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ON COLOUR-CRITICAL GRAPHS

by

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Notations. Since this paper deals with colouring problems, the graphs, considered by us are undirected graphs without loops and multiple edges. If G^n is a graph, n denotes the number of its vertices. $\chi(G^n)$ is the chromatic number of G^n. Let x be a vertex of G^n, st x denotes the star of x, i.e. the set of vertices, joined to x; $\sigma(x)$ is the valence of x. Let e be an edge (or x be a vertex) of G^n, then G^n-e (or G^n-x) denotes the graph obtained from G^n omitting e (or omitting x and the edges, incident with it).

Introduction

The concept of the critical graphs was introduced by G. DIRAC, [1], [2]. Let G be a k-chromatic graph and e be an edge of it. e is said to be critical, if $\chi(G-e) = k-1$. The graph G is critical, if each edge of it is critical. In our case G is called a critical k-chromatic or shortly a k-critical graph.

The following two problems of T. GALLAI are investigated in this paper:

(A) For given k, n and m how many independent vertices of valence $\geqq m$ can be contained by a k-critical graph of n vertices?

The maximum will be denoted by $i(k, n, m)$.

(B) $\sigma(G)$ denotes the minimum valence in the graph G. For given k and n how large can $\sigma(G^n)$ be if G^n is k-critical.

The first part of this paper contains the following results:

THEOREM 1. *Let* $4 \leqq k \leqq m+1 \leqq n$, *then*

$$n-i(k,n,m) \geqq \frac{1}{2}\sqrt[k-1]{(k-2)!\,nm}\,. \qquad (1)$$ [1]

Clearly $i(k,n) = i(k,n,k-1)$ is the maximum number of independent vertices a k-critical graph of n vertices can have. Theorem 1 implies

$$n-i(k,n) \geqq \frac{1}{2}\sqrt[k-1]{(k-1)!\,n}\,. \qquad (2)$$

This result is not too far from the best possible in the following sense:

[1] Clearly, this theorem gives an estimation for *every* k-critical graph, while theorems 2, 4, 5 are constructions „only".

THEOREM 2. *Let $k \geqq 4$. Then for infinitely many values of n*

$$n-i(k,n) = O\left(n^{\frac{1}{2\left[\frac{k-1}{3}\right]}}\right). \tag{3}$$

If $k=4$, one can improve Theorem 1 in the following way:

THEOREM 3. *Let $n \geqq m+1 \geqq 4$. There exists a constant $c_1>0$ such that*

$$n-i(4,n,m) \geqq c_1 (nm)^{2/5}. \tag{4}$$

The technique applied to prove Theorem 3 gives also some sharpening in the case $k>4$, but the value of k is greater, the obtained result is relatively the worse and the proof becomes very complicated; therefore this case will not be investigated.

The second part of the paper contains a construction of a 4-critical graph W^n depending on some parameters. Choosing these parameters in two different suitable ways we obtain the following theorems:

THEOREM 4. *Let n be an even integer, large enough. Then*

$$n-i(4,n,m) \leqq 20\sqrt{nm}\,. \tag{5}$$

THEOREM 5. *Let n be an even integer, large enough. Then there exists a 4-critical graph W^n such that*

$$\sigma(W^n) \geqq \frac{\sqrt[3]{n}}{6}. \tag{6}$$

This result can be sharpened. Let ec (G) denote the edgeconnectivity of a graph G, i.e. the least integer ϑ such that omitting ϑ suitable edges of G we may obtain a disconnected graph from it. Clearly ec $(G) \leqq \sigma(G)$.

I. JACOBSEN asked, what can be said about the edge-connectivity of a 4-critical graph? The example, proving Theorem 5 also proves

THEOREM 6. *Let n be an even integer, large enough. There exists a 4-critical graph W^n such that*

$$\mathrm{ec}\,(W^n) \geqq \frac{\sqrt[3]{n}}{6}. \tag{7}$$

Here I have to make some remarks on the history of these results. W. G. BROWN and J. W. MOON gave a construction [3] which proved for infinitely many n that

$$n-i(4,n) = O(\sqrt{n}).$$

Applying Lemma 1 to a 6-critical graph, constructed by G. DIRAC, which had $2n$ vertices and n^2+2n edges, I could prove only

$$n-i(6,n) = O(\sqrt{n}) \tag{3*}$$

(for infinitely many n) but this result is essentially weaker than the result of BROWN and MOON. Last September (1969) B. TOFT constructed a 4-critical graph $\Gamma(n)$ which had $4n$ vertices and $\approx n^2$ edges. Applying Lemma 1 to TOFT's graph I could already prove (3*). T. GALLAI pointed out that the set of many independent vertices in both proofs consists of vertices of valence 3. This note led B. TOFT and me to Problem (A). We solved this problem and besides Problem (B) as well independently from each other at the same time and in very similar ways: both our proofs apply my splitting method (i.e. Lemma 1) in a modified form to the original graph of B. TOFT. The most important differences between our constructions are that

(1) Instead of Lemma 1 B. TOFT applies two similar lemmas and applying them succesively to the graph $\Gamma(n)$ he builds up the desired graphs. In my proof no splitting method is used explicitely, I construct a graph W^n directly and prove, that it is 4-critical. (I used the splitting method only to find the graph W^n.) From this point of view B. TOFT's proof is algorithmic, while mine is more direct, constructive.

(2) My graph consists of 3 similar parts, each of which is very similar to the graph of B. TOFT corresponding to Problem (B). From this point of view my construction is a little more complicated. The reason, why I had to "stick" together three such blocks is, that I split the vertices of TOFT's original graph into vertices, having disjoint stars, while B. TOFT did not.

Both the methods have their advantages and thus B. TOFT and I decided to publish both of them and almost together: the next paper of this journal contains a short description of B. TOFT's results [4]. The reader can also find the description of B. TOFT's original graph in it. An other source to find $\Gamma(m)$ is [6]. Finally, one can obtain $\Gamma(n)$ from the block Q constructed in the second part of this paper taking $p=q=1$, $a=d=n$.

I recommend the reader to think over, that this graph is 4-critical, because this will help to understand many things, connected with my construction.

Finally I remark that in the proofs I shall never verify that a given colouring of a given graph is good or not. The reader can easily prove in this cases, that no two vertices of the same colour are joined.

Added in proof. (February, 1973.) Last Summer I gave a lecture on a ,,Working Sminar on Hypergraphs" (Columbus, Ohio, USA) the bases of which was the hypergraphtheoretical part of this paper. It turned out, that W. G. BROWN, P. ERDŐS and V. T. SÓS [AP 1] also proved Lemma 2 for the special case $s=2$. At the same time I succeded to prove that Lemma 2 is sharp for $s=3$ as well (AP 2]. Finally, what is perhaps the most important, L. LOVÁSZ [AP 3], improving the methods of this paper and the construction of [3] recently has proven that there exist two positive constants c_1 and c_2 such that

$$c_1 n^{1/k-2} \leqq n-i(k,n) \leqq c_2 n^{1/k-2}$$

(The corresponding 3 papers are sumbitted but not published yet.)

[AP1] BROWN, W. G., ERDŐS, P., SÓS, V. T., On the existence of triangulated spheres in 3-graphs, *Periodica Math. Hung.*

[AP2] SIMONOVITS, M., *Note on a Hypergraph Extremal Problem,* Proc. of Working Seminar on Hypergraphs, Columbus, Ohio, 1972, Springer Verlag, Lecture Notes.

[AP3] LOVÁSZ, L., Independent sets in critical chromatic graphs. *Studia Sci. Math. Hungar* **8** (1973) ·

§ 1. The upper bounds

First I introduce the concept of splitting a vertex x into the vertices $x_1, \ldots, x_\nu$.

Definition. Let G and $\tilde{G}$ be two graphs and $x \in G$, $x_1, \ldots, x_\nu \in \tilde{G}$ be vertices given in them such that

$$G - x = \tilde{G} - x_1 - x_2 - \cdots - x_\nu$$

and $\operatorname{st} x = \bigcup_{i=1}^{\nu} \operatorname{st} x_i$ hold. We shall say that $\tilde{G}$ can be obtained from G by splitting x into $x_1, \ldots, x_\nu$.

The following lemma is of great importance in our investigations:

LEMMA 1. *Let G be a k-critical graph and x be a vertex of it. There exists a k-critical graph $\tilde{G}$, which can be obtained from the graph G by splitting x into $\nu \geqq \frac{\sigma(x)}{k-1}$ new vertices $x_1, \ldots, x_\nu$. Besides $\sigma(x_i) = k-1$, $i=1, \ldots, \nu$.*

PROOF. Let $s = \binom{\sigma(x)}{k-1}$ and let us consider s new vertices $x_1, \ldots, x_s$. Let us join x_i to $k-1$ vertices of $\operatorname{st} x$ in the graph $G-x$ so that $\operatorname{st} x_i \neq \operatorname{st} x_j$ unless $i=j$. Thus we split x into $x_1, \ldots, x_s$. This procedure does not increase the chromatic number, since $x_1, \ldots, x_s$ are independent. Thus the obtained graph G^* is at most k-chromatic. We prove that $\chi(G^*)=k$.

If $\chi(G^*) \leqq k-1$ held, we could colour G^* by $1, 2, \ldots, k-1$. Since each subset of $G-x$ consisting of $k-1$ elements is joined to an x_i, it must be coloured by at most $k-2$ colours. So $\operatorname{st} x$ is coloured by at most $k-2$ colours. Restricting this colouring of G^* to $G-x$ we can extend it to G giving to x the colour from $\{1, \ldots, k-1\}$ which does not occur in $\operatorname{st} x$. This implies $\chi(G) \leqq k-1$ contradicting $\chi(G)=k$. Hence $\chi(G^*)=k$.

Each k-chromatic graph contains a k-critical subgraph and this subgraph contains all the critical edges and vertices of the graph (where a vertex is called critical if omitting it we obtain a $k-1$-chromatic graph).

Let $\tilde{G}$ be a k-critical subgraph of G^*.

a) *$\tilde{G}$ contains $G-x$.* In order to prove this it is enough to prove that each edge of $G-x$ is critical. If e is an edge of $G-x$, $\chi(G-e) = k-1$ since G is k-critical. G^*-e can be obtained from $G-e$ by splitting x into $x_1, \ldots, x_s$, therefore $\chi(G^*-e) \leqq \leqq \chi(G-e) = k-1$, i.e. e is a critical edge in G^*. Thus e belongs to $\tilde{G}$.

b) *If y is a vertex of* $\operatorname{st} x$, *then there exists an x_i joined to y (in $\tilde{G}$).* Otherwise $\tilde{G}$ could be obtained from $G-(x, y)$ by splitting x into $x_1, \ldots, x_s$ and then omitting some vertices and edges from the resulting graph. This would imply $\chi(G^*) \leqq k-1$. But this is a contradiction showing that $\operatorname{st} x = \bigcup_{x_i \in \tilde{G}} \operatorname{st} x_i$ where the stars $\operatorname{st} x_i$ are counted in $\tilde{G}$.

c) Now the proof is completed. Indeed, we know that $\tilde{G}$ is k-critical. a) and b) together state that $\tilde{G}$ can be obtained from G by splitting x into some x_i's. Since a k-critical graph does not contain vertices of valence $< k-1$, thus $\sigma(x_i) = k-1$ if $x_i \in \tilde{G}$. According to b) $\sigma(x) \leqq \sum_{x_i \in \tilde{G}} \sigma(x_i)$, i.e. the number of x_i's, belonging to $\tilde{G}$, is at least $\frac{\sigma(x)}{k-1}$. Q.e.d.

PROOF of Theorem 1. We have to prove that if G is a k-critical graph and $x_1, \ldots$ $\ldots, x_t$ are independent vertices of valence $\geqq m$ in it, then

$$t \leqq n - \frac{1}{2} \sqrt[k-1]{(k-2)!\,mn}\,.$$

Let us split the vertices $x_1, \ldots, x_t$ into the vertices $\{x_{i,j}\}_{\substack{i \leqq t \\ j \leqq v_i}}$ successively: the graph, obtained in the $(i-1)$th step contains x_i and x_i has the same valence in it as in the original graph. We split x_i into $x_{i,1}, \ldots, x_{i,v_i}$ so that the resulting graph is also k-critical and $v_i \geqq \dfrac{k-1}{\sigma(x_i)}$. Since $x_1, \ldots, x_t$ are independent, the vertices $x_{i+1}, \ldots, x_t$ remain untouched.

In the last step we obtain a k-critical graph G^N. Since G^N is k-critical, $\operatorname{st} x_{i,j} \neq$ $\neq \operatorname{st} x_{k,l}$ unless $(i,j)=(k,l)$. Further, $\operatorname{st} x_{i,j}$ is a subset of $G^n - x_1 - \cdots - x_t$, consisting of $k-1$ elements. Hence

$$\binom{n-t}{k-1} \geqq \sum v_i \geqq \frac{1}{k-1} \sum \sigma(x_i) \geqq \frac{1}{k-1}\, mt. \tag{8}$$

$$n - t \geqq \sqrt[k-1]{(k-2)!\,mt} \tag{9}$$

follows immediately from (8) and if $n \leqq 2^{k-1} t$, (8) implies (1). If $n > 2^{k-1} t$, then

$$n^{k-1}\left(1-\frac{1}{2^{k-1}}\right)^{k-1} > \left(\frac{n}{2}\right)^{k-1} > nm\,\frac{n^{k-3}}{2^{k-1}} > nm\,\frac{k^{k-3}}{2^{k-1}} > nm\,\frac{(k-2)!}{2^{k-1}}$$

which gives (1) also in this case:

$$n - t \geqq n\left(1 - \frac{1}{2^{k-1}}\right) > \sqrt[k-1]{nm(k-2)!} \cdot \frac{1}{2} \quad \text{Q.e.d.}$$

Since (8) is a very rough estimation, one can try to improve it. As we mentioned already in the Introduction, this can be done, though for $k>4$ it is rather complicated. Thus we shall improve Theorem 1 only for $k=4$ and only later, since first we prove Theorem 2, showing, that Theorem 1 is sharp in a certain sense.

PROOF of Theorem 2. Let G^{n_l} be a k_l-critical graph of n_l vertices for $l=1, \ldots, T$ and $I_l = \{x_{i,l}\}_{i \leqq \xi_l}$ be a set of independent vertices of valence $k_l - 1$ in G^{n_l}. We construct a $\Sigma(k_l-1)+1$-critical graph G^N.

Let $\tilde{G}^N$ be the following graph: we join each vertex of $G^{n_l} - I_l$ to each vertex of $G^{n_m} - I_m$ for $1 \leqq l < m \leqq T$. Then we consider $\Pi \xi_l$ new vertices $P(i_1, \ldots, i_T)$ where $i_l = 1, 2, \ldots, \xi_l$. The vertices $P(i_1, \ldots, i_T)$ form a set of independent vertices of valence $\Sigma(k_l-1)$: $P(i_1, \ldots, i_T)$ is joined to a vertex $u \in G^{n_l} - I_l$ if and only if $x_{i,l}$ is joined to u in G^{n_l}:

$$\operatorname{st} P(i_1, \ldots, i_T) = \bigcup_{l=1}^{T} \operatorname{st} x_{i_l, l}\,.$$

1. $\chi(\tilde{G}_N) \geqq \Sigma(k_l-1)+1$. Indeed, $G^{n_l}-I_l$ is a k_l-1-chromatic graph and each vertex of $G^{n_l}-I_l$ is joined to each one of $G^{n_m}-I_m$ if $l \neq m$. Thus the subgraph G^*, spanned by the graphs $G^{n_l}-I_l$ must be coloured by at least $\Sigma(k_l-1)$ colours. If G^* is coloured by exactly $\Sigma(k_l-1)$ colours, then each $G^{n_l}-I_l$ is coloured by k_l-1 colours. Since $\chi(G^{n_l})=k_l$, there exists an $x_{\tau_l, l}$ such that st $x_{\tau_l, l}$ is coloured by at least k_l-1 colours. Hence st $P(\tau_1, \ldots, \tau_T)$ is coloured by exactly $\Sigma(k_l-1)$ colours, i.e. $\chi(\tilde{G}^N) \geqq \geqq \Sigma(k_l-1)+1$.

2. Now we prove that each $P(i_1, \ldots, i_T)$ is critical. Let us consider a $P(\tau_1, \ldots, \tau_T)$. We have to prove that

$$\chi(\tilde{G}^N - P,(\tau_1 \ldots, \tau_T)) = \sum (k_l-1).$$

Since $\chi(G^{n_l}-x_{\tau_l l}) = k_l-1$, $G^{n_l}-I_l$ can be coloured by k_l-1 colours so that each st $x_{i,l}$ but st $x_{\tau_l l}$, is coloured by $\leqq k_l-2$ colours. Let us fix such a colouring for each $G^{n_l}-I_l$. Now st $P(i_1, \ldots, i_T)$ is coloured by at most $\Sigma(k_l-1)-1$ colours unless $i_1=\tau_1, \ldots, i_T=\tau_T$. Thus the colouring of G^* by $\Sigma(k_l-1)$ colours can be extended onto $\tilde{G}^N-P(\tau_1, \ldots, \tau_T)$, i.e. $\chi(\tilde{G}^N-P(\tau_1, \ldots, \tau_T)) = \Sigma(k_l-1)$.

3. Let now G^M be a $\Sigma(k_l-1)+1$-critical subgraph of $\tilde{G}^N$. According to 2 G^M contains all the vertices $P(i_1, \ldots, i_T)$. This proves

$$M - i(\sum (k_l-1)+1, M) \leqq \sum_{i=1}^{T} (n_l-\xi_l). \tag{10}$$

Let now G^n be a 4-critical graph with $n-O(\sqrt{n})$ independent vertices. According to [3] or according to Theorem 4 there exist such graphs for infinitely many n. Setting $G^{n_l}=G^n$ and $k_l=4$ we obtain a $3T+1$-critical graph G^M. Here

$$n_l-\xi_l = O(\sqrt{n_l}) = O(\sqrt{n})$$

and from this and (10) we get

$$M-i(3T+1, M) = O(\sqrt{n}).$$

Since $M \geqq \prod \xi_l \approx n^T$,

$$M-i(3T+1, M) = O(M^{\frac{1}{2T}}).$$

This proves Theorem 2 if $k = 3T+1$. In order to obtain Theorem 2 in the other cases we apply the trivial inequality

$$i(k+1, n+1) \geqq i(k, n) \tag{11}$$

Q.e.d.

Remarks. 1. If we apply our construction to an odd circuit, we get for infinitely many M

$$M-i(k, M) = O\left(M^{\frac{1}{\left[\frac{k-1}{2}\right]}}\right) \tag{12}$$

which is only slightly weaker, than Theorem 2; on the other hand we do not need BROWN and MOON's result in this case.

2. *The graph $\tilde{G}^N$ in the proof of Theorem 2 is a critical graph, i.e. $\tilde{G}^N=G^M$.* This can be proved very easily.

Now we give the mentioned sharpening of Theorem 1. As we have mentioned in the introduction, we consider only the case $k=4$. In this case Theorem 1 gives

$$n-i(4,n,m) \geqq \frac{1}{2}\sqrt[3]{2mn} \tag{13}$$

while Theorem 4 gives only

$$n-i(4,n,m) = O(\sqrt{nm}) \tag{14}$$

(where n is even and sufficiently large). Improving (8) we shall prove

THEOREM 3. *Let $4 \leqq m+1 \leqq n$, then there exists a constant $c>0$ such that*

$$n-i(4,n,m) \geqq c(nm)^{2/5}. \tag{4}$$

PROOF. I shall not introduce the concept of triangle-graphs but refer to [5]. Some parts of the proof will be omitted.

Definition. $C_{3,s,t}$ denotes the following triangle-graph: the vertices of $C_{3,s,t}$ are $u_1, \dots, u_s, v_1, \dots, v_t$ and the triangles of it are the triplets $(u_i v_j v_{j+1})$, where $v_{t+1}=v_1$ and $i=1, \dots, s$; $j=1, \dots, t$.

LEMMA 2. *If G^m is a triangle-graph of m vertices which does not contain a $C_{3,s,t}$ for $t=3, 4, \dots$, then G^m has at most*

$$\left(\frac{1}{3}+o(1)\right)m^{3-\frac{1}{s}} \tag{15}$$

triangles.

The proof of Lemma 2 will be given later. Now let us consider the vertices of $G^{n-t} = G^n - x_1 - \dots - x_t$ in the proof of Theorem 1 and the triplets st $x_{i,j}$; $i=1, \dots, t$, $j=1, \dots, v_i$. These vertices and triplets define a triangle-graph of Σv_i triangles. We prove that this triangle-graph does not contain a $C_{3,s,t}$, thus Lemma 2 gives $\Sigma v_i = O((n-t)^{5/2})$ instead of (8). From this we obtain Theorem 3 by the same way as we obtained Theorem 1 from (8). We call a triangle-graph "good" if for each triangle of it we can colour its vertices by 3 colours so that this triangle is coloured by 3 colours but all the others by at most two ones. The reader can easily prove that $C_{3,s,t}$ is not "good". Since a subgraph of a "good" graph is also "good", a "good" graph cannot contain $C_{3,s,t}$, thus a "good" graph of m vertices has at most $O(m^{5/2})$ triangles, (Lemma 2). Thus it is enough to prove that the triangle-graph, constructed on the vertices of $G^n - x_1 - \dots - x_t$ is "good". Since G^N is 4-critical, it has a 4-colouring, such that the colour of $x_{i,j}$ is not used in $G^N - x_{i,j}$. In this case st $x_{i,j}$ is coloured by 3 colours and st $x_{k,l}$ is coloured by at most two colours, if $(k,l) \neq (i,j)$. $G^n - x_1 - \dots - x_t$ is coloured by 3 colours. Thus the considered triangle-graph is "good". Q.e.d.

PROOF of Lemma 2. The Lemma is similar to Theorem 1 in a paper of P. ERDŐS [5] and the proof is also almost the same.

Let G^m be a triangle-graph, $1, \dots, m$ be its vertices and t denote the number of its triangles; let $A(i,j)$ denote the set of vertices k such that $\{i,j,k\}$ is a triangle of the graph. Clearly

$$3t = \sum_{1 \leqq i < j \leqq m} |A(i,j)| \tag{16}$$

A set $u_1, \ldots, u_s$ and a pair (i, j) will be called a "flower" if $u_l \in A(i, j)$ $(l=1, \ldots, s)$. Since (i, j) is contained in $\binom{|A(i,j)|}{s}$ "flowers",

$$F = \sum_{1 \leq i < j \leq m} \binom{|A(i,j)|}{s} \tag{17}$$

is the number of "flowers" contained in G^m.

On the other hand, if $u_1, \ldots, u_s$ are given, at most n pairs (i, j) form a "flower" with $u_1, \ldots, u_s$, otherwise there would be a cycle $(v_1, \ldots, v_t, v_1)$ such that $u_1, \ldots, u_s$ formed a "flower" with each pair $(v_l v_{l+1})$ when $l=1, \ldots, t$, $v_{t+1}=v_1$. Thus the vertices $u_1, \ldots, u_s, v_1, \ldots, v_t$ would determine a $C_{3,s,t}$ in G^m. This is a contradiction and hence

$$n\binom{n}{s} \geq F. \tag{18}$$

(16), (18) and the convexity of $\binom{x}{s}$ for $x \geq s$ imply

$$t \leq \left(\frac{2^{1/s}}{6} + o(1)\right) n^{3-\frac{1}{s}} \qquad \text{Q.e.d.}$$

Remark. One can prove that Lemma 2 is sharp for $s=2$: if c is a positive, but sufficiently small constant and we select each triangle-graph having m vertices and $cm^{3-\frac{1}{2}}$ triangles with the same probability, then the selected graph will not contain any $C_{3,2,t}$ with probability, tending to 1 (when m tends to infinity). I do not know, whether Theorem 3 is sharp or not.

Now we construct a graph proving Theorems 4, 5, 6.

§ 2. The lower bounds for $i(k, n, m)$

We restrict our investigations to the case $k=4$, because

a) If G^n is a 4-critical graph and we join the vertices of a complete $(k-4)$-graph to each vertex of G^n, then we obtain a k-critical graph. Hence our constructions give also some lower bounds for the general case.

b) The problem (B) is not too interesting for $k \geq 6$: Let $\gamma(n)$ denote a circuit of n vertices. If we join each vertex of a $\gamma(n)$ to each vertex of another $\gamma(n)$ and n is odd, then we obtain a 6-critical graph of $2n$ vertices with minimum valence $n+2$. (This construction is due to G. DIRAC.)

The desired results will be obtained by a construction: we construct a 4-critical graph depending on many different parameters and consisting of 3 or more similar blocks.

The block **Q**.

The vertices of the graph **Q** can be divided into four parts, which will be called the stories of the block. Let a, d, p, q be given odd integers.

The *second story* consists of apq independent vertices, denoted by $B(i, j, x)$, where $i=1, \ldots, p$; $j=1, \ldots, a$; $x=1, \ldots, q$.

The *third story* consists of *dpq* vertices, denoted by $C(k, l, y)$, where $k=1, \dots, q$; $l=1, \dots, d$; $y=1, \dots, p$. $B(i, j, k)$ is joined to $C(k, l, i)$ for every i, j, k and l. The set $\{B(i,j,x)\}_{j,x}$ will be called a class and denoted by C_i. Similarly, $\{C(k, l, y)\}_{l,y}$ is the class $\bar{C}_k$. The sets $\{B(i,j,x)\}_x$ will be called groups and denoted by $G_{i,j}$. Similarly, $\{C(k, l, y)\}_y$ is the group $\bar{G}_{k,l}$.

The *first story* consists of the vertices $A(i,j)$; $i=1, \dots, p$; $j=1, \dots, a$. These vertices form a circuit $\gamma(ap)$: $A(i,j)$ is joined to $A(i',j')$ if $i=i'$ and $|j-j'|=1$ or if $i'=i+1$, $j=a$, $j'=1$ (or conversely) and $A(p, a)$ is joined to $A(1, 1)$. Fixing i we obtain the arcs α_i of the circuit $\gamma(ap)$. The vertices of the first story are also joined to some vertices of the second one: $A(i,j)$ is joined to $B(i,j,x)$ for $x=1, \dots, q$. The fourth story is similar to the first one: it consists of the vertices $D(k, l)$; $k=1, \dots, q$; $l=1, \dots, d$ which determine the circuit $\bar{\gamma}(dq)$ consisting of the arcs $\bar{\alpha}_k$. $D(k, l)$ and $C(k, l, y)$ are joined.

This is the block (graph) **Q**. First we investigates its 3-colourings. We need the following definition:

Definition. If γ is an odd circuit, a 3-colouring of it is called elementary if one of the three colours is used only once. This colour and the corresponding vertex are called the exceptional colour and vertex respectively. If γ is divided into arcs, a 3-colouring of it is called periodic if each arc is coloured by two colours.

LEMMA 3. *Let be given a 3-colouring of* **Q**. *Then either $\gamma(ap)$ or $\bar{\gamma}(dq)$ is coloured periodically.*

Remark. In the case $p=q=1$ $a=d=n$ we obtain B. TOFT's graph of $4n$ vertices and $\approx n^2$ edges. Since neither $\gamma(a)$ nor $\bar{\gamma}(d)$ has periodic colourings, $\chi(\mathbf{Q}) \geqq 4$. It is easy to prove that **Q** is a 4-critical graph.

PROOF of Lemma 3. If neither $\gamma(ap)$ nor $\bar{\gamma}(dq)$ is coloured periodically, then for an i and a k both $\gamma_i(a)$ and $\gamma_k(d)$ are coloured by 3 colours. Hence both the sets $\{B(i,j,k)\}_j$ and $\{C(k, l, i)\}_l$ are coloured by at least two colours. These sets span a complete bipartite graph, therefore the colours, used at the two sets are different, i.e. Q is coloured by at least 4 colours. This contradiction proves Lemma 3.

Lemma 3 concerns with the question, how Q can not be coloured by 3 colours. The next question is, how it can.

LEMMA 4. *For given i, j and k, if $q \geqq 3$, then* **Q** *can be coloured by* 1, 2 *and* 3 *so that the only vertex of $\gamma(ap)$ coloured by* 1 *is $A(i,j)$ and if $D(k', l)$ is coloured by* 2, *then $k'=k$.*

PROOF. Let us consider the following colouring of **Q**:

a) $\gamma(ap)$ is coloured by 1, 2 and 3 elementary: $A(i,j)$ is the only vertex coloured by 1.

b) $B(i,j,k)$ is coloured by 2, $B(i,j,x)$ by 3 $(x \neq k)$ and $B(i^*, j^*, x^*)$ by 1 if $(i^*, j^*) \neq (i,j)$.

c) $C(k^*, l^*, y^*)$ is coloured by 3 if $k^*=k$ and by 2 if $k^* \neq k$.

d) Finally we have to colour $\bar{\gamma}(dq)$. Let us colour $D(k, l^*)$ by 2 if l^* is odd and by 1 otherwise. The other vertices of $\bar{\gamma}(dq)$ span a path, which can be coloured by 1 and 3 since the vertices of the third story, joined to the vertices of this path are coloured by 2. The obtained 3-colouring proves Lemma 3.

Now we turn to the

Problem: How can $\mathbf{Q}-e$ be coloured by 3 colours if e is an edge of $\mathbf{Q}$?

LEMMA 5. *Let e and f be two edges of $\mathbf{Q}$:*

$$e=(B(i,j,x);\ C(k,l,y)),\ (i=y, x=k)$$

and

$$f=(A(i,j):\ B(i,j,x)).$$

Then both $\mathbf{Q}-e$ and $\mathbf{Q}-f$ can be coloured by 3 colours so that $\gamma(ap)$ and $\bar{\gamma}(dq)$ are coloured elementary, $A(i,j)$ and $D(k,l)$ are the exceptional vertices.

If g is an edge of $\gamma(ap)$, then any given 3-colouring of $\bar{\gamma}(dq)$ can be extended onto $\mathbf{Q}-g$ so that the path $\gamma(ap)-g$ is coloured by two colours.

PROOF. A) Let us consider the following colouring of $\mathbf{Q}-e$: $\gamma(ap)$ and $\bar{\lambda}(dq)$ are coloured elementary, the exceptional vertices are $A(i,j)$ and $D(k,l)$ which are coloured by 1 and 2 respectively. $\mathbf{G}_{i,j}$ and $\overline{\mathbf{G}}_{k,l}$ are coloured by 3, all the other vertices of the second story are coloured by 1 and all the other vertices of the third story are coloured by 2. Trivially this is a good 3-colouring of $\mathbf{Q}-e$. If we colour $B(i,j,x)$ by 1 instead of 3, we obtain a good colouring of $\mathbf{Q}-f$. This proves the first part of our lemma.

B) Let be given a 3-colouring of $\bar{\gamma}(dq)$ by 1, 2 and 3. Now we colour the second story by 3, the first one by 1 and 2. Up to now no edge joins vertices of the same co our. This 3-colouring can be extended onto the third story, since each $C(k,l,y)$ is joined to some vertices of the second story, coloured by 3 and to one vertex of the fourth story. Thus $C(k,l,y)$ can be coloured either by 1 or by 2. Q.e.d.

We need also the following trivial lemma:

LEMMA 6. *Let $\gamma(ap)$ be coloured periodically. Then the vertices $A(i,1)$ and $A(i,a)$ are coloured by the same colour and the set $\{A(i,1)\}_i$ is coloured by exactly 3 colours.*

PROOF. Since a is odd and the arc $\{A(i,j)\}_j$ is coloured by 2 colours, $A(i,1)$ and $A(i,a)$ have the same colour. Let us consider an odd circuit $\gamma(p)$ and colour its ith vertex by the colour of $A(i,1)$. Since $A(i,a)$ has the same colour as $A(i,1)$ and is joined to $A(i+1,1)$, thus $A(i,1)$ and $A(i+1,1)$ have different colours. Thus we obtained a good 3-colouring of $\gamma(p)$ and therefore $\{A(i,1)\}_i$ is coloured by exactly 3 colours. Q.e.d.

Graphs with many independent vertices of high valence.

Let $\mathbf{Q}$ be a block, defined above, where

$$p=1,\quad d=m,\quad a=qm.$$

Let E be a new vertex and let us join each $D(k,1)$ to it. Thus we obtain a 4-chromatic graph $\tilde{S}^n$ which contains a 4-critical subgraph S^n.

1. $\chi(\tilde{S}^n)\geqq 4$. Indeed, if $\mathbf{Q}$ is coloured by 1, 2 and 3, $\bar{\gamma}(dq)$ must be coloured periodically, since $\gamma(a)$ has no periodic colouring. According to Lemma 6 $\{D(k,1)\}_k$ is coloured by 3 colours, hence E must have another, fourth colour.

2. Let $e=(B(1,j,x);\ C(x,l,1))$ and $l\neq 1$. Then $\chi(\tilde{S}^n-e)\leqq 3$. Indeed, according to Lemma 5 $\mathbf{Q}-e$ can be coloured by 1, 2 and 3 so that only $D(k,l)$ is coloured by 2 in $\bar{\gamma}(dq)$. Thus E can be coloured by 2. Similarly, one can prove that $\chi(\tilde{S}^n-f)\leqq 3$

and $\chi(\tilde{S}^n - g) \leqq 3$ if $f=(A(1,j);\, B(1,j,x))$ and g is an edge of $\gamma(a)$ or $\bar{\gamma}(dq)$. (The last two assertions will not really be needed.)

3. Since $\chi(\tilde{S}^n) \geqq 4$, $\tilde{S}^n$ contains a 4-critical subgraph S which contains all the edges of $\tilde{S}^n$ mentioned in 2. This and $\chi(\mathbf{Q})=3$ imply that S contains all the vertices of $\tilde{S}^n$. The vertices $B(1,j,x)$ are aq independent vertices of valence $\geqq m$. Thus $i(4,n,m) \geqq aq$. Here $a=qm$. Therefore

$$(19)\quad n - i(4,n,m) \leqq \big(a(q+1)+2mq+1\big) - dq = a+2mq+1 = 3mq+1 \leqq 3\sqrt{mn}.$$

This proves Theorem 4 for every m for infinitely many n. Later we shall prove Theorem 4 for every even and sufficiently large value of n.

The 4-*critical graph* W^n.

First we define a 4-chromatic graph $\tilde{W}^n$ which has a lot of critical edges and then we select a 4-critical subgraph W^n of it which will be just the desired graph. (I.e. W^n will prove Theorems 5, 6.)

Let t be an odd integer and let us consider t blocks $\mathbf{Q}_1, \dots, \mathbf{Q}_t$ with the parameters

$$a_\tau, d_\tau, p_\tau, q_\tau, \qquad \tau = 1, \dots, t.$$

These blocks will be connected to each other in the following way:

$$E_\tau(i), \quad F_\tau(i), \quad G_\tau(k), \quad H_\tau(k)$$

are new vertices. $E_\tau(i)$ is joined to $A_\tau(i,1)$, $F_\tau(i)$ is joined to $A_\tau(i,a)$, $G_\tau(k)$ is joined to $D_\tau(k,l)$ and $H_\tau(k)$ is joined to $D_\tau(k,d)$. (Here e.g. $A_\tau(1,a)$ is the abbreviation of $A_\tau(i,a_\tau)$; generally the index showing, which block is meant will be omitted where it causes no confusion.) Let us join each $E_\tau(i)$ to each $F_{\tau+1}(j)$ and each $G_\tau(k)$ to each $H_{\tau+1}(l)$ for $\tau=1,\dots,t$, where $t+1 \equiv 1$. The obtained graph will be denoted by $\tilde{W}^n$.

Investigating the colouring properties of the graph $\tilde{W}^n$ first we shall colour the blocks $\mathbf{Q}_\tau$ by 3 colours and then extend this colouring onto the whole graph. The following assertion deals with the possibility of this extension:

(+) Let us suppose that some vertices of $\tilde{W}^n$ are coloured by 3 colours and no edge joins vertices of the same colour. Let us also suppose that for a fixed τ the vertices of $\gamma_\tau(ap)$ and $\gamma_{\tau+1}(ap)$ are coloured, the vertices of $\{E_\tau(i)\} \cup \{F_{\tau+1}(j)\}$ are not. We can extend this colouring onto $\{E_\tau(i)\} \cup \{F_{\tau+1}(j)\}$ if and only if either $\{A_\tau(i,1)\}_i$ or $\{A_{\tau+1}(j,a)\}_j$ is coloured by (at most) two colours.

Indeed, if both $\{A_\tau(i,1)\}$ and $\{A_{\tau+1}(j,a)\}$ are coloured by 3 colours, then both $\{E_\tau(i)\}$ and $\{F_{\tau+1}(j)\}$ must be coloured by at least 2 colours. Since each $E_\tau(i)$ is joined to each $F_{\tau+1}(j)$, the set $\{A_\tau(i,1)\} \cup \{A_{\tau+1}(j,a)\} \cup \{E_\tau(i)\} \cup \{F_{\tau+1}(j)\}$ must be coloured by at least 4 colours.

On the other hand, if e.g. $\{A_\tau(i,1)\}$ is coloured by 1 and 2, then we colour $E_\tau(i)$ by 3. Since each $F_{\tau+1}(j)$ is joined to exactly one vertex of $\gamma_{\tau+1}(ap)$, it can be coloured either by 1 or by 2. This completes the proof of (+).

2. Now we prove that $\chi(\tilde{W}^n) \geqq 4$. If we colour the blocks by 1, 2 and 3, then at least t circuits $\gamma_\tau(ap)$ and $\bar{\gamma}(dq)$ must be coloured periodically. Without the loss of generality we may assume that at least $\frac{t+1}{2}$ of the circuits $\gamma_\tau(ap)$ are coloured

periodically and therefore there exists a τ such that $\gamma_\tau(ap)$ and $\gamma_{\tau+1}(ap)$ are coloured periodically. According to Lemma 6 both $\{A_\tau(i,1)\}$ and $\{A_{\tau+1}(j,a)\}$ are coloured by 3 colours and this and $(+)$ give that $\chi(\tilde{W}^n)\geqq 4$.

3. Now we show that almost all the edges of $\tilde{W}^n$ are critical. Because of the symmetry of $\tilde{W}^n$ we may assume that the considered edges belong to the first block or are of the form $\big(G_t(k_0), H_1(k_1)\big)$ or $\big(H_1(k_1), D_1(k_1, d)\big)$. Let us colour the blocks $\mathbf{Q}_3, \mathbf{Q}_5, \dots, \mathbf{Q}_t$ by 1, 2 and 3 so that $\bar{\gamma}_\tau(dq)$ be coloured periodically and the colour 2 be used only on the arc $\bar{\alpha}_{\tau,k_0}$; $\gamma_\tau(ap)$ be coloured elementarily and only $A_\tau(1,2)$ be coloured by 1. $\mathbf{Q}_1$ is coloured similarly: in $\gamma_1(ap)$ only $A_1(1,2)$ is coloured by 1 and in $\bar{\gamma}_1(dq)$ only some vertices of $\bar{\alpha}_{1,k_1}$ have the colour 3. $\mathbf{Q}_2, \mathbf{Q}_4, \dots, \mathbf{Q}_{t-1}$ are coloured conversely: in $\bar{\gamma}_{2\nu}(dp)$ only $D_{2\nu}(k_2, 2)$ has the colour 1 and only on α_{τ,i_0} is used the colour 2. According to Lemma 4 these colourings do exist. The following scheme illustrates the situation:

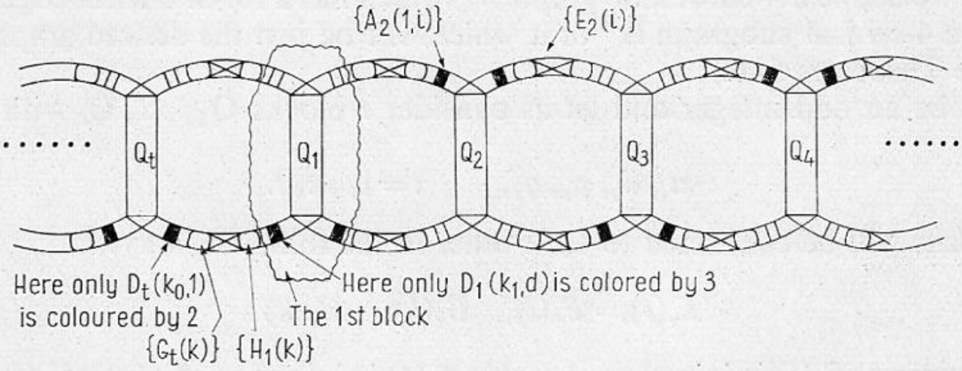

Fig. 1

Now we try to extend the given 3-colouring of the blocks onto the whole graph. $(+)$ quarantees that the sets $\{E_\tau(i)\cup\{F_{\tau+1}(j)\}$ and $\{G_\tau(k)\}\cup H_{\tau+1}(l)\}$ can be coloured by 1, 2 and 3 except in the case of $\{G_t(k)\}\cup\{H_1(k)\}$. Now we colour $\{G_t(k)\}_{k\neq k_0}$ by 2 and $\{H_1(k)\}_{k\neq k_1}$ by 3 and colour $G_t(k_0)$ by 1. The only vertex of $\tilde{W}^n$ which is not coloured yet, is $H_1(k_1)$ and no edge joins vertices of the same colour. If we colour $H_1(k_1)$ by 1, then only $\big(G_t(k_0); H_1(k_1)\big)$ has endpoints of the same colour; if we colour $H_1(k_1)$ by 3, then only $\big(H_1(k_1); D_1(k_1, d)\big)$ has endpoints of the same colour. Thus both $\big(G_t(k_0); H_1(k_1)\big)$ and $\big(H_1(k_1), D_1(k_1 d)\big)$ are critical edges in $\tilde{W}^n$.

Because of the symmetry all the edges, not belonging to the blocks are critical. (Besides, $\chi(\tilde{W}^n-(G_0(k_0); H_1(k_1))=3$ implies $\chi(\tilde{W}^n)\leqq 4$. This and 2 give $\chi(\tilde{W}^n)=4$.)

4. Now we turn to the following question: which edges of a block, say of $\mathbf{Q}_1$ are critical?

Let h be an edge of $\mathbf{Q}_1$ and the blocks $\mathbf{Q}_\tau$ be coloured as in 3 if $\tau\neq 1$. The block $\mathbf{Q}_1-h$ is coloured by 1, 2 and 3 in the way, described in Lemma 5. Because of $(+)$ all the vertices of $\tilde{W}^n$ but the vertices of $\{E_1(i)\}\cup\{F_2(j)\}$ and $\{G_t(k)\}\cup\{H_1(l)\}$ can be coloured by 1, 2 and 3 so that no edge joins vertices of the same colours. This colouring can be extended onto $\{E_1(i)\}\cup\{F_2(j)\}$ and $\{G_t(k)\}\cup\{H_1(l)\}$ if and only if both $\{D_1(k,d)\}_k$ and $\{A_1(i,1)\}_i$ are coloured by two colours. According to Lemma 5 this can be achieved if

a) h is an edge belonging to $\gamma_1(ap)$ or to $\bar{\gamma}_1(dq)$.

b) $h=\big(A_1(i,j); B_1(i,j,x)\big)$, $j\neq 1$.

c) $h=(B(i,j,k);\,C_1(k,l,i))$, $j\neq 1$ and $l\neq d_1$. Thus the edges described in a), b), c) are critical. Because of the symmetry the following edges are also citical:

b′) $h=(A_1(i,j);\,B_1(i,j,x))$, $j\neq d_1$.

c′) $h=(B_1(i,j,k);\,C_1(k,l,i))$, $j\neq a_1$ and $l\neq 1$. Thus all the edges of $\mathbf{Q}_1$ are critical except perhaps some edges of form $(B_1(i,j,k);\,C_1(k,l,i))$ where either $j=l=1$ or $j=a_1$, $l=d_1$.

5. Let W^n be a 4-critical subgraph of $\tilde{W}^n$. 3. and 4. give that W^n contains all the edges of $\tilde{W}^n$ except a few one of form $(B_\tau(i,j,k),\,C_\tau(k,l,i))$ where $j=l=1$ or $j=a_\tau$ $l=d_\tau$. Thus

$$\sigma(W^n)\geqq\sigma(\tilde{W}^n)-1=\min_\tau\min(p_\tau,q_\tau,a_\tau,d_\tau)=:m+1. \tag{20}$$

Here we applied that the valence of vertices $E_1(i)$, $A_1(i,j)$, $B_1(i,j,x)$ are at least p_2+1, q_2+2, d_1+1 respectively and the valence of the other vertices can be estimated from below similarly because of the symmetry.

If ec (G) denotes the edge-connectivity of the graph G, then trivially ec $(G)\leqq\sigma(G)$. In our case

$$\mathrm{ec}\,(W^n)=\sigma(W^n). \tag{21}$$

For the sake of simplicity we prove only

$$\mathrm{ec}\,(W^n)\geqq\sigma(W^n)-1=m. \tag{21*}$$

To prove this we need the following trivial notice:

$(++)$ Let G be a graph and omit $m-1$ edges of it. Let G^* denote the obtained graph. If K is a connected component of G^* and A is a vertex of G, joined by m independent paths to m vertices of K in the graph G, then A also belongs to K.

Let us omit $m-1$ edges of W^n and denote the obtained graph by W^*. Let K_τ be the component of W^* containing $E_\tau(1)$. In order to prove (21*) we have to prove, that K_τ contains all the vertices of W^*.

The paths $E_\tau(1)-F_{\tau+1}(i)-E_\tau(i_0)$ are independent in W^n, thus $E_\tau(i_0)\in K_\tau$. The paths $F_{\tau+1}(i_0)-E_\tau(i)$ are also independent, hence $F_{\tau+1}(i_\tau)\in K_\tau$. Considering the paths

$$D_\tau(k,l)-C_\tau(k,l,i)-B_\tau(i,2,k)-A_\tau(i,2)-A_\tau(i,1)-E_\tau(i)$$

for $i=1,\dots,m$ we obtain that $D_\tau(k,l)\in K_\tau$.

Because of the symmetry $D_\tau(k,l)$ belongs not only to the component of $E_\tau(1)$ but to the component of $F_\tau(1)$, i.e. to $K_{\tau-1}$ as well. Thus $K_\tau=K_{\tau-1}$, i.e. the components K_τ are identical with each other. This common component will be denoted by K hereafter. The paths $G_\tau(k_0)-H_{\tau+1}(k)-G_\tau(k)-D_\tau(k,1)$ for $k\leqq m$, $k\neq k_0$ and $G_\tau(k_0)-D_\tau(k_0,1)$ show that $G_\tau(k_0)\in K$. Similarly $H_\tau(k_0)\in K$, i.e. each vertex not belonging to the blocks belongs to K and the same holds for the vertices of the fourth stories. Because of the symmetry the vertices of the first stories also belong to K. The only assertion we have to prove is that $C_\tau(k_0,l_0,i_0)\in K$ too.

Since the paths $C_\tau(k_0,l_0,i_0)-B_\tau(i_0,j,k_0)-A_\tau(i_0,j)$ and $C_\tau(k_0,l_0,i_0)-D_\tau(k_0,l_0)$ are independent, $C_\tau(k_0,l_0,i_0)\in K$ and this completes the proof of (21*). (If $l_0=1$ or $l_0=d_\tau$, one of these paths is not contained by W^n!)

PROOF of Theorem 6. Let us consider W^* with the following parameters:

$$t=3, \quad p_\tau=q_\tau=a_\tau=d_\tau=v.$$

Trivially the number of vertices of W^n is

$$n = 6(v^3+v^2+2v) \tag{22}$$

while $\mathrm{ec}\,(W^n)\geqq n$. This proves Theorem 6 for infinitely many integers.

If we wish to prove Theorem 6 for every sufficiently large even integer, we can do it in the following way:

First we suppose that n is not divisible by 4. Let us consider a W^n with 19 blocks and let

$$p_\tau = v+\varepsilon_\tau, \quad q_\tau = v-\varepsilon_\tau, \quad a_\tau = d_\tau = v$$

for $1\leqq\tau\leqq 4$, where t is odd, ε_τ is even,

$$p_\tau = v, \quad q_\tau = v+2, \quad a_\tau+d_\tau = 2v$$

for $5\leqq\tau\leqq 19$. Since W^n has

$$n = \sum p_\tau q_\tau(a_\tau+d_\tau)+(p_\tau a_\tau+q_\tau d_\tau)+2(p_\tau+q_\tau) \tag{23}$$

vertices in the general case, now it has

$$(38v^3+98v^2+76v+60)-2v\sum_{\tau=1}^{4}\varepsilon_\tau^2+2\sum_{\tau=5}^{19}d_\tau \tag{24}$$

vertices. Here the first and second terms are divisible by 4 while the third one is not. Therefore (24) is not divisible by 4.

Now we fix the least odd integer such that $38v^3\geqq n$. Clearly

$$98v^2 \leqq 38v^3+98v^2+76v-60-n = O(v^2).$$

Since every integer is the sum of four square numbers, we can achieve

$$n-20v \leqq 38v^3+98v^2+76v+60-2v\sum_{1}^{4}\varepsilon_\tau^2 \leqq n-12v.$$

Finally we select d_τ from $\left[\frac{12}{15}v, \frac{20}{15}v\right]$ so that (24) be equal to n. Since $\varepsilon_\tau=O(\sqrt{v})$ and $a_\tau, d_\tau\geqq\frac{10}{15}v$ thus Theorem 6 (and Theorem 5 too) is proved.

The case, when n is divisible by 4 can be treated similarly. The only change is that we take 5 blocks of the first type and 14 blocks of the second type. Thus W^n has

$$n = 38v^3+94v^2+76v+60-2v\sum_{\tau=1}^{5}\varepsilon_\tau^2+2\sum_{\tau=6}^{19}d_\tau$$

vertices, what is divisible by 4. This completes the proof.

A second proof of Theorem 4. We consider a W^n with 21 blocks. The first block has the parameters

$$q_1=q, \quad d_1=m, \quad a_1=qm \quad \text{and} \quad p_1=3.$$

Let $\tilde{\mathbf{Q}}_\tau$ denote the subgraph of W^n spanned by $\mathbf{Q}_\tau$ and the vertices $E_\tau(i)$, $F_\tau(j)$, $G_\tau(k)$, $H_\tau(l)$. The number of vertices of $\tilde{\mathbf{Q}}_1$ is

$$(25) \qquad 3q(qm+q)+3qm+qm+4(q+3) = 3q^2m+7qm+4q+12 =: f(q, m)$$

The proof of Theorem 6 shows that if n_1 is a sufficiently large even integer: $n_1 > n_0$, then the other parameters can be chosen so that the number of vertices of the blocks $\tilde{\mathbf{Q}}_2, \dots, \tilde{\mathbf{Q}}_{20}$ is exactly n_1. Let us choose q so that

$$f(q, m) \leqq n - n_0 \leqq f(q+1, m).$$

In this case

$$0 \leqq (n-n_0) - f(q, m) \leqq f(q+1, m) - f(q, m) = 6mq + 10m + 4.$$

If the parameters of the other blocks are chosen suitably, the graph W^n has exactly n vertices. Since the second story of $\mathbf{Q}_1$ contains $3q^2m$ independent vertices of valence $\geqq m$, we conclude

$$(26) \qquad n - i(4, n, m) = O(qm) + n_0 = O\left(\sqrt{nm}\right)$$

(here we applied $qm = \sqrt{q^2m} \cdot \sqrt{m} = O\sqrt{nm}$.)

Final remarks. Theorem 4 holds not only for even integers, e.g. the first proof contains a construction having odd number of vertices. If we consider the same block $\mathbf{Q}$ as in the first proof of Theorem 4 and $E_1, E_2, \dots, E_q, F_1, \dots, F_s, G_1, \dots, G_t$ are new vertices, where $\{G_k\}_k$ span an odd circuit, F_l is joined to one of the vertices G_k and each F_l is joined to each E_i, finally E_i is joined to $D(i, 1)$, then we obtain a graph $\tilde{S}^n$ containing a 4-critical subgraph S^n, which proves Theorem 4 for every sufficiently large n.

It would be interesting to have some non-trivial upper bounds for (B). Finally I remark that the multiplicative constants can be improved in many statements of this paper. Since the order of magnitude of the upper and lower bounds are different, I was not interested in the constants.

REFERENCES

[1] Ore, O.: *Theory of graphs.* Amer. Math. Soc. Coll. Publ. Vol. XXXVIII. 1962, § 14. 3.
[2] Dirac, G. A.: Circuits in critical graphs, *Monatsh. Math.* **59** (1955), 178—187.
[3] Brown, W. G. and Moon, J. W.: Sur les ensemble de sommets indépendants dans les graphe chromatiques minimaux. *Can. J. Math.* **21** (1969), 274—278.
[4] Toft, B.: Two theorems on critical 4-chromatic graphs (the next paper in this volume).
[5] Erdős, P.: On extremal problems of graphs and generalised graphs. *Israel J. of Math.*, **2** (1964), 184—190.
[6] Toft, B.: On the maximal number of edges of critical k-chromatic graphs. *Studia Sci. Math. Hungar.* **5** (1970), 461—470.

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