $G_n$ contains a given $L$ or not. Obviously, we cannot decide this for sure – using only a few questions – unless we allow some errors in the answer: if we can change a few edges in $G_n$ to get a $\tilde{G}_n \in \mathcal{P}$ then we accept a YES. Some of the earliest questions of this type came from Paul Erdős, though in somewhat different form. In the papers of Alon and Shapira it turned out that – in the reasonable cases – one can decide the question if one can decide it by applying the regularity lemma to $G_n$ and then considering the densities between the partition classes.

### 6.6. Blow-up lemma

In many cases we embed a small graph $L$ into a large one, $G_n$. There are some exceptions, when we wish to find in $G_n$ a Hamiltonian cycle, or a spanning tree of given structure, .... In these cases mostly (a) we have to assume some sparseness condition on $L$, say a bound on $d_{\max}(L)$. (b) Even if we can embed $L$ into $G_n$, if $v(L) = n$, then we have to struggle with finding places for the last few vertices.

To solve this problem Komlós, G. Sárközy and Szemerédi [240] established a special "extension" of the Regularity Lemma, called the *Blow-Up Lemma*. Komlós has a survey [245] on early successes of the Blow-Up Lemma. This survey very nicely describes the classification of embedding problems [25] and lists several conjectures solved with the help of the Blow-Up Lemma.

We call a pair $(X, Y)$ of vertex-sets in $G_n$ $(\varepsilon, \tau)$-super-regular if $|X| \approx |Y|$, it is $\varepsilon$-regular, $d(X, Y) \geq \tau$ and the minimum degree of $G(X, Y)$ is also at least $(d(X, Y) - \varepsilon)|X|$.[26]

**Theorem 6.5** (Blow-Up Lemma, short form). *For every $\delta, \Delta > 0$ there exists an $\varepsilon_0 > 0$ such that the following holds. Given a graph $H_\nu$, and a positive integer $m$, and $G_n$ and $U_n$ are obtained by replacing every vertex of $H_\nu$ by $m$ or $m-1$ vertices, and replacing the edges of $H_\nu$ with $(\varepsilon, \delta)$-super-regular pairs and by complete bipartite graphs, respectively. If $L_n \subseteq U_n$ and $d_{\max}(L_n) \leq \Delta$, then $L_n \subseteq G_n$.*

| |
|---|
| The meaning of this is that if we do not have large degrees in $L_n$ and small degrees in $G_n$ and we apply the Regularity Lemma to $G_n$, and replace each of the $\varepsilon$-regular $\tau$-dense pairs by complete bipartite graphs, then, if we can embed $L_n$ into the so obtained $U_n$, then we can embed $L_n$ into the original, much sparser $G_n$ as well. |

The basic idea was (i) first to use a randomized greedy embedding algorithm for most of the vertices of the graph to be embedded and (ii) then take care of the remaining ones by applying a König–Hall type argument [240].

The Blow-Up Lemma successfully solved several open problems, see e.g., Komlós, Sárközy, and Szemerédi, proving the Pósa–Seymour conjecture, [246], the Alon–Yuster conjecture [243], .... Here the Pósa–Seymour conjecture asks about ensuring the $k^{th}$ power of a Hamiltonian cycle (meaning that we have a Hamiltonian cycle, where all the vertices are joined whose distance on this $H$ is at most $k$).

The randomization was later eliminated by Komlós, Sárközy and Szemerédi and the embedding became an algorithmic one [242]. An alternative "derandomized" proof was also given by Rödl and Ruciński [300]. This approach turned out to be extremely successful. The blow-up lemma was also extended to hypergraphs, see Keevash [217].

When using the Regularity Lemma, or the Blow-Up Lemma, we often apply some "classical" result to the Cluster Graphs. Here we often need the famous

[25] fixed size $L$, $o(n)$ size $L$, $v(L) = cn$, $v(L) = n$

[26] We could define this basic notion also slightly differently.

**Theorem 6.6** (Hajnal–Szemerédi [244]). *If $n$ is divisible by $p$ and*

$$d_{\min}(G_n) \geq \left(1 - \frac{1}{p}\right) n,$$

*then $V(G_n)$ can be covered by vertex-disjoint copies of $K_p$.*

When Hajnal and Szemerédi proved this conjecture of Erdős, that was an enormous technical achievement, but I do not think that most people in the surrounding new that this would also be an important "tool".

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**Further sources to read:** Several related results discuss how one can get rid of applying the Blow-Up Lemma (or variants of the Regularity Lemma, see, e.g. Levitt, Sárközy and Szemerédi [247]). Kühn and Osthus have a related survey [255], and Rödl and Ruciński another one [301]. See also Alon, Rödl and Ruciński [19], B, Csaba [96].

## 7. Arithmetic structures and combinatorics

This will be the shortest section of this survey. Clearly, writing of the influence of Turán in Discrete Mathematics one cannot avoid the Erdős–Turán conjecture, nowadays Szemerédi's $r_k(n)$-theorem. This asserts that

**Theorem 7.1** (Szemerédi [348]). *For any fixed $k$, if a sequence $A$ of integers does not contain $k$-term arithmetic progressions, then it has only $o(n)$ elements in $[1, n]$.*

This theorem was very strongly connected to combinatorics. Szemerédi proved and used an earlier, weaker version of his Regularity Lemma, to prove Theorem 7.1. Vera Sós has a paper describing the origins of this conjecture [340] (based on the letters exchanged by Erdős and Turán, during the war).

**Remarks 7.2.** (a) Szemerédi's theorem is one of the roots of many results that connect Combinatorics (Graph Theory?) and Combinatorial Number Theory. Beside this it also connects Ergodic Theory and Combinatorial Number Theory, since Fürstenberg [181] gave an ergodic theoretic proof of it, then Fürstenberg, Katznelson [182] and others gave several generalizations, using ergodic theoretic methods. The reader is recommended to read e.g. the corresponding chapter of the book of Graham, Rothschild and Spencer [190]. At the same time, there are fascinating approaches to this field using deep analysis, due to Gowers, and others,[27] see recent papers of Gowers [186], or an even newer paper of Gowers [189] on these types of problems, on arithmetic progressions.

[27] This approach originates from Roth.

(b) Historically it may be interesting to read the first, fairly weak results of Erdős and Turán on this topic, in [150]. They start with proving that $r_3(n) < \frac{1}{2}n$. Then they prove a slight improvement, and formulate a conjecture of Szekeres which turned out to be false.

One of the most famous conjectures of Erdős was

**Conjecture 7.3.** If $A = (a_1, \dots, a_n, \dots)$ is a sequence of integers with

$$\sum \frac{1}{a_i} = \infty,$$

then, for any $k$, $A$ contains a $k$-term arithmetic progression.

One motivation of this conjecture is that it would imply

**Theorem 7.4** (Green–Tao [191]). *For arbitrary $k$ there exist $k$-term arithmetic progressions in the set of primes.*

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**Further sources to read:** Elek and Szegedy on the nonstandard methods in this area, [104, 105].

## 8. Multigraph and digraph extremal problems

Here I formulate only the digraph problem, which includes the multigraph case. Let $r$ be fixed and consider digraphs in which for any two vertices at most $r$ arcs of the same orientation can join them. (Hence the number of arcs joining two vertices is at most $2r$.) The problem is obvious:

> For a given family $\overrightarrow{\mathcal{L}}$ of digraphs what is the maximum number of arcs a digraph $\overrightarrow{D}_n$ can possess without containing any $\overrightarrow{L} \in \overrightarrow{\mathcal{L}}$?

The concepts of $\mathbf{ex}(n, \overrightarrow{\mathcal{L}})$ and $\mathbf{EX}(n, \overrightarrow{\mathcal{L}})$ are defined in the obvious way. Brown and Harary [84] started investigating multigraph extremal problems. Several general theorems were proved by W. G. Brown, P. Erdős and M. Simonovits [78], [79], [80], [81]. Some results concerning directed multi-hypergraphs can also be found in a paper of Brown and Simonovits [85]. For the Erdős conference in 1999 we wrote a longer survey on the topic [86]. The case $r = 1$, at least, the asymptotics of $\mathbf{ex}(n, \overrightarrow{\mathcal{L}})$ in this case, is sufficiently well described. Below we formulate only one theorem, indicating that the whole theory of digraph extremal problems is strongly connected to the theory of matrices with nonnegative integer entries.

**Brown–Erdős–Simonovits theorem [78].** *Let us consider digraphs where any two vertices are joined by at most one arc in each direction. Let $\overrightarrow{\mathcal{L}}$ be a given family* *of forbidden digraphs. Then there exists a* 0-1 *matrix $A$ (of say $t$ rows and columns) such that:*

*(a) If we partition $n$ vertices into $t$ classes $U_1, \dots, U_t$, and for $i \neq j$ join each vertex of $U_i$ to each vertex of $U_j$, by an arc oriented from $U_i$, to $U_j$, iff $a_{i,j} = 1$, and put transitive tournaments into the classes $U_i$ iff $a_{i,i} = 1$ (otherwise these are independent vertices) then the resulting digraph does not contain subdigraphs from $\overrightarrow{\mathcal{L}}$.*

*(b) One can partition $n$ vertices into $t$ classes $U_1, \dots, U_t$ in such a way that the resulting digraphs $\overrightarrow{D}_n$ form an almost extremal sequence: $e(\overrightarrow{D}_n)/\mathbf{ex}(n, \overrightarrow{\mathcal{L}}) \to 1$ (and $\overrightarrow{D}_n$ contains no forbidden subdigraphs).*

The meaning of this theorem is that for $r = 1$ we can always find an almost extremal graph sequence of fairly simple structure, where the structure itself excludes the containment of forbidden subgraphs.

**Example 8.1.** (a) Let $r = 1$. Let $L_3$ be the following digraph: $a$ is joined to $b$ and $c$ by two arcs of opposite directions and $b$ is joined to $c$ by one arc. The extremal structure is a $\overrightarrow{G_n}$ obtained from $\mathcal{T}_{n,2}$ replacing each edge by two arcs of opposite direction. Any tournament $\overrightarrow{T}_n$ is also an almost-extremal graph, and there are many other extremal graphs, see [86].

(b) There are digraph families for which the structure in Figure 5 (a) is extremal, and for some other family $\overrightarrow{\mathcal{L}}$ the structures in Figure 5 (b)–(e) forms an extremal sequence, respectively.

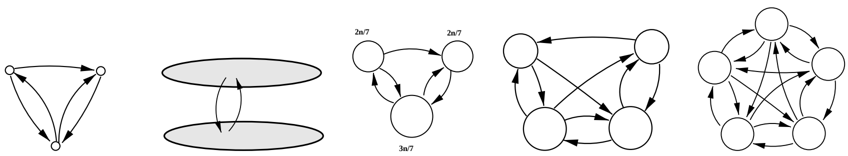


Figure 5. (a) Excluded (b), (c), (d) and (e) extremal structures for some $\overrightarrow{\mathcal{L}}$.

Brown, Erdős, and myself had conjectures asserting that most of the results for $r = 1$ can be generalized to any fixed $r$, however, most of our conjectures were "killed" by some counter-examples of Sidorenko [317] and then of Rödl and Sidorenko [304].

## 9. Hypergraph extremal problems

Just to emphasize that we are speaking of hypergraphs, hyperedges, …, we shall use script letters, and occasionally an upper index indicates the $r$-ity: $\mathcal{H}_n^{(r)}$ denotes an $r$-uniform hypergraph on $n$ vertices.

Given two positive integers $h$ and $r$, we may consider $h$-uniform $r$-multi-hypergraphs, that is, $h$-uniform hypergraphs, where the edges may have some multiplicities $\leq r$. Obviously, given a family of such multi-hypergraphs, $\mathbf{ex}(n, \mathcal{L})$ is defined as the maximum number of $h$-tuples (counted with multiplicity) such a multi-hypergraph on $n$ vertices can have without containing some members of $\mathcal{L}$ as submulti-hypergraphs. Some results on such general extremal graph problems were obtained by W. G. Brown and M. Simonovits [85], but for the sake of simplicity we shall confine our considerations to $r = 1$, that is, to ordinary $h$-uniform hypergraphs. Even for $h = 3$ most of the problems we meet prove to be hopeless or at least extremely hard. Therefore we shall mostly restrict our considerations to 3-uniform hypergraphs.

## 9.1. Degenerate hypergraph problems

Let $\mathcal{K}_h^{(h)}(m)$ be the following $h$-uniform hypergraph: it has $hm$ vertices partitioned into disjoint $m$-tuples $U_1, \dots, U_h$, and the edges are those $h$-tuples which have exactly one vertex from each $U_i$.

**Theorem 9.1** (Erdős' theorem [111]). *There exist two constants $c = c_h > 0$ and $A = A_h$ such that*

$$n^{h-cm^{-(h-1)}} < \mathbf{ex}(n, \mathcal{K}_h^{(h)}(m)) < An^{h-m^{-(h-1)}}.$$

Note 4: Please insert a reference to Figure 6 in the text

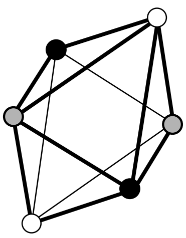

**Fig. 6:** Octahedron hypergraph.

Clearly, $\mathcal{K}_2^{(2)}(m) = K_2(m, m)$, and the above theorem is a generalization of the Kővári–T. Sós–Turán theorem. For the sake of simplicity, Theorem 9.1 was given only for the case when the sizes of classes of the excluded $h$-uniform $h$-partite graph were equal. One annoying feature of this theorem is that we do not have matching lower and upper bounds for the exponents even in the simplest hypergraph case $h = 3$ and $m = 2$.[28] At this point, it is worth defining two different chromatic numbers of hypergraphs.

**Definition 9.2** (Strong-Weak chromatic number). A hypergraph $\mathcal{H}$ is strongly $t$-colorable, if $V(\mathcal{H})$ can be $t$-colored so that each hyperedge uses each color at most once; the strong chromatic number $\chi_s(\mathcal{H})$ is the smallest such $t$.

A hypergraph $\mathcal{H}$ is weak $t$-colorable if we can $t$-color its vertices so that each of them gets at least 2 colors; $\chi(\mathcal{H})$ is the smallest such $t$.

This way we see, by Theorem 9.1, that for $r$-uniform hypergraphs $\mathbf{ex}(n, \mathcal{L}^{(r)}) = o(n^r)$ if and only if there is an $\mathcal{H}^{(r)} \in \mathcal{L}^{(r)}$ that is strongly $r$-colorable. This extends from $r = 2$ to $r > 2$, which we already knew from Section 3.3.

[28] This is the octahedron hypergraph, defined by the triangles of an octahedron.

Let $\mathcal{L}_{k,t}$ denote the family of 3-uniform hypergraphs of $k$ vertices and $t$ edges. Brown, Erdős and T. Sós [82] started investigating the function $f(n,k,t) = \mathbf{ex}(n, \mathcal{L}_{k,t})$.[29] The problem of finding good estimates of $f(n,k,t)$ is sometimes relatively simple, for some other values of $k$ and $t$ it seems to be extremely hard. One case which they could not settle was if $f(n,6,3) = o(n^2)$. Ruzsa and Szemerédi [311] proved the following surprising result.

**Ruzsa–Szemerédi theorem.** *Let $r_k(n)$ denote the maximum number of integers one can choose in $[1,n]$ so that no $k$ of them form an arithmetic progression.*[30] *Then there exists a constant $c > 0$ such that*

$$cnr_3(n) < f(n,6,3) = o(n^2).$$

It is known that

**Theorem 9.3** (Behrend [45], and Roth [309]).

$$n^{1-\frac{c}{\sqrt{\log n}}} < r_3(n) < c^* \frac{n}{\log\log n}.$$

The upper bound was recently improved by Tom Sanders [312] to

$$r_3(n) < c^{**} n \frac{(\log\log n)^5}{\log n}.$$

So, among others, the Ruzsa–Szemerédi theorem is surprising, since it shows the nonexistence of an $\alpha \in (1,2)$ such that $C_1 n^\alpha < f(n,6,3) < C_2 n^\alpha$. Another surprising feature is that it implies that $r_3(n) = o(n)$, which was considered a beautiful result of K. F. Roth [308, 309], though superseded by the famous result of Szemerédi:

**Theorem 9.4** (Szemerédi on arithmetic progressions [348]). *For every fixed $k$, as $n \to \infty$, $r_k(n) = o(n)$.*

For some related generalizations, see Alon and Shapira [20].

## 9.2. The "simplest" hypergraph extremal problem?

Next we turn to a hypergraph extremal problem which has a very simple extremal structure. G. O. H. Katona conjectured and Bollobás proved that

[29] The same question was investigated in some sense by Dirac [100] and in several papers of Erdős, and of Simonovits, see also Griggs, Simonovits and Thomas [192].

[30] We have already considered this problem in Section 7.

**Theorem 9.5** (Bollobás [52]). *If $\mathcal{H}^{(3)}_{3n}$ is a 3-uniform hypergraph with $n^3+1$ triples, then it contains three triples where one contains the symmetric difference of the other two.*

This can be viewed as a possible generalization of Turán's theorem: $K_3$ has three pairs and the symmetric difference of two of them is contained in the third one. To understand a statement like Theorem 9.5, one always has to consider the conjectured extremal structure. Now this is the complete 3-partite 3-uniform hypergraph with (almost) equal class sizes. For us it is much more interesting that such a simple nice-looking extremal problem exists for hypergraphs.

### 9.3. Turán's hypergraph conjecture

We finish this part with the famous unsolved problem of P. Turán [361]:

Given a $p$, we define the complete $h$-uniform $p$-graph $\mathcal{K}^{(h)}_p$ as the $h$-uniform hypergraph on $p$ vertices and with all the $\binom{p}{h}$ hyperedges. What is the maximum number of hyperedges in an $h$-uniform hypergraph $\mathcal{H}^{(h)}_n$ if it does not contain $\mathcal{K}^{(h)}_p$ as a subhypergraph?

For $h=3$ Turán formulated some plausible conjectures. The conjectured extremal hypergraphs differed in structure for the cases if $p$ was even or odd. For the sake of simplicity we formulate them only for $p=4$ and $p=5$.

(a) For $p=4$ let us consider the 3-uniform hypergraph obtained by partitioning $n$ vertices into 3 classes $U_1$, $U_2$ and $U_3$ as equally as possible and then taking all the triples of form $(x,y,z)$ where $x,y$, and $z$ belong to different classes; further, take all the triplets $(x,y,z)$ where $x$ and $y$ belong to the $i^{th}$ class and $z$ to the $(i+1)^{th}$, $i=1,2,3$, and $U_4 := U_1$.

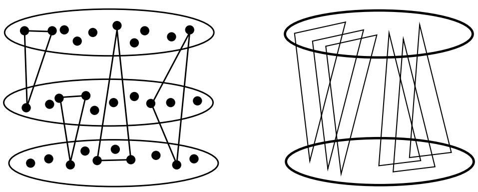

Figure 7. The conjectured extremal hypergraphs for $\mathcal{K}^{(3)}_4$ and $\mathcal{K}^{(3)}_5$.

**Note 5:** Please insert a reference to Figure 7 in the text

(b) For $p=5$ Turán had a construction with 4 classes and another one with 2 classes. The one with 2 classes is simple: we take all the triples having two vertices in one class and the third vertex in the other class. V. T. Sós observed that the construction with 2 classes can be obtained from the construction with 4 classes by moving some triples in some simple way. Later J. Surányi found a construction showing that Turán's conjecture for $p=5$ is false for $n=9$. As far as I know Kostochka has found a generalization of Surányi's construction: counter-examples for every $n=4k+1$. Still Turán's conjecture may be asymptotically sharp.

(c) Let us return to the case of $L = \mathcal{K}_4^{(3)}$. Even in this simple case Turán's conjecture seems to be very hard, even if we look only for asymptotics, that is, for $\lim \mathbf{ex}(n, \mathcal{K}_4^{(3)})/n^3$. There are no counter-examples to the conjecture, however, first Katona, Nemetz and Simonovits [215] have found some other constructions, slightly different from Turán's one, and only for $n = 3k+1$ and $n = 3k+2$. Later W. G. Brown [77] gave another construction without $\mathcal{K}_4^{(3)}$ and with the same number of triples, having 6 classes, depending on one parameter and containing Turán's construction as a special case. Finally Kostochka [251] has found a construction with $t$ parameters, $3t$ classes, for arbitrary $t$, and having the same number of triples as Turán's one, without containing $\mathcal{K}_4^{(3)}$. His construction was a generalization of Brown's one. In these new constructions $n = 3k$, which seems to be the most interesting case. Next Fon der Flaass [160] gave a characterization of all of Kostochka's (3,4)-graphs, "explaining" why the Kostochka constructions do work. Recently Andrew Frohmader [166] found some new constructions. As to numerical estimates, see e.g. Chung and Lu [92].

Some people include intersection results into extremal hypergraph theory. I prefer to distinguish between them. Yet, I will include here a very famous problem of Erdős and Rado.

**Problem 1** (Delta-systems, [130], [124].). Let us call a system of sets, $A_1, \dots, A_k$ a strong $\Delta$-system, if the intersection of any two of them is the same. Is it true that if $\mathcal{A}$ is a system of $r$-tuples on an $n$-element set, without a $k$-Delta-system, then $|\mathcal{A}| < C_r^m$, for some constant $C_r > 0$.

## 9.4. Do hypergraphs jump?

**Definition 9.6** (Jumping constants). The number $\alpha \in [0, 1)$ is a jump for $r$ if for any $\varepsilon > 0$ and integer $m \geq r$, any $r$-uniform hypergraph $\mathcal{H}_n^{(r)}$ with $n > n_o(\varepsilon, m)$ vertices and at least $(\alpha + \varepsilon)\binom{n}{r}$ edges contains a subhypergraph $\mathcal{H}_m^{(r)}$ with at least $(\alpha + c)\binom{m}{r}$ edges, where $c = c(\alpha)$ does not depend on $\varepsilon$ and $m$.

By the Erdős–Stone–Simonovits theorem, for ordinary graphs (i.e. $r = 2$) every $\alpha$ is a jump. Erdős asked [111] whether the same is true for $r \geq 3$. For the sake of simplicity we restrict ourselves to 3-uniform hypergraphs. For such a hypergraph $\mathcal{H}_n^{(3)}$ define the triple density as

$$\zeta(\mathcal{H}_n^{(3)}) = \frac{e(\mathcal{H}_n^{(3)})}{\binom{n}{3}}.$$

Theorem 9.1 of Erdős shows that if for a three-uniform hypergraph sequence $(\mathcal{H}_n^{(3)})$ the triple-density $\zeta(\mathcal{H}_n^{(3)}) > \alpha > 0$,[31] then there exist some subgraphs $\mathcal{H}_{m(n)}^{(3)} \subset \mathcal{H}_n^{(3)}$

[31] We may define the density dividing by $n^r$ and by $\binom{n}{r}$.

with $m(n) \to \infty$, for which

$$\zeta(\mathcal{H}^{(3)}_{m(n)}) \geq \frac{6}{27} \quad \text{as } n \to \infty.$$

This means that – in this sense – the density jumps from $\alpha = 0$ to $\alpha' = 2/9$. It seems to me that Erdős wanted to know if this minimum density, $2/9$ (i.e. the density of $K_3^{(3)}(m)$) is a jumping constant. However, he formulated his question in a more general form and that was disproved (by a "random graph construction"), by Frankl and Rödl:

**Theorem 9.7** (Frankl and Rödl [163]). *Suppose that* $r \geq 3$ *and* $\ell > 2r$. *Then* $1 - \frac{1}{\ell^{r-1}}$ *is not a jumping constant.*

**Theorem 9.8** (Baber–Talbot [30]). *If* $\alpha \in [02299, 02316]$, *then* $\alpha$ *is a jump for* $r = 3$.

These are the first non-trivial jumping constants. The proof uses Razborov's flag algebra method. Theorem 9.8 follows from that for an appropriately chosen family $\mathcal{F}$ of 3-uniform hypergraphs $\mathbf{ex}(n, \mathcal{F}) < 0.2299\binom{n}{3} + o(n^3)$.

**Remark 9.9.** The jumping constant problem came up slightly differently (perhaps earlier) in the digraph extremal problems, in the following form: "prove that the extremal densities form a well-ordered set under the ordinary relation '<' ". Actually, a YES answer implies that the corresponding digraph extremal problems can algorithmically be solved. For the details we refer the reader to [81, 86]. The answer was YES for $r = 1$ and NO for large values of $r$, see Sidorenko [317], and Rödl and Sidorenko [304].

## 9.5. The story of the Fano problem

Consider the 3-uniform hypergraph defined by the "lines" of the Fano geometry (see Figure 8 (a)). This hypergraph has 7 vertices and 7 triples and any two (distinct) of them intersect in exactly 1 vertex. This is the smallest finite geometry. As a hypergraph, it will be denoted by $\mathcal{F}_7$.

Figure 8. (a) Fano hypergraph (b) Fano extremal graph.

Vera Sós asked what is the extremal graph for $\mathcal{F}_7$, and conjectured [339] that it is the complete bipartite 3-uniform graph shown in Figure 8 (b). Why is this conjecture natural? [32]

(i) Because $\mathcal{F}_7$ is 3-chromatic, by Definition 9.2,
(ii) however, deleting any triple of $\mathcal{F}_7$ we get a 2-chromatic hypergraph;
(iii) $\mathcal{F}_7$ is relatively sparse.

**Theorem 9.10** (de Caen and Füredi [90]).

$$\mathrm{ex}_3(n, \mathcal{F}_7) = \frac{3}{4}\binom{n}{3} + O(n^2).$$

**Theorem 9.11** (Füredi–Simonovits [179], Keevash–Sudakov [219]). *For $n > n_0(\mathcal{F}_7)$ the complete bipartite 3-uniform hypergraph is the only extremal hypergraph for $\mathcal{F}_7$.*

Actually, in [179] a stronger, stability result was proved, easily implying Theorem 9.11. Observe that the degrees of the conjectured extremal graph are around $\frac{3}{4}\binom{n}{2}$.

**Theorem 9.12.** *There exist a $\gamma_2 > 0$ and an $n_2$ such that the following holds. If $\mathcal{H}$ is a triple system on $n > n_2$ vertices not containing the Fano configuration $\mathcal{F}_7$ and*

$$\deg(x) > \left(\frac{3}{4} - \gamma_2\right)\binom{n}{2}$$

*holds for every $x \in V(\mathcal{H})$, then $\mathcal{H}$ is bipartite, $\mathcal{H} \subseteq \mathcal{H}(X, \overline{X})$ for some $X \subseteq V(\mathcal{H})$.*

This result is a distant relative of Theorem 4.25 (of Andrásfai, Erdős and T. Sós).

**Remark 9.13** (Tools). These proofs heavily use some multigraph extremal results of Füredi and Kündgen [174]: the basic approach is that one finds first a $\mathcal{K}_4^{(3)} \subset \mathcal{H}_n^{(3)}$. If its vertices are $a, b, c, d$, then one considers the four link-graphs of these vertices, where the link-graph of an $x$ in a 3-uniform hypergraph is the pairs $uv$ forming a 3-edge with $x$.[33] These link-graphs define a (colored) multigraph on $V(\mathcal{H}_n^{(3)}) - \{a, b, c, d\}$. We apply a multigraph extremal theorem of [174] to get an $\mathcal{F}_7 \subset \mathcal{H}_n^{(3)}$. The boundedness of multiplicities is trivial.

There are a few further cases where we have sharp results on hypergraph extremal problems. I mention here e.g. Füredi, Pikhurko and Simonovits [176, 177, 178], where the last one refers to 4-hypergraphs. Other sharp results can be found on 4-hypergraph cases in Füredi, Mubayi, Pikhurko [175].

[32] We used the complete $k$-chromatic graph for Theorem 9.1 in a slightly different way. Actually, there we considered the strong chromatic number, here the weak one.
[33] Actually, we use only the three largest ones of them.

### 9.6. Co-degree problems

For hypergraphs we have several options to define degrees. Below we restrict our considerations again to the 3-uniform case and instead of degrees we consider co-degrees: the co-degree of two vertices $x$ and $y$ is the number of triples of $\mathcal{H}_n^{(3)}$ containing both of them.

**Theorem 9.14** (Mubayi [280]). *For every $\varepsilon > 0$ there exists an $n_0$ such that for $n > n_0$, if for any pair of vertices $x, y \in V(\mathcal{H}_n^{(3)})$ their co-degree is at least $(\frac{1}{2} + \varepsilon)n$ then $\mathcal{F}_7 \subset \mathcal{H}_n^{(3)}$.*

Mubayi conjectured that $\varepsilon = 0$ would be sufficient to ensure a Fano subgraph. Mubayi and Zhao remark in [281] that for co-degree problems many questions have answers different from that of the ordinary hypergraph extremal problems. One such case is the problem of jumping constants (see Section 9.4). The co-degree densities are defined in the obvious way, thus the jumping constants are defined almost the same way as for hyperedge densities.

**Theorem 9.15** (Mubayi–Zhao [281]). *For co-degree problems every $c \in (0, 1)$ is a non-jumping constant.*

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**Further sources to read:** We close this section mentioning some references on hypergraph extremal theorems: Balogh, Bohman, Bollobás, and Yi Zhao: [33], Frankl and Füredi [162], Keevash and Sudakov [220].

## 10. Ramsey–Turán theory

Vera Sós [338] and then Erdős and Vera Sós [143] initiated a whole new research field, the Ramsey–Turán theory. We shall concentrate primarily on the most recent results, since a longer survey of Vera Sós and myself [335] covers the earlier results well.

The extremal configuration in Turán's original theorem is too regular. This is why one could feel that perhaps better estimates could be achieved by replacing Turán's original theorem by some version of it, where the too regular configurations are somehow excluded. One way to exclude regular patterns is to assume that $G$ does not contain too many independent vertices – Turán's extremal graph does. This means that we exclude large complete graphs in the complementary graphs. This is, how we arrive at problems which, as a matter of fact, are combinations of Ramsey and Turán type problems. Very soon after the first results of Erdős and Vera T. Sós [143, 144, 145] were published, many others joined to this research.

As we mentioned, Turán's original theorem was motivated by Ramsey's theorem. It would have been quite natural to ask sooner or later, whether the two results could be combined. The questions thus arising would have been interesting on their own, too. However, only much later, in connection with the applications discussed in Section 13 did the Ramsey–Turán problems emerge.

We denote by $RT(n, L, m)$ the maximum number of edges a graph $G_n$ can have if $L \not\subseteq G_n$ and $\alpha(G_n) \le m$. Setting $m = n$ we arrive at Turán's extremal theorem. On the other hand, if $m$ is too small then, by Ramsey's theorem, there are no graphs in the considered class. The first problems and results in this field can be found in Sós [338], generalized by Burr, Erdős and Lovász [87].

As we shall see in Section 13, if we wish to apply Turán's theorem to find lower bounds on "geometric sums" of type (13.1), then we use many different graphs on the same vertex set, simultaneously. We know that the first one contains no complete $p_1$-graph, the second one contains no complete $p_2$-graph, and so on. We would like to find some estimate on some weighted sum of the number of their edges. The simplest case is, when these weights are equal. This is how Vera T. Sós arrived in [338] at the following question:

> Partition the edges of a $K_n$ into $k$ sets, thus obtaining the graphs $G_1, \dots, G_k$ on $V(K_n)$. We know that for $i = 1, \dots, k$, $G_i$ contains no complete $p_i$-graph. What is the maximum of $e(G_1) + \dots + e(G_{k-1})$?

Of course, if $k$ and $p_1, \dots, p_k$ are fixed and $|V|$ is too large, then such graphs simply do not exist. This is just Ramsey's theorem. However, in the cases interesting for us $p_1, \dots, p_{k-1}$ are fixed and $p_k$ tends to infinity. We assume only that $p_k = o(n)$, or more generally, that $p_k = o(f(n))$. Thus we could use the notation

$$RT(n, L_1, \dots, L_{k-1}; o(f(n)) \le cn^2$$

or $RT(\dots) = o(f(n))$ where the left-hand side means that we consider a graph sequence $(G_n)$ with $\alpha(G_n) = o(f(n))$.

Surprisingly enough, such questions sometimes prove to be extremely difficult. The simplest tractable case was when we had two graphs, $G_n$ and its complementary graph $H_n$ and wanted to maximize $e(G_n)$ under the assumption that $G_n$ contains no $K_{p+1}$ and the largest complete graph in $H_n$ is of size $o(n)$. The first real breakthrough was

**Theorem 10.1** (Erdős and Sós [143]).

$$RT(n, K_{2p+1}, o(n)) = e(T_{n,p}) + o(n^2). \tag{10.1}$$

So the estimate of $RT(n, K_m, o(n))$ was solved by Erdős and V. T. Sós [143] for the case when $m$ is odd. The case of even $p$'s was much more difficult. Thus e.g. it was a longstanding problem whether for $p = 4$ $e(G_n) = o(n^2)$ or not. Finally Szemerédi proved that

**Theorem 10.2** ([350]). $RT(n, K_4, o(n)) < \frac{1}{8}n^2 + o(n^2)$.

Later Bollobás and Erdős [58] constructed graphs, showing that Szemerédi's estimate is sharp.

**Theorem 10.3** ([58]). $RT(n, K_4, o(n)) = \frac{1}{8}n^2 + o(n^2)$.

The next breakthrough was when Erdős, Hajnal, V. T. Sós and Szemerédi, [129], determined (among others) the limit of $RT(n, K_{2p}, o(n))/n^2$, (thus generalizing Theorem 10.3). Ramsey–Turán theory is one of the areas of Extremal Graph Theory where many new results were proved lately. In [127] Erdős, Hajnal, Simonovits, Sós, and Szemerédi asked:

**Problem 2.** Does there exist a $c > 0$ for which $RT(n, K_4, \frac{n}{\log n}) < (\frac{1}{8} - c)n^2$?

One step to answer Problem 2 was

**Theorem 10.4** (Sudakov [342]). *If $\omega(n) \to \infty$, and $f(n) = n/e^{\omega(n)\sqrt{\log n}}$, then $RT(n, K_4, f(n)) = o(n^2)$.*

Then Problem 2 was answered in the negative by

**Theorem 10.5** (Fox, Loh and Zhao [161]). *For $\sqrt{\frac{\log\log^3 n}{\log n}} \cdot n < m < \frac{1}{3}n$,*

$$RT(n, K_4, m) \geq \frac{1}{8}n^2 + \left(\frac{1}{3} - o(1)\right) mn.$$

On the other hand,

**Theorem 10.6** (Fox, Loh and Zhao [161]). *There is an absolute constant $c > 0$, such that for every $n$, if $e(G_n) > \frac{1}{8}n^2$, and $K_4 \not\subseteq G_n$, then*[34]

$$\alpha(G_n) > c\frac{n}{\log n} \log\log n.$$

*In other words, if $\tilde{c} > 0$ is small enough, then*

$$RT\left(n, K_4, \tilde{c}\frac{n \log\log n}{\log n}\right) \leq \frac{1}{8}n^2.$$

In addition, they proved that

**Theorem 10.7** (Fox, Loh and Zhao [161]).

$$RT(n, K_4, \alpha) \leq \frac{1}{8}n^2 + 10^{10}\alpha n.$$

[34] Let us use binary log here, but assume that $\log n > 1$.

J. Balogh, Ping Hu, and M. Simonovits [40] proved (among many other results) the following phase transition phenomenon.

**Theorem 10.8.** $RT(n, K_5, o(\sqrt{n \log n})) = o(n^2)$.

One difficulty in this area is that there are no known Erdős–Stone–Simonovits type results (though there are some related conjectures in [129]). Thus, e.g. if $L(t)$ is a blown-up version of $L$, $RT(n, L, o(n))$ and $RT(n, L(t), o(n))$ may behave completely differently, even for $L = K_3$. We close this part with a related construction of V. Rödl. Erdős asked if

$$RT(n, K(2,2,2), o(n)) = o(n^2). \tag{10.2}$$

Rödl modified the Bollobás–Erdős construction [58]; his version still did not decide if (10.2) holds, however, it answered another question of Erdős:

**Theorem 10.9** (Rödl [299]). *There exist graphs $G_n$ with $e(G_n) > \frac{1}{8}n^2 - o(n^2)$ edges and with $\alpha(G_n) = o(n)$, however, not containing $K_4$, nor $K(3,3,3)$.*

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**Further sources to read:** Erdős and Sós [143, 144].

## 10.1. Sparse Ramsey–Turán problems

Starting out from completely different problems, Ajtai, Komlós and Szemerédi also arrived at Ramsey–Turán type problems. To solve some number theoretical and geometry problems, they arrived at the following Ramsey–Turán theorem:

**Theorem 10.10** ([5, 1, 6]). *If the average degree of $G_n$ is $d$ and $K_3 \not\subseteq G_n$ then*

$$\alpha(G_n) > c\frac{\log d}{d}. \tag{10.3}$$

This means a $\log d$ improvement over the ordinary Turán theorem. Another interpretation of this is that excluding a triangle in the complementary graph makes $G_n$ random-looking. These and similar results, e.g. [1], were used to improve earlier estimates in some problems in Geometry [239], [238], Combinatorial Number Theory [6] and Ramsey Theory [5]. We skip the details.

## 10.2. $\alpha_p$-independence problems

We close this very short part with two relatively new results of Balogh and Lenz [39]. Hypergraph Ramsey–Turán problems motivate the following problem:

> Given two sample graphs $H$ and $L$, and two integers $n$, and $m$. How many edges can a graph $G_n$ have if any induced $G_m \subseteq G_n$ contains an $H$ and $G_n$ does not contain $L$.

For $H = K_2$ we get back the ordinary $RT(n, L, m)$, while for $H = K_p$ we call the maximum $m$ in the condition $\alpha_p$-independence and denote it by $\alpha_p(G_n)$. Several related results can be found in [127, 128], and for newer results see Balogh and Lenz [39]. We mention here just one of them:

**Theorem 10.11** (Balogh–Lenz). *For* $t \geq 2$ *and* $2 \leq \ell \leq t$, *let* $u = \lceil t/2 \rceil$. *Then* $RT_t(n; K_{t+\ell}, o(n)) \geq \frac{1}{2}\left(1 - \frac{1}{\ell}\right) 2^{-u^2} n^2$.

This is a breakthrough result, answering our earlier questions, where we [128] wanted to decide, for which $\ell$ is $RT_t(n; K_{t+\ell}, o(n)) \geq c(\ell, t)n^2$ for some constant $c(\ell, t) > 0$. Balogh and Lenz found important "generalizations" of the Bollobás–Erdős construction [58].

**Further sources to read:** Balogh and Lenz [38].

## 11. Anti-Ramsey theorems

Anti-Ramsey problems[35] (in the simplest case) have the following form: Given an arbitrary coloring of a graph, we call a subgraph $H$ Totally Multicolored (TMC) or Rainbow if all its edges have distinct colors.[36]

**Problem 3.** We have a "sample graph" $H$. Let $\mathbf{AR}(n, H)$ be the maximum number of colors $K_n$ can be colored with without containing a TMC $H$.

The problem of determining $\mathbf{AR}(n, H)$ is connected not so much to Ramsey theory but to Turán type problems. For a given family $\mathcal{H}$ of finite graphs, the general result corresponding to Theorem 3.3 is

**Theorem 11.1** (Erdős–Simonovits–Sós [141]). *Let*

$$d + 1 := \min_{e \in E(H)} \{\chi(H - e) \; : \; e \in E(H)\}. \tag{11.1}$$

*Then*

$$\mathbf{AR}(n, H) = e(T_{n,d}) + o(n^2), \qquad \textit{if} \qquad n \to \infty. \tag{11.2}$$

[35] I heard this expression "anti-Ramsey" first from Richard Rado and it is also the title of his paper [292] on sequences. There the topic is analogous but not really connected to our problems.

[36] Originally we called it TMC, later Erdős and Tuza started calling such an $H$ "rainbow" colored, and some people would call it heterochromatic.

**The reason for this Transfer Principle:** Assume that $H-e$ has the minimum chromatic number in (11.1). Consider an edge-coloring of $K_n$ and choose one edge from each color. This way we get a TMC graph $G_n$. Now, $e(G_n) > \mathbf{ex}(n, H-e)+\varepsilon n^2$ would guarantee $cn^{v(H)}$ copies of $H-e$. Hence some pair $uv$ would be contained in $c'n^{v(H)-2}$ copies of $H-e$, yielding with $uv$ this many copies of $H$. We could choose two of them having no common vertices but $u$ and $v$. Since all the colors in this union are distinct, whichever way we color $uv$, we get a TMC copy of $H$. □

## 11.1. Path, cycles and further related results

The above approach gives a good asymptotic if $d > 1$ in (11.1). On the other hand, for $d = 1$ new problems have to be overcome. The anti-Ramsey problem of $P_\ell$ was solved by Simonovits and Sós [334]. The question of $C_\ell$ was much more complicated.

**Problem 4** (Erdős–Simonovits–Sós [141]). How many colors ensure a totally multicolored (Rainbow) $C_\ell$ with some $\ell > k$.

One immediately sees that this problem is an analog of the Erdős–Gallai problem on cycles. One of the important open problems in this area was the problem of Rainbow cycles.

**Conjecture 11.2** (Erdős, Simonovits and Sós [141]). Fix a cycle length $\ell$. Consider the following edge-coloring of $K_n$. First we cover the vertices by complete subgraphs of $\ell-1$ vertices each and a remainder smaller one, $K_r$ (they form an extremal graph for $P_\ell$.) Give a "private color" to these edges. Enumerate the complete subgraphs as $H_1, \dots, H_m, \dots$ and color the edges between $H_i$ and $H_j$ by the new color $c_i$ if $i < j$. One can easily see that this coloring of $K_n$ has no totally multicolored (rainbow) $C_\ell$. Show that this is the maximum number of colors one can use:

$$\mathbf{AR}(n, C_\ell) = \frac{1}{2}(\ell-2)n + \frac{n}{\ell-1} + O(1).$$

The conjecture is easy for $K_3$, was proved for $C_4$ by Noga Alon [8], then for $\ell = 5, 6$ independently by Schiermeyer [313] and by Jiang Tao and Doug West [206], and finally the problem was completely settled by Montellano-Ballesteros and Neumann-Lara [274].

## 11.2. Other types of anti-Ramsey graph problems

In the results of the previous section typically some colors are used very many times but the others only once. To eliminate this, Erdős and Tuza counted the "color-degrees":

**Theorem 11.3** (Erdős and Tuza [151]). *Consider an arbitrary coloring of $K_n$. Denote by $k(i)$ the number of colors at the $i^{th}$ vertex. If $K_n$ does not contain TMC (rainbow) triangles, then $\sum 2^{-k(i)} > 1$.*

They consider the cases when the color distribution is forced to be uniform in some sense and list several problems and provide further theorems.

**Theorem 11.4** (Frieze–Reed [165]). *If $c > 0$ is a sufficiently small constant, $n$ is large, and the edges of $K_n$ are colored so that no color appears more than $k = c\frac{n}{\log n}$ times, then $K_n$ has a TMC Hamilton cycle.*

We close this part with mentioning results stating that there are very sparse graphs having the anti-Ramsey property. In the next two theorems – instead of assuming that the number of colors used is large – we assume that they form a proper coloring.

**Theorem 11.5** (Rödl and Tuza [305]). *There exist graphs $G$ with arbitrarily high girth such that every proper edge coloring of $G$ contains a cycle all of whose edges have different colors.*

The proof of the above results was probabilistic. Haxell and Kohayakawa proved that the Ramanujan graphs constructed by Lubotzky, Phillips and Sarnak [267] also have this property.

**Theorem 11.6** ([200]). *For every positive integer $t$, every real $\delta$ such that $0 < \delta < 1/(2t+1)$, and every $n$ sufficiently large with respect to $t$ and $\delta$, there is a graph $G_n$ such that (i)* $\mathrm{girth}(G) = t + 2$, *and*

*(ii) for any proper edge-coloring of $G_n$ there is a rainbow $C_\ell \subset G_n$ for all $2t+2 \le \ell \le n^\delta$ .*

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**Further sources to read:** Babai and Sós [29], Babai [27], Alon, Lefmann and Rödl [17], Hahn and Thomassen [198], Axenovich and Kündgen [26], Burr, Erdős, Graham, Sós, Frankl [89, 88] ....

## 12. Turán-like Ramsey theorems

Considering Ramsey theorems for ordinary graphs we may observe the following "dichotomy":

**(a) Pseudo-random graphs:** In many cases the Ramsey extremal graphs look as if they were random graphs.[37]

[37] A famous conjecture of V. T. Sós suggests that (at least for complete graphs) these are quasi-random graphs.

**(b) Canonical structures:** In other cases the Ramsey extremal structures look like (almost?) *Canonical Graph Sequences*: $n$ vertices are partitioned into $q$ classes $U_1, U_2, \dots, U_q$ and the graphs $G[U_i]$ are monochromatic cliques, the bipartite graphs $G[U_i, U_j]$ are also monochromatic complete bipartite graphs, and the sizes of these classes may vary. (However, in our cases it may happen that Canonical Sequences are Ramsey-extremal, but there are also some other almost-canonical graph sequences that are Ramsey-extremal: we can change the colors of a negligible number of edges without creating monochromatic forbidden subgraphs.)

Denote by $R_k(L_1, L_2, \dots, L_k)$ the Ramsey number corresponding to $L_1, L_2, \dots, L_k$: the minimum $N$ for which, if we $k$-edge-color $K_N$, then for some $i$ the $i^{th}$ color will contain an $L_i$.

**Conjecture 12.1** (Bondy–Erdős). If $n$ is odd, then

$$R_k(C_n) := R_k(C_n, C_n, \dots, C_n) = 2^{k-1}(n-1) + 1. \tag{12.1}$$

The background of this conjecture is that for two colors, according to the Bondy–Erdős theorem [65], or the Faudree–Schelp [154] or Rosta theorems [306] the conjecture is true. The sharpness can be seen if we take two complete BLUE $K_{n-1}$'s and join them completely by RED edges.

Now, if we have a construction on $N = 2^{k-1}(n-1)$ vertices, $k$-colored, without monochromatic $C_n$, then we may take two copies of this construction and a new color $k$ and join the two copies completely by this new color. This provides the lower bound in (12.1).

For $k \geq 3$, the conjecture seemed to be harder to prove. Łuczak [269] proved that if $n$ is odd, then $R_3(C_n) = 4n + o(n)$, as $n \to \infty$. Later, Kohayakawa, Simonovits and Skokan (adding some fairly involved stability arguments to Łuczak's original one) showed that

**Theorem 12.2** (Kohayakawa, Simonovits and Skokan, [231], [232]). *There exists an* $n_0$ *for which for* $n > n_0$,

$$R_3(C_n, C_n, C_n) = 4n - 3. \tag{12.2}$$

The special case $n = 7$ of (12.2) was proved in [152]. Conjecture 12.1 is still open for $k > 3$. Bondy and Erdős [65] remarked that they could prove $R_k(C_n) \leq (k+2)!n$ for $n$ odd. The next result improves this:

**Theorem 12.3** (Luczak–Simonovits–Skokan [272]). *For every odd* $k \geq 4$,

$$R_k(C_n) \leq k3^{k-1}n + o(n), \qquad \textit{as} \qquad n \to \infty.$$

The following conjecture is unknown even for $k = 4$:

**Conjecture 12.4** (Kohayakawa, Simonovits, Skokan). If $n_1, n_2, \dots, n_k$ are fixed, then there are asymptotically Ramsey-extremal graphs $U_N$ for the corresponding Ramsey problem of finding $R_k(C_{n_1}, C_{n_2}, \dots, C_{n_k})$, where $V(U_N)$ can be partitioned into a bounded number $O_k(1)$ of classes and – apart from $O_k(N)$ edges – the color of each edge depends only on the classes it joins.

The case of even cycles has a slightly different answer, since the construction described above contains long monochromatic even cycles. Related results can be found in e.g. Łuczak [270], Figaj and Łuczak, Benevides and Skokan [47]. For further related results see the 3-color-Path results of Gyárfás, Ruszinkó, Sárközy, and Szemerédi [194], [195].

Slightly different, yet related questions are discussed in the paper of Faudree and Simonovits [156].

# 13. Applications of Turán's graph theorem

## 13.1. Distance distribution

Here we shall discuss very briefly some applications of Turán's graph theorem to the distribution of distances in metric spaces. Perhaps Erdős noticed first that Turán's theorem can be applied to distance distributions.

**Theorem 13.1** (Erdős [107]). *If we have a set $X$ of $n$ points in the plane, $X = \{P_1, \dots, P_n\}$ and the diameter of $X$ is at most 1, then at least*

$$\binom{n}{2} - \mathbf{ex}(n, K_4) \approx \frac{1}{3}\binom{n}{2}$$

*of the distances $P_iP_j$ is at most $1/\sqrt{2}$.*

To prove this, observe that for any 4 points – by an easy argument – at least one of the 6 distances is $\le 1/\sqrt{2}$. So the graph $G_n$ defined by the distances $> 1/\sqrt{2}$ contains no $K_4$. Hence $e(G_n) \le \mathbf{ex}(n, K_4)$. □

Obviously, this result is sharp: if we fix an equilateral triangle of diameter 1 and put $n/3$ points into each of its vertices, then roughly 1/3 of the $\binom{n}{2}$ distances will be 0 and all the others are equal to 1.

Fourteen years later Turán pointed out that a slight generalization of this simple observation may yield far-reaching and interesting results (estimates) in geometry, analysis and some other fields, too. Turán's basic observation was as follows: Instead of $d = 1/\sqrt{2}$, we can apply the same idea simultaneously to several distances. We define the corresponding *Packing Constants*:

**Definition 13.2.** Given a metric space $\mathbb{M}$ with the metrics $\rho(x, y)$ and an integer $k$, let

$$d_k := \max_{\mathbf{diam}\{P_1,\dots,P_k\}\le 1} \min_{i\ne j} \rho(P_i, P_j).$$

(If $|\mathbb{M}| = \infty$, it may happen that we have to replace the min by inf.)

Now, the above argument shows that if the $\rho$-diameter of an $n$-element set is at most 1, then it contains at least $\binom{n}{2} - \mathbf{ex}(n, K_k)$ distances $\rho(P_i, P_j) \le d_k$. Using Abel summation, we may obtain good estimates on sums of the form

$$\sum f(\rho(P_i, P_j)). \tag{13.1}$$

This way, through distance distribution results, Turán [363], V. T. Sós [338], and later Erdős, Meir, V. T. Sós and Turán [132, 133, 134] could give estimates on certain integrals, potentials, certain parameters from functional analysis, and other geometric sums. In [132] the authors write:

> In what follows, we are going to discuss systematic applications of graph theory – among others – to geometry, potential theory and to the theory of function spaces . . . These applications show that suitably devised graph theorems act as flexible logical tools (essentially as generalizations of the pigeon hole principle) . . . We believe that the applications given in this sequence of papers do not exhaust all possibilities of applications of graph theory to other branches of mathematics. Scattered applications of graph theory, (mostly via Ramsey theorem) existed already in the papers of Erdős and Szekeres [149] and Erdős [106], [116].

**Remarks 13.3.** These lines are 40 years old, however, the development of Discrete Mathematics really shows that Discrete Mathematics became a very applicable theory in very many areas of mathematics. Strangely enough, or perhaps because Turán died too soon, not too many results were published on the application of extremal graph results to distance distribution after Turán's death.

However, two further areas were strongly connected to this approach. The first one was the application of Turán type graph results in estimating distributions in Probability Theory. This area was pioneered by G. O. H. Katona. He was able to prove some inequalities concerning the distribution of certain random variables [213]–[216]. Next several important results of the field were proved by A. Sidorenko. This volume has a separate article on this topic, by Katona [214]. I would risk the opinion that among the several steps that led to the theory of graph limits one important step was this: introducing integrals in areas related to extremal graph theory.

The other one is *Ramsey–Turán theory* discussed in Section 10.

### 13.2. Application to geometry

Given $n$ points in the space (or in any bounded metric space), for every $c > 0$ we can define a graph $G^{(c)}$ by joining the points $P$ and $Q$ iff $PQ > c$. By establishing some appropriate geometric facts, we may ensure that $G^{(c)}$ contains no complete $p = p(c)$-graph. Hence we know (by Turán's theorem) that the number of pairs $(P, Q)$ with $PQ > c$ is at most $\mathbf{ex}(n, K_{p(c)})$.

Assume that we apply this method with several constants $c_1 > c_2 > \cdots > c_k > 0$. If $f(x)$ is a monotone decreasing function in (13.1), then we may obtain lower bounds on this expression by replacing all the distances between $c_i$ and $c_{i+i}$ by $c_i$. The 'only' problems to be solved are:

How to choose the constants $c_1 > c_2, > \cdots > c_k > \cdots > 0$?

How to choose the integers $p_k$ for the constants $c_k$, to get good results?

This was the point where the packing constants (depending largely on the geometric situation) came in. Their investigation goes back at least to a dispute between Newton and Gregory, see Turán [364]. It was also somewhat surprising that not all packing constants count in our application. It is enough to regard those ones, where $c_k > c_{k+1}$. It is not worth giving a detailed description of the results obtained this way, since the Introduction of [134] does it. We make only one critical remark on a side issue:

In [364] Turán remarks that perhaps his method, implemented on a good computer, would help to decide problems such as the one in the Newton–Gregory dispute. Namely, it could decide whether $c_t = c_{t+1}$ or not.

This is not quite so. First of all, such an algorithm can never give a positive answer. Further, even if the answer is in the negative, and that could be proved by the method suggested by Turán, then probably that could be decided also without using Turán's method.

## 13.3. Other applications

An old unsolved problem is that if we have $n$ points in the $k$-dimensional Euclidean space, how many unit distances can occur. For the plane Erdős observed that the graph given by the unit distances cannot contain a $K_2(2, 3)$. Hence – by the Kővári–T. Sós–Turán theorem – the number of unit distances is $O(n^{3/2})$. A similar argument works in $\mathbb{R}^3$: the 3-space, but for higher dimension the situation changes. Unfortunately, the application of Turán type theorems is not enough to get the conjectured bounds: to prove that the number of unit distances is at most $O(n^{1+\varepsilon})$.

(b) Some other type of applications of hypergraph extremal problems are found in the works of Simonovits [322] and Lovász [261] yielding sharp bounds on some questions related to color-critical graphs. For more details see either the original papers or the Füredi and Simonovits survey [180].

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**Further sources to read:** Erdős [116], Erdős and Simonovits [142], ....

## 14. Extremal subgraphs of random graphs

What happens if, instead of considering all the $\mathcal{L}$-free graphs $G_n$, we consider only $\mathcal{L}$-free subgraphs $G_n$ of some host-graphs $R_n$ and maximize their number of edges. One of the most investigated subcases of this problem is when $R_n$ is a random graph with some given distribution. The maximum is $\mathbf{ex}(R_n, \mathcal{L})$, however this is a random number, depending on the random graph $R_n$. So we can state only that certain events will hold with high probability.

Rödl and Schacht wrote very recently an excellent survey [303] on this topic, so we shall give only a very short introduction to this area.

Assume that $R_n$ is a random graph of binomial distribution, with given edge probability: $R_n \in \mathcal{G}_{n,p}$. The phenomena to be discussed are

> If $L$ is a sample graph, $k = \chi(L) - 1$, and we take a random graph $R_n \in \mathcal{G}_{n,p}$ with edge probability $p > 0$,
>
> (a) is the subgraph $F_n \subseteq R_n \in \mathcal{G}_{n,p}$ not containing $L$ and having the maximum number of edges $k$-chromatic with probability $1 - o(1)$?
>
> (b) if (a) does not hold, is it true that at least we can delete $o(e(R_n))$ edges from $R_n$ to get a $k$-chromatic graph, almost surely?

An early result in this area was

**Theorem 14.1** (Babai–Simonovits–Spencer [28]). *There exists a $p_0 < \frac{1}{2}$ for which in a random $R_n \in \mathcal{G}_{n,p}$, almost surely, the maximum size $K_3$-free subgraph, $F_n \subseteq R_n$ is bipartite.*

Several generalizations of this were proved in [28], however, in those days no "Sparse Regularity Lemma" was known, and the proofs of Babai, Simonovits and Spencer used the (ordinary) Szemerédi Regularity Lemma [349] and the stability method. Hence [28] could cover only the case when the edge probability was $p > p_0 > 0$. As soon as the Kohayakawa–Rödl version of the Regularity Lemma was proved and became known, the possibility to generalize the results of [28] became possible. First Brightwell, Panagiotou and Steger [75] proved that Theorem 14.1 holds under the much weaker condition that $p > n^{-1/250}$ and very recently B. De Marco and Jeff Kahn [97] proved that

**Theorem 14.2.** *There exists a $C > 0$ such that if the edge probability is $p > C\sqrt{\log n/n}$, then every maximum triangle-free subgraph of $G_{n,p}$ is bipartite, with probability tending to 1, as $n \to \infty$.*

This is best possible.

Let

$$d_2(H) = \max\left\{\frac{e(H')}{v(H')}\ :\ H' \subseteq H,\ \text{and}\ v(H') \geq 3\right\}.$$

**Conjecture 14.3** (Kohayakawa–Rödl–Schacht [230]). Let $v(H) \geq 3$ and $e(H) > 0$. Let $G = G_{n,p}$ be a random graph with edge probability $p = p_n$ where $p_n n^{1/d_2(H)} \to \infty$. Then

(i) almost surely (as $n \to \infty$),

$$\mathbf{ex}(G, H) = \left(1 - \frac{1}{\chi(H) - 1}\right) e(G) + o(e(G)).$$

(ii) Further, for $\chi(H) \geq 3$, a stability phenomenon also holds: almost surely, deleting $o(e(G_{n,p}))$ edges, one can make $G_{n,p}$ $(\chi(H) - 1)$-colorable.

The above conjecture is proved for several cases. Thus, e.g., for cycles it was proved by Haxell, Kohayakawa and Łuczak [201] and [202], while the paper of Kohayakawa, Łuczak and Rödl [227] contains a proof of (i) for $H = K_4$.

## 15. Typical structure of $L$-free graphs

Here we consider the following problem:

> What is the typical structure of $L$-free graphs? Or, more generally, we have a Universe (graphs, hypergraphs, multigraphs, permutations, ordered sets, $\ldots$) and a property $\mathcal{P}$, can we say something informative about the typical structures in $\mathcal{P}$?

This question has basically two subcases: the exclusion of some $L$ as a not necessarily induced subgraph and the exclusion of some induced subgraphs.

### 15.1. Starting in the middle

In this part, excluding $L \subset G_n$ we do not assume that (only) the induced subgraphs are excluded. The difference can be seen already for $C_4$: If we define a complete graph on $A$ and an independent set on $B$ and join them arbitrarily, the resulting $G_n$ contains many $C_4$'s but no induced $C_4$. So first we consider the case of not necessarily induced subgraphs.

First we assume that the forbidden graphs are non-bipartite, and return to the degenerate case in the next, very short subsection. Denote by $\mathcal{P}(n, \mathcal{L})$ the family of $n$-vertex graphs without subgraphs from $\mathcal{L}$. Since all the subgraphs of any $S_n \in \mathbf{EX}(n, \mathcal{L})$ belong to $\mathcal{P}(n, \mathcal{L})$, therefore

$$|\mathcal{P}(n, \mathcal{L})| \geq 2^{\mathbf{ex}(n,\mathcal{L})}. \tag{15.1}$$

This motivated

**Conjecture 15.1** (Erdős).

$$|\mathcal{P}(n, \mathcal{L})| = 2^{\mathbf{ex}(n,\mathcal{L})+o(n^2)}. \tag{15.2}$$

Of course, the meaning of this is that $\mathcal{P}(n,\mathcal{L})$ cannot be much larger than the right-hand side of (15.1). This was confirmed first for $K_{p+1}$. The result for $K_3$ was much sharper than for the general case.

**Theorem 15.2** (Erdős–Kleitman–Rothschild [131]). *(i) Almost all triangle-free graphs $G_n$ are bipartite.*

*(ii) In general,*

$$|\mathcal{P}(n,K_{p+1})| \le 2^{\left(1-\frac{1}{p}\right)n+o(n^2)}.$$

Later Erdős, Frankl, and Rödl proved the original Erdős conjecture.

**Theorem 15.3** (Erdős, Frankl, and Rödl [125]).

$$|\mathcal{P}(n,\mathcal{L})| \le 2^{\mathbf{ex}(n,\mathcal{L})+o(n^2)}.$$

As we have already pointed out, the finer structure in the extremal graph problems depends on the "Decomposition family" $\mathbb{M}$ of $\mathcal{L}$. So Balogh, Bollobás and myself improved Theorem 15.3 in several steps. First, in [34] we improved the error term $o(n^2)$ of Theorem 15.3 to $O(n^{2-c})$.

**Theorem 15.4.** *For every $\mathcal{L}$, if $\mathbb{M}$ is the decomposition family of $\mathcal{L}$ and $\mathbb{M}$ is finite, then*

$$|\mathcal{P}(n,\mathcal{L})| \le n^{\mathbf{ex}(n,\mathbb{M})+c_{\mathcal{L}}\cdot n} \cdot 2^{\frac{1}{2}\left(1-\frac{1}{p}\right)n^2}, \tag{15.3}$$

*for some sufficiently large constant $c_{\mathcal{L}} > 0$.*

This was an improvement, indeed: if $L \in \mathcal{L}$ and $v = v(L)$ is of minimum chromatic number, then we can choose a bipartite $M \subseteq L$ from $\mathbb{M}$. Hence $\mathbf{ex}(n,\mathbb{M}) < c \cdot n^{2-\frac{2}{v}}$, yielding a better error term in the exponent in (15.3).

Our next result yields also structural information.

**Theorem 15.5** (Balogh, Bollobás, Simonovits [35]). *Let $\mathcal{L}$ be an arbitrary finite family of graphs. Then there exists a constant $h_{\mathcal{L}}$ such that for almost all $\mathcal{L}$-free graphs $G_n$ we can delete $h_{\mathcal{L}}$ vertices of $G_n$ and partition the remaining vertices into $p$ classes, $U_1,\dots,U_p$, so that each $G[U_i]$ is $\mathbb{M}$-free.*

For some particular cases we can provide even more precise structural information. A good test case is when the octahedron graph is excluded. In our main result below we describe the structure of almost all octahedron-free graphs. We say that a graph $G$ has property $\mathcal{Q} = \mathcal{Q}(C_4,P_3)$ if its vertices can be partitioned into two sets, $U_1$ and $U_2$, so that $C_4 \not\subseteq G[U_1]$ and $P_3 \not\subseteq G[U_2]$. If $G \in \mathcal{Q}$ then $G$ does not contain $O_6$. It was proved by Erdős and Simonovits [137] that for $n$ sufficiently large every $O_6$-extremal $G_n$ has property $\mathcal{Q}$. The typical structure of $O_6$-free graphs is described by

**Theorem 15.6** (Balogh, Bollobás, Simonovits [36]). *The vertices of almost every $O_6$-free graph can be partitioned into two classes, $U_1$ and $U_2$, so that $U_1$ spans a $C_4$-free graph and $U_2$ spans a $P_3$-free graph.*

A similar, slightly simpler, result is the following. Denote $\mathcal{P}(n;a,b)$ the family of graphs $G_n$ for which no $a$ vertices of $G_n$ span at least $b$ edges. In some sense, G. Dirac started investigating such problems [100]. Several results of Erdős and Simonovits are related to this topic, and they became very important for hypergraphs, see e.g., Brown, Erdős and T. Sós [82], or Ruzsa and Szemerédi [311]. Much later, Griggs, Simonovits and Thomas [192] proved that, for $n$ sufficiently large, the vertex set of any $\mathcal{P}(n,6,12)$-extremal graph $G_n$ can be partitioned into $U_1$ and $U_2$ so that the induced subgraphs, $G[U_1]$ is $\{C_3,C_4\}$-free and $G[U_2]$ is an independent set. Note that if $G_1$ is $\{C_3,C_4\}$-free and $e(G_2)=0$ then $G_1 \otimes G_2$ is $(6,12)$-free.

**Theorem 15.7** (Balogh, Bollobás, Simonovits [36]). *The vertex set of almost every graph in $\mathcal{P}(n;6,12)$ can be partitioned into two classes, $U_1$ and $U_2$, so that $U_1$ spans a $\{C_3,C_4\}$-free graph and $U_2$ is an independent set.*

To avoid technicalities, we formulated only this special case. Another line is the problem of critical edges.

**Theorem 15.8** (Prömel and Steger [291]). *For every $L$ having a critical edge, almost all $L$-free graphs have chromatic number $\chi(L)-1$.*

This is sharp, since no graph with chromatic number $\chi(L)-1$ contains $L$ as a subgraph, (see also Hundack, Prömel, and Steger [203].) To demonstrate the power of our methods we proved a generalization of their result. Denote by $sH$ the vertex-disjoint union of $s$ copies of $H$. Let the excluded graph be $L=sH$, where $H$ has a critical edge, and $\chi(H)=p+1\geq 3$. Simonovits [321] proved that for $n$ sufficiently large, the unique $L$-extremal graph is $H(n,p,s)$, see Theorem 4.15. Observe that if one can delete $s-1$ vertices of a graph $G_n$ to obtain a $p$-partite graph, then $G_n$ is $L$-free.

**Theorem 15.9** (Balogh, Bollobás, Simonovits [36]). *Let $p$ and $s$ be positive integers and $H$ be a $p+1$-chromatic graph with a critical edge. Then almost every $sH$-free graph $G_n$ has a set $S$ of $s-1$ vertices for which $\chi(G_n-S)=p$.*

## 15.2. Degenerate cases

One could think that if $L$ is bipartite but not a tree, then (15.2) remains valid:

$$|\mathcal{P}(n,L)| < 2^{\mathbf{ex}(n,L)(1+o(1))}. \tag{15.4}$$

Yet, this is not known even in the simplest case, for $L=C_4$. The first important result in this area was

**Theorem 15.10** (Kleitman–Winston [224]).

$$2^{(\frac{1}{2}-o(1))n\sqrt{n}} \le |\mathcal{P}(n,C_4)| < 2^{cn\sqrt{n}} \qquad \textit{with} \qquad c = 1.082.$$

The result itself is highly non-trivial. The next result in this direction was

**Theorem 15.11** (Kleitman–Wilson [372]).

$$|\mathcal{P}(n,C_6)| < 2^{cn\sqrt[3]{n}}.$$

The corresponding results for $C_{2k}$ for $k \ge 4$ are still open. Balogh and Samotij also have analogous results for $K_{t,t}$, and – more generally, – for $K_{s,t}$.

**Theorem 15.12** (Balogh and Samotij [42, 43]). *For $L = K_{s,t}$, there exist a constant $c = c_L$ for which*

$$|\mathcal{P}(n,L)| \le 2^{c\mathbf{ex}(n,L)}.$$

Their method also implies that

**Theorem 15.13** (Balogh and Samotij [42, 43]). *For $L = K_{2,t}$, there exists a constant $\tilde{c} = \tilde{c}_L$ for which for almost all $L$-free $G_n$, we have*

$$\frac{1}{12}\mathbf{ex}(n,L) \le e(G_n) \le (1-c)\mathbf{ex}(n,L).$$

Several of the related papers contain a "mini-survey" of the situation, so we stop here.

## 15.3. Typical hypergraph structures

As we have mentioned, for many years there were only a few hypergraph extremal results. In the last few years this changed dramatically. As we have seen in Section 9, several interesting extremal hypergraph theorems were proved lately. Also some corresponding "typical structure results" were obtained, e.g. [41]. Here we give only a few examples. The first one is connected to the Fano results [179] and [219].

**Theorem 15.14** (Person and Schacht [287]). *Almost all $\mathcal{F}_7$-free 3-uniform hypergraphs are 2-chromatic.*

Call the following three edges a triangle: $(u,v,w)$, $(u,v,x)$, $(x,y,w)$. The following result extends the sharper version of Theorem 15.2, at least for triangles.

**Theorem 15.15** (Balogh and Mubayi [41]). *Almost all triangle-free 3-uniform hypergraphs are tripartite.*

The following result attacks already the general case, extends the Erdős–Frankl–Rödl theorem to 3-uniform hypergraphs.

**Theorem 15.16** (Nagle and Rödl [283]). *For any fixed 3-uniform hypergraph $L$,*

$$|\mathcal{P}(n,L)| < 2^{\mathbf{ex}(n,L)+o(n^3)}.$$

This was extended to $k$-uniform graphs by Nagle, Rödl and Schacht [284].

**Other structures.** There are some other structures where analogous results were proved fairly early, showing that some specific structures dominate (in number) the others. Here we mention some results of Kleitman and Rothschild [222] on the number of partially ordered sets on $n$ elements.

Consider $\mathcal{Q}(n)$, the family of partial orders of the following structures: $n$ vertices are distributed in three classes $L_1, L_2$, and $L_3$, where $|L_1| = n/4 + o(n)$, $|L_2| = n/2 + o(n)$, $|L_3| = n/4 + o(n)$. Define a partial order by its Hasse diagram. Define the partial order $Q$ as follows: the arcs go from $L_i$ to $L_{i+1}$, $i = 1, 2$, and if we forget about the orientations, we get a $\frac{1}{2}$-quasi-random graph between $L_i$ and $L_{i+1}$. Kleitman and Rothschild proved that [222]]

**Theorem 15.17** (Kleitman and Rothscild [222], see also [221]).

$$|\mathcal{P}_n| = \left(1 + O\left(\frac{1}{n}\right)\right)|\mathcal{Q}_n|.$$

*Thus*

$$|\mathcal{P}_n| = 2^{n^2/4+o(n^2)}.$$

See also Kleitman, Rothschild and Spencer [223].

## 15.4. Induced subgraphs?

If instead of excluding some not necessarily induced subgraphs, we exclude induced subgraphs, the situation completely changes. The first results in this direction were proved by Prömel and Steger [289], [290] .... Several extensions were proved by Alekseev, Bollobás and Thomason, and others.

**Definition 15.18.** The sub-coloring number $p_c(\mathcal{P})$ of a hereditary graph property $\mathcal{P}$ is the maximum integer $p$ for which if we put complete graphs into some classes of a $T_{n,p}$ (somehow), and delete some original edges, the resulting graph cannot have property $\mathcal{P}$.

**Example 15.19.** Let the property $\mathcal{P}$ be that $G_n$ contains an induced $C_4$. Consider a complete graph $K_\ell$ and a set $I_m$ of independent vertices (with disjoint vertex sets) and join them arbitrarily. The resulting graph will not contain induced $C_4$'s. It is easy to see that here $p_c(\mathcal{P}) = 2$.

**Theorem 15.20** (Alekseev [7], Bollobás–Thomason [61]). *If $\mathcal{P}$ is a hereditary property of graphs, and $\mathcal{P}(n,\mathcal{L})$ denotes the family of $n$-vertex graphs of property $\mathcal{P}$, and $p := p_c(L)$ then*

$$|\mathcal{P}(n,\mathcal{L})| = 2^{\frac{1}{2}\left(1-\frac{1}{p}\right)n^2+o(n^2)}.$$

This was improved in [12].

**Definition 15.21.** Given an integer $k$, the universal graph $U(k)$ is the bipartite graph with parts $A=\{0,1\}^k$ and $B=\{1,\dots,k\}$, where $j\in B$ is joined to a $k$-tuple $X$ if $j\in X$, (i.e., the $j^{\text{th}}$ coordinate of $X$ is 1).

**Theorem 15.22** (Alon, Balogh, Bollobás, Morris [12]). *Let $\mathcal{P}$ be a hereditary property of graphs, with coloring number $\chi_c(\mathcal{P})=p$. Then there exist constants $k=k(\mathcal{P})\in\mathbb{N}$ and $\varepsilon=\varepsilon(\mathcal{P})>0$ such that the following holds. For almost all graphs $G_n\in\mathcal{P}$, there exists a partition $(A,S_1,\dots,S_p)$ of $V(G_n)$, such that:*
*(a) $|A|<n^{1-\varepsilon}$ ,*
*(b) $G[S_j]$ is $U(k)$-free for every $j\in[p]$.*
*Moreover, if $\mathcal{P}_n$ is the family of $n$-vertex graphs of $\mathcal{P}$, then*

$$2^{(1-1/p)\binom{n}{2}} < |\mathcal{P}_n| \le 2^{(1-1/p)\binom{n}{2}+n^{2-\varepsilon}}$$

*for every sufficiently large $n\in\mathbb{N}$.*

There are several further interesting results in [12], but we stop here.

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**Further sources to read:** Bollobás [56].

## 15.5. Counting the colorings

Some of the above results are strongly connected to estimating

$$c_{r,F}(\mathcal{H}) := \#\{r-\text{colorings of }\mathcal{H}\text{ without monochromatic copies of }F\}$$

Estimating $c_{r,F}$ is strongly connected to the extremal problem of $F$, i.e. determining $\mathbf{ex}(n,F)$ and also with Erdős–Frankl–Rödl type theorems, first of all, with Theorems 15.2 and 15.3 Erdős and Rothschild conjectured that

**Conjecture 15.23.**

$$c_{2,K_\ell}(G_n)\le 2^{\mathbf{ex}(n,K_\ell)}.$$

For triangles this was proved by Yuster [374]. This was extended to arbitrary complete graphs by Alon, Balogh, Keevash and Sudakov [11]. A similar coloring-counting theorem was proved by Lefmann, Person, Rödl and Schacht [259], also explaining the connection of these results to each other. We skip the details.

# 16. “Random matrices”

This part is devoted to random $\pm 1$ matrices, where the questions are:

(i) How large is the determinant of a random matrix,

(ii) what is the probability that a random matrix is singular,

(iii) what can be said about the eigenvalues of a random matrix.

Recently very many new results were obtained in this field. Below I shall mention some of them and provide some references, and also refer the reader to the excellent survey paper of Van Vu [370].

Szekeres and Turán [347] were primarily interested in (i), more precisely, in the average of the absolute value of the determinant of a $\pm 1$ matrix. Later Turán continued this line, Szekeres went into another direction.

## 16.1. Hadamard matrices

According to the famous theorem of Hadamard, given a matrix $A = (a_{ij})$, $|\det(A)|$ can be estimated from above by the product of the lengths of the row vectors. Equality holds iff the row vectors are pairwise orthogonal. If the entries of the matrix are 1’s and -1’s, then Hadamard’s result yields that

$$|\det(A)| \le n^{n/2}. \tag{16.1}$$

It is natural to ask whether the equality in (16.1) can be achieved for $\pm 1$ entries. In other words, are there orthogonal $n \times n$ matrices with $\pm 1$ entries? Such matrices are called Hadamard matrices. The smallest ones are $(1)$ and $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. One can easily prove that if for some $n > 2$ such a matrix does exist, then $n$ is divisible by 4. It is a very famous, old and widely investigated but still open conjecture that

**Conjecture 16.1.** Hadamard matrices exist for every $n$ divisible by 4.

One can easily construct Hadamard matrices for $n = 2^k$ and it is not too difficult to construct them for $n = 4k$ if $n - 1$ is a prime.

## 16.2. Szekeres–Turán approach

In connection with the Hadamard problem, Gy. Szekeres and P. Turán arrived at the following question [347]:

**Problem 5.** Consider all the $\pm 1$ matrices $A$ of $n$ rows and columns. How large is the average of $|\det(A)|^k$, as a function of $n$?

They proved in [347] that

**Theorem 16.2.** *The average of* $|\det(A)|^2$ *for the* $n \times n$ $\pm 1$ $A$*, is* $n!$.

They simply calculated the sum of the squares of the determinants of all the $n \times n$ $\pm 1$ matrices. Their proof was very simple and elegant. They also calculated the sum of the fourth powers of these determinants, proving that this is $(n!)^2 \cdot \varphi(n)$, where $\varphi(n)$ is a function defined by the recursion

$$\varphi(1) = 1, \qquad \varphi(2) = 2, \qquad \varphi(n) = \varphi(n-1) + \frac{2}{n}\varphi(n-2). \tag{16.2}$$

**Remark 16.3.** For every $c > 0$, $\varphi(n)$ is between $n^{2-c}$ and $n^2$, if $n$ is sufficiently large. This means that the average of the squares and fourth powers of these determinants are (in some weak sense) fairly near to the desired maximum. Geometrically, if we take $n$ $\pm 1$ vectors independently, at random, they will be roughly orthogonal to each other.

**Remark 16.4.** Superficially we could think that the main goal of the Szekeres–Turán paper was to prove the existence of a good approximation of Hadamard matrices, using Random Matrix methods. Maybe, originally this was their purpose. However, as they remarked, Erdős had pointed out[38] that the following direct construction provides a much better result on the maximum value of the determinant:

Find a prime $p = 4k - 1 < n$ sufficiently near to $n$ and then build a Hadamard matrix for this $\tilde{n} = 4k$. Using the monotonicity of the maximum, one gets a much better estimate than by the Szekeres–Turán argument.

Is this result more than merely answering an important and interesting mathematical problem in an elegant way? YES, in the following sense:

Here we can see one of the first applications of stochastic methods instead of giving constructions for some optimization problem in Discrete Mathematics. Later this method was applied many times and proved to be one of our most powerful methods. (In combinatorics and graph theory it was Paul Erdős who started applying probabilistic methods *systematically*.) From this point of view the Szekeres–Turán paper was definitely among the pioneering ones.

## 16.3. Turán's and Szekeres' continuation

Later both Turán [357, 360, 362] and Szekeres [344, 345] returned to these questions. They generalized their original results in various ways. However, they did not really succeed in estimating the average of the $2k^{th}$ power of the considered determinants.[39] (The average of the odd powers is, by symmetry, 0!) Turán seemed to be more interested in finding analytically various averages of $\pm 1$ determinants. Szekeres went basically into two directions:

[38] This was remarked in the paper of Turán and Szekeres and also, e.g., in the "problem collection paper" of Erdős [108].

[39] As I see, they could not estimate the average of the $6^{th}$ powers.

(a) He considered the so called *skew Hadamard matrices*, restricted the averaging to these matrices i.e., where for $i \neq j$ $a_{i,j} = -a_{j,i}$. For them the averaging method [344] gave higher average.

(b) Also, Szekeres invented new combinatorial/algebraic constructions of Hadamard Matrices, Skew Hadamard Matrices [345]. He also used computer searches to find "small" examples. e.g. for $n = 52, 92$.

## 16.4. Expected or typical value?

The paper of Szekeres and Turán determines the average and the square average of $\det(A)^2$. In many cases the typical values of some random variable $\xi$ are very near to its expected values. This is e.g. the case in Turán's "Hardy–Ramanujan" paper [356]. In case of the $\pm 1$ determinants the situation is different.

**A correction/historical remark.** Here I have to make a "Correction": Writing my notes for Turán's Collected Papers [368] I "overstated" Theorem 16.2. I wrote that Szekeres and Turán proved that the determinant of almost all $A$ in Theorem 16.2 is near to the average $\sqrt{n!}$. This holds only in some fairly weak logarithmic sense. In the ordinary sense, not only they did not state this, but – as it turns out below – this is not even true.

Of course, Szekeres and Turán did not speak of "probability". The point is that they did not use Chebishev inequality, and they did not calculate the standard deviation. (Slightly earlier, Turán, in his proof of the Hardy–Ramanujan theorem, without speaking of probabilities, calculated the mean and the standard deviation of the number of prime divisors and then applied Chebishev inequality.) Theorem 16.6 below implies that for a positive percentage of the considered random matrices the determinant is above $(1 + c)\sqrt{n!}$, for some fixed $c > 0$.

This question, when $\xi$ is noticeably above $\mathbb{E}(\xi)$ (where $\mathbb{E}$ denotes the expected value), is discussed in e.g. in

**Theorem 16.5** (Schlage–Puchta [314]). *Let $\xi$ be a nonnegative real random variable, and suppose that $\mathbb{E}(\xi) = 1$ and $\mathbb{E}(\xi^2) = a > 1$. Then the probability $P(\xi \geq a)$ is positive, and for every $b < a$ we have $\int_{|\xi|>b} \xi^2 \geq a - b$.*

The paper remarks that this theorem is nearly a triviality, but it has several interesting corollaries. One of them is a lower estimate for $|\det(A)|$ in the Szekeres–Turán problem. Since the $4^{th}$ moment is much larger than the $2^{nd}$, (by (16.2)), Theorem 16.5 is applicable here.

## 16.5. The Hadamard "goodness" of random matrices

Denote the (Euclidean) norm of $\mathbf{a}$ by $||\mathbf{a}||$. Let $A$ be an $n \times n$ matrix with column vectors $\mathbf{a}_i$, $(i = 1, \dots, n)$. Define its "Hadamard goodness" as

$$h(A) = \frac{\det(A)}{\prod ||\mathbf{a}_i||},$$

if the denominator does not vanish, otherwise define $h(A) = 0$.

John Dixon [101] wrote a nice and interesting paper on the above discussed question, primarily on the typical goodness of the random method in the "Hadamard approach". He wrote that for him a paper of Cabay and Lam suggested that (logarithmically, in some natural settings) the values of the determinants of random matrices are close to their maximum. He proved that this is not so: the logarithmic distance is typically what is suggested in the Szekeres–Turán theorem: $\det(A)^{1/n} \approx (\sqrt{n!})^{1/(2n)} \approx \sqrt{n/e}$.

The question investigated by Dixon [101] is, how large the expected value of $h(A)$ is if $A$ is a random matrix, where the distribution of entries obey some weak smoothness conditions. The conclusion of Dixon's results is that typically $h(A)^{1/n} \approx 1/\sqrt{e}$.

**Condition (D1)** If $\mathbf{a}_1, \dots, \mathbf{a}_n$ are the columns of $A$, then the density of the distribution at $A$ depends only on the values of $||\mathbf{a}_1||, \dots, ||\mathbf{a}_n||$.

**Condition (D2)** The probability that $\det(A) \neq 0$ is 1.

**Theorem 16.6** (Dixon [101]). *Let $A$ be a random matrix whose distribution satisfies (D1) and (D2). Denote by $\mu_n$ and $\sigma_n^2$ the mean and variance of the random variable $\log h(A)$. Then*

*(i) $\mu_n = -\frac{1}{2}n - \frac{1}{4}\log n + O(1)$, and $\sigma_n^2 = \frac{1}{2}\log n + O(1)$, as $n \to \infty$;*

*(ii) For each $\varepsilon > 0$, the probability that*

$$n^{-\frac{1}{4}-\varepsilon}e^{-\frac{1}{2}n} < h(A) < n^{-\frac{1}{4}+\varepsilon}e^{-\frac{1}{2}n}$$

*tends to 1 as $n \to \infty$.*

## 16.6. Probability of being singular

In this section we are discussing the upper bounds for the probability that $\det(A) = 0$. For a reader interested in more details, the following sources are suggested: Komlós [237], Kahn, Komlós, and Szemerédi [211], or some more recent papers of Van Vu [370], Terry Tao and Van Vu [352].

Obviously, for continuous distributions this probability is 0. One can easily see that this probability must be the largest for $\pm 1$ matrices, where both values are taken with equal probabilities.

**Theorem 16.7** (Komlós, [237]). *Let $A = (a_{ij})$ be an $n \times n$ matrix whose entries are random independent variables, taking values $\pm 1$ with probability $\frac{1}{2}$. Then* $\det(A) \neq 0$ *with probability $p_n \to 1$ as $n \to \infty$.*

A more general result is

**Theorem 16.8** (Komlós, [237]). *Let $A = (\xi_{ij})$ be an $n \times n$ matrix whose entries are random independent variables, with common, non-degenerate distribution.*[40] *Then* $\det(A) \neq 0$ *with probability $p_n \to 1$ as $n \to \infty$.*

**Conjecture 16.9.** Let $P_n$ be the probability that a random $n \times n$ matrix with elements $\pm 1$ is singular. Then $P_n = (1 + o(1))n^2 2^{1-n}$.

The first breakthrough was

**Theorem 16.10** (Kahn, Komlós and Szemerédi [211]). *There is a positive constant $\varepsilon$ for which $P_n < (1 - \varepsilon)^n$.*

This is a considerable improvement on the best previous bound, $P_n = O(1/\sqrt{n})$ given by Komlós in 1977.

## 16.7. Eigenvalues of random matrices

This field is again a very wide one, with many interesting results. The beginnings of this part heavily relies on the Füredi–Komlós paper [173].

Investigating the distribution of the eigenvalues of matrices goes back to E. P. Wigner (1955), who was motivated by quantum mechanics. The following generalization is due to L. Arnold [25].

**Theorem 16.11** (Wigner, semicircle law.). *Assume that $A$ is a random symmetric matrix with random independent entries $a_{ij}$ for $i \geq j$. Let the distribution of these entries be $F$ for $i \neq j$ and $G$ for $i = j$. Assume that $\int |x|^k \, dF < \infty$, $\int |x|^k \, dG < \infty$ for $k = 1, 2, \ldots$ and set $D^2 a_{ij} = \mathbf{Var}\, a_{ij} = \sigma^2$. Let $W_n(x)$ be the empirical distribution of the number of eigenvalues of $A$ not exceeding $xn$. Let*

$$W(x) = \begin{cases} \frac{2}{\pi}\sqrt{1 - x^2} & \text{for } |x| \leq 1, \\ 0 & \text{for } |x| > 1. \end{cases}$$

*Then*

$$\lim_{n \to \infty} W_n(2\sigma\sqrt{n} \cdot x) = W(x).$$

[40] A distribution is degenerate if with probability 1, its outcome is the same.

This implies that for $c > 2\sigma$ with probability $1-o(1)$, all but $o(n)$ of the eigenvalues belong to $[-c\sqrt{n}, c\sqrt{n}]$. Yet, this does not give information on the largest eigenvalues. Ferenc Juhász [209] gave some weak estimates on this and those were improved to much better ones by the Füredi–Komlós theorems which basically assert that

**Theorem 16.12** (Füredi, Komlós [173]). *Let $A = (a_{ij})_{n\times n}$ be an $n \times n$ symmetric matrix where $a_{ij}$ are independent (not necessarily identically distributed) random real variables bounded with a common bound $K$, for $i \geq j$. Assume that, for $i > j$, $a_{ij}$ have a common expectation $\mu$ and variance $\sigma^2$. Further, assume that $\mathbb{E}(a_{ii}) = \nu$. (Here $a_{ij} = a_{ji}$.) The numbers $K$, $\mu, \sigma^2, \nu$ will be kept fixed as $n \to \infty$.*

*If $\mu > 0$ then the distribution of the largest eigenvalue of $A = (a_{ij})$ can be approximated in order $1/\sqrt{n}$ by a normal distribution of expectation*

$$(n-1)\mu + \nu + \sigma^2/\mu \tag{16.3}$$

*and variance $2\sigma^2$. Further, with probability tending to $1$,*

$$\max_{i\geq 2} |\lambda_i(A)| < 2\sigma\sqrt{n} + O(\sqrt{n}\log n), \tag{16.4}$$

*where $\lambda_i$ is the $i^{th}$ eigenvalue of $A$.*[41]

**Remark 16.13.** The semi-circle law implies that $\max_{i\geq 2} |\lambda_i(A)|$ cannot be much smaller than $2\sigma\sqrt{n}$.

## 16.8. Singularity over finite fields

One could ask what happens if we take the entries of a random $n \times n$ matrix from a finite field $\mathcal{F}$.

**Theorem 16.14** (Jeff Kahn, J. Komlós [210]). *The probability that a random square matrix of order $n$, with entries drawn independently from a finite field $F(q)$ according to some distribution, is nonsingular is asymptotically (as $n \to \infty$) the same as for the uniform distribution (excepting certain pathological cases, see below):*

$$\Pr(M_n \textit{ is nonsingular }) \to \prod_{i\geq 1}\left(1 - \frac{1}{q^i}\right) \qquad \textit{as} \qquad n \to \infty. \tag{16.5}$$

**What is pathological?** Kahn and Komlós write that if the entries of the random matrix $M_n$ are chosen independently and uniformly from $\mathcal{F}$, that is enough to ensure (16.5) and this was fairly widely known. Among others in [91] (see also [253, 254]) it is proved that

[41] $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

**Theorem 16.15.** *Let $M_n$ be a random $n \times n$ $\mathcal{F}$-matrix with entries chosen according to some fixed non-degenerate probability distribution $\mu$ on $\mathcal{F}$. Then (16.5) holds if and only if the support of $\mu$ is not contained in any proper affine field of $\mathcal{F}$.*

We skip the details here, again.

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