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Note

On the minimum degree forcing F-free graphs to be (nearly) bipartite

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Abstract

Let $\beta(G)$ denote the minimum number of edges to be removed from a graph G to make it bipartite. For each 3-chromatic graph F we determine a parameter $\xi(F)$ such that for each F-free graph G on n vertices with minimum degree $\delta(G) \geqslant 2n/(\xi(F)+2)+o(n)$ we have $\beta(G)=o(n^2)$, while there are F-free graphs H with $\delta(H) \geq \lfloor 2n/(\xi(F)+2) \rfloor$ for which $\beta(H)=\Omega(n^2)$.

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1. Introduction

A well-known theorem of Turán [13] states that a K_{p+1}-free graph on n vertices with maximum number of edges is p-partite (for $p=2$ this fact was already proved by Mantel [8]). This result was subsequently generalized in different directions (see Bollobás' excellent monograph [2] devoted to this subject, or the surveys [10,11]).

The Erdős–Simonovits theorem [4], easily following from the Erdős–Stone theorem [6], generalizes Turán's theorem for all graphs F with chromatic number $p+1$. It says that for any $\eta > 0$, any graph G with n vertices and at least $(1-1/p)\binom{n}{2}+\eta n^2$ edges contains F, provided n is large enough. Hence, the 'critical' density of F-free graphs depends only on the chromatic number $\chi(F)$ of F. Erdős and Simonovits [3,9] showed also that for such an F and every $\alpha > 0$ there exists an $\eta(\alpha) > 0$ such that for n large enough every graph G with n vertices and at least $(1-1/p)\binom{n}{2}-\eta n^2$ edges can be made p-partite by omitting at most αn^2 edges.

For $F=K_{p+1}$ the chromatic number of dense F-free graphs has been investigated in more detail. The main result on this problem is due to Andrásfai et al. [1] who proved that

$$\max\{\delta(G):\, v(G)=n,\, \chi(G)\geqslant p+1,\, K_{p+1}\nsubseteq G\} = \left(1-\frac{1}{p-(1/3)}\right)n+\mathrm{O}(1), \tag{1}$$

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where $v(G)$ denotes the number of vertices of a graph G. Observe that in the above result the 'density' of a graph G is measured by its minimum degree $\delta(G)$ rather than the number of edges. This approach is motivated by the fact that if we try to maximize the number of edges in the K_{p+1}-free graphs, $p \geqslant 2$, then the additional condition that $\chi(G)$ is large decreases the number of edges only by $O(n)$ (for instance one can boost the chromatic number of a K_{p+1}-free graph by adding to it a triangle-free component of large chromatic number). For the minimum-degree problem, the change in the critical value is more noticeable and leads to a non-trivial structural result. Erdős and Simonovits [5] generalized (1) to arbitrary F with at least one critical edge, i.e. with $e \in E(F)$ for which $\chi(F - e) < \chi(F)$.

In this paper we study the conditions under which, for a given (small) 3-chromatic graph F, all F-free graphs with high minimum degree can be made bipartite by omitting $o(n^2)$ edges. The main result of this note, Theorem 1, states that for every $\alpha > 0$ and $\eta > 0$ there exists n_0 such that if the minimum degree of a F-free graph G on $n > n_0$ vertices is greater than $a(F)n + \alpha n$, for an appropriate parameter $a(F) > 0$, then G can be made bipartite by omitting at most ηn edges. The lower bound for $\delta(G)$ is, basically, best possible, since there is a family of 'blown-up odd cycles' H_n, such that H_n is F-free, has minimum degree $\delta(H_n) = \lfloor a(F)n \rfloor$, and cannot be made bipartite by removing fewer than $\Omega(n)$ edges. In fact, we show that our result is 'stable' and each 'extremal' large graph G is 'close' to such a 'blown-up odd cycle' H_n. We also remark that, in the case of the general 3-chromatic graphs F, in the assertion of Theorem 1 ηn cannot be replaced by $n^{1-\varepsilon}$ for some $\varepsilon > 0$ (Example 5).

Finally, we mention that recently Győri et al. [7] studied dense F-graphs with large minimum degree when F is an odd cycle, obtaining much more precise information on the structure of extremal graphs in this special case.

2. Notation

All graphs we consider in this paper, usually denoted by F, G, H with or without sub- and superscripts, are simple graphs without loops and multiple edges. For a graph G, by $V(G)$ we denote the set of its vertices, by $E(G)$ the set of the edges of G, $v(G) = |V(G)|$, $e(G) = |E(G)|$, and $\delta(G)$ stands for the minimum degree of G. We write $F \subseteq G$ when F is a subgraph of G, not necessarily induced. If $F \nsubseteq G$, then G is F-free.

A graph G can be homomorphically mapped into F, if there exists a map $\phi : V(G) \to V(F)$ such that for each $\{v, w\} \in E(G)$ we have also $\{\phi(v), \phi(w)\} \in E(F)$; in such a case we write $G \to F$. For a graph F and a natural number m, a new graph $F^{(m)}$ is obtained from F by 'blowing-up' each vertex to the size m, i.e. by replacing the vertices $v_1, \ldots, v_n$ of F by sets $V_1, \ldots, V_n$, $|V_1| = \cdots = |V_n| = m$, and each edge $\{v_i, v_j\} \in E(F)$ by a complete bipartite graph $K_{m,m}$ with bipartition (V_i, V_j). Therefore, for instance, $F^{(m)} \to F$ and the chromatic number $\chi(F)$ of F can be defined as

$$\chi(F) = \min\{p : F \to K_p\} = \min\{p : F \subseteq K_p^{(v(F))}\}.$$

Observe that if G can be homomorphically mapped into H, and H can be homomorphically mapped into F, then, clearly, G can be homomorphically mapped into F. Thus, for instance, if G is homomorphically mapped into an odd cycle C_{2k+1}, $k \geqslant 2$, then it can also be homomorphically mapped into C_{2k-1}. We also remark that a bipartite G can be homomorphically mapped into K_2 and so it can be homomorphically mapped into every non-empty graph, in particular, into any odd cycle.

For a 3-chromatic graph F we define another parameter $\xi(F)$ setting

$$\begin{aligned}\xi(F) &= \max\{k : k \text{ is odd and } F \to C_k\}\\ &= \max\{k : k \text{ is odd and } F \subseteq C_k^{(v(F))}\}.\end{aligned}$$

Note that $\xi(F)$ cannot be larger than $g_{\mathrm{odd}}(F)$, the length of the shortest odd cycle contained in F.

Finally, by $\beta(G)$ we denote the minimum number of edges that must be deleted from G to make it bipartite.

3. Main theorem

The main result of this note can be stated as follows.

Theorem 1. *Let F be a 3-chromatic graph. Then for every $\alpha, \eta > 0$, there exists an n_0 such that for every F-free graph G with $v(G) = n \geqslant n_0$ and*

$$\delta(G) \geqslant \left\lfloor \frac{2n}{\xi(F)+2} \right\rfloor + \eta n, \tag{2}$$

we have $\beta(G) \leqslant \alpha n^2$.

Furthermore, for every $\alpha > 0$ there exist an $\bar{\eta} > 0$ and an $\bar{n}_0$ such that each F-free graph G with $v(G) = n \geqslant \bar{n}_0$ and

$$\delta(G) \geqslant \left\lfloor \frac{2n}{\xi(F)+2} \right\rfloor - \bar{\eta} n, \tag{3}$$

contains a subgraph G' with at least $e(G) - \alpha n^2$ edges such that $G' \to C_{\xi(F)+2}$.

The proof of Theorem 1 is based on the following simple observation on graphs not containing short odd cycles (cf. [1]).

Lemma 2. *Let G be a graph on n vertices not containing odd cycles shorter than $2\ell + 2$.*

(i) *If*

$$\delta(G) > \left\lceil \frac{2n}{2\ell+3} \right\rceil, \tag{4}$$

then G is bipartite.

(ii) *If*

$$\delta(G) > \left\lceil \frac{2n}{2\ell+3} \right\rceil - \eta n, \tag{5}$$

for some η, $0 < \eta < 1/(18\ell^2)$, then G contains an induced subgraph G' such that $G' \to C_{2\ell+3}$ and $v(G') \geqslant v(G) - (2\ell+3)(\eta n + 3)$, so that $e(G') \geqslant e(G) - (2\ell+3)(\eta n + 3)n$.

Proof. Let us assume that G is not bipartite and let $C_{2k+1} = v_1 v_2 \dots v_{2k+1} v_1$, $k \geqslant \ell + 1$, be a shortest odd cycle contained in G. Then, clearly,

$$e(V(C_{2k+1}), V(G) \setminus V(C_{2k+1})) \geqslant (2k+1)(\delta(G) - 2). \tag{6}$$

On the other hand, since C_{2k+1} is the shortest odd cycle in G, each vertex from $v \in V(G) \setminus V(C_{2k+1})$ must have at most two neighbors in C_{2k+1} and so

$$e(V(C_{2k+1}), V(G) \setminus V(C_{2k+1})) \leqslant 2(n - (2k+1)). \tag{7}$$

Note that if (4) holds, then (6) contradicts (7), so (i) follows.

Now, let us consider the case when the minimum degree of G fulfills the weaker condition (5). If v_1 denotes the number of vertices adjacent to exactly one vertex of C_{2k+1} then, from (6),

$$v_1 + 2(n - v_1) \geqslant (2k+1)(\delta(G) - 2) + (2k+1),$$

and so

$$v_1 \leqslant \frac{4n(\ell - k + 1)}{2\ell + 3} + (2\ell+3)(\eta n + 3).$$

Consequently, we infer that $k = \ell + 1$ and all but at most $(2\ell+3)(\eta n + 3)$ vertices of G are adjacent to two vertices of $C_{2k+1} = C_{2\ell+3}$. Note also that if $v \in V(G) \setminus V(C_{2k+1})$ has two neighbors in $C_{2\ell+3}$, then, to avoid odd cycles shorter than $2\ell + 3$, they must lie at distance two in $C_{2\ell+3}$. Hence, all but $(2\ell+3)(\eta n + 3)$ vertices of G can be partitioned into sets $V_1, \dots, V_{2\ell+3}$, where each $w_i \in V_i$ is adjacent to $v_{i-1}, v_{i+1} \in V(C_{2\ell+3})$ (here and below we add modulo

$2\ell+3$). Since $g_{\text{odd}}(G)=2\ell+3$, the only edges contained in $V_1\cup\cdots\cup V_{2\ell+3}$ are those joining V_i and V_{i+1} for some $i=1,2,\ldots,2\ell+3$. Hence, for the subgraph G' spanned in G by $V_1\cup\cdots\cup V_{2\ell+3}$ we have $G'\to C_{2l+3}$. □

Proof of Theorem 1. Since the proof is based on a standard application of the Regularity Lemma [12], here we only sketch the argument.

We start with a simple observation. Consider a graph H whose vertex set can be partitioned into $W_1,\ldots,W_k$, where k is an odd number not larger than $\xi(F)$, $|W_i|=m$, and the bipartite graph induced by each pair (W_i,W_{i+1}) is ε-regular and contains at least $\sqrt{\varepsilon}m^2$ edges. One can easily argue that for sufficiently large m and sufficiently small $\varepsilon=\varepsilon(k)>0$ such a graph always contains a copy of $C_k^{(v(F))}$ and thus a copy of F as well.

Now, let us take a sufficiently large F-free graph G that fulfills (2) and apply to it the Regularity Lemma with an $\varepsilon>0$ that has the above property and is much smaller than α and η. Consider the graph G^* with $v(G^*)>1/\varepsilon$, whose vertices are the sets of the partition and two sets are joined by an edge of G^* if they form an ε-regular pair with density at least $\sqrt{\varepsilon}$. Since G is F-free, a standard argument shows that the graph G^* contains no odd cycles of length $\xi(F)$ or shorter. We could choose $\varepsilon>0$ small enough so that $v(G)$ (and thus also $v(G^*)$) is sufficiently large and the difference between the densities of G^* and G is much smaller than η given in (2), say, smaller than η^5. But then, G^* contains an induced subgraph $\hat{G}$, with at least $(1-\eta^2)v(G^*)$ vertices and the minimum degree at least $(2/(\xi(F)+2)+\eta/2)n$. Now, the first part of the assertion follows from Lemma 2(i). The second part of the assertion can be deduced from Lemma 2(ii) in a similar way. □

4. Final remarks

An edge $e\in E(F)$ is critical, if $\chi(F-e)<\chi(F)=3$. We note that for 3-chromatic graphs F with at least one critical edge the value of $\xi(F)$ is equal to a 'more natural' parameter: the length of the shortest odd cycle $g_{\text{odd}}(F)$.

Claim 3. *If* $\chi(F)=3$ *and* F *has a critical edge, then* $\xi(F)=g_{\text{odd}}(F)$.

Proof. Let $g_{\text{odd}}(F)\geqslant 2\ell+1$ and let $e=\{v,w\}$ be a critical edge of F. Clearly, without loss of generality, we can assume that F is connected. We partition the set of vertices of F into sets $U_0=\{v\},U_1,\ldots,U_{2\ell-1},U_{2\ell}=\{w\}$ in the following way. The sets U_i, $i=1,\ldots,2\ell-3$, are the vertices that lie at the distance i from v in $F-e$. A vertex $x\in V(F)\backslash\bigcup_{i=1}^{2\ell-3}U_i$ we put in $U_{2\ell-2}$ if its distance from w in $F-e$ is even; otherwise it goes to $U_{2\ell-1}$. From the condition $g_{\text{odd}}(F)\geqslant 2\ell+1$ it follows that each of the sets U_i, $i=0,1,\ldots,2\ell-3$ is non-empty. Moreover, one can easily verify that each edge of F different from e joins U_i,U_{i+1} for some $i=0,\ldots,2\ell-1$, and so $F\to C_{2\ell+1}$. Now, to complete the proof, it is enough to observe that for no k such that $2k+1<g_{\text{odd}}(F)$ we have $F\to C_{2k+1}$. □

On the other hand, we remark that we may have $\xi(F)=3$ even for graphs F with arbitrarily large girth.

Example 4. Note that there exists a 3-partite graph $F(m;k)$ with vertex set $V_1\cup V_2\cup V_3$, where $|V_1|=|V_2|=|V_3|=m$ such that $F(m;k)$ contains no cycles shorter than k but each pair of subsets $W_i\subseteq V_i$, $W_j\subseteq V_j$, $i\neq j$, $|W_i|,|W_j|\geqslant 0.01m$, is joined by an edge. Then $g_{\text{odd}}(F(m;k))\geqslant k$ but, clearly, $\xi(F)=3$. The existence of $F(m;k)$ follows from an elementary probabilistic argument. It is enough to generate a tripartite random graph putting each edge with probability, say, $\log m/m$, and then remove from it all the edges which belong to cycles shorter than k; with positive probability we end up with a graph $F(m;k)$ which has all required properties.

It is easy to observe that the assertion of Theorem 1 does not hold if we set in (2) $\alpha=0$. Indeed, the appropriately 'blown-up' cycle $C_{\xi(F)+2}$ on n vertices has the minimum degree $\lfloor 2n/(\xi(F)+2)\rfloor$, but in order to make this graph bipartite one needs to delete at least $\lfloor n/(\xi(F)+2)\rfloor^2$ edges. It is perhaps slightly more surprising that, as the following example shows, one cannot replace $\beta(G)\leqslant\alpha n$ by $\beta(G)\leqslant n^{1-\varepsilon}$ for any $\varepsilon>0$, even if F contains critical edges.

Example 5. For given integers ℓ, and m, let $F(2\ell+1,m)$ be a graph with vertex set $V(F(2\ell+1,m))=\bigcup_{i=1}^{2\ell+1}V_i$, where $|V_1|=|V_2|=1$, $|V_i|=m$ for $i=3,\ldots,2\ell+1$, and each of the pairs $(V_1,V_{2\ell+1})$ and (V_i,V_{i+1}), $i=1,\ldots,2\ell$, spans a complete bipartite graph. By $G(n;m)$ we denote a graph whose vertex set consists of seven sets $W_1,\ldots,W_7$, where $|W_i|\geqslant\lfloor n/7\rfloor$, for $i=1,2,\ldots,7$. The pairs (W_1,W_2), (W_2,W_3), (W_3,W_4), (W_5,W_6), and (W_6,W_7) induce complete bipartite graphs; the pairs (W_4, W_5) and (W_1, W_7) span $K_{m,m}$-free bipartite graphs B with minimum degree $\Omega(n^{1-2/(m+1)})$. The existence of the bipartite graphs B follows easily from the probabilistic method, provided n is large enough. Indeed, it is enough to generate a random bipartite graph $\mathbf{G}$ of size $2\lfloor n/7 \rfloor$, in which probability of an edge is $cn^{-2/(m+1)}$ for some small constant $c > 0$, and check that in such a graph, with probability at least $1/2$ when n is large enough, every vertex is incident to at least $0.01cn^{1-2/(m+1)}$ edges which are contained in no copies of $K_{m,m}$. Thus, removing all copies of $K_{m,m}$ from $\mathbf{G}$ leads to a graph B which, with positive probability, fulfills our requirements.

Now note that $G(n; m)$ is an $F(2\ell + 1, m)$-free graph with minimum degree larger than $n/7$, but to make it bipartite one has to omit $\Omega(n^{2-2/(m+1)})$ edges.

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