How to deduce sharp extremal graph results from general theorems?

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Corrected and slightly extended version of the lecture slides.

Introduction

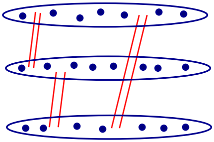
www.renyi.hu/~p_erdos = Erdős papers up to 1989 See also [0]
www.renyi.hu/~p_miki (several related surveys)

Today: Ordinary graphs, no loops, no multiple edges

Turán type extremal graph problems

Given a family $\mathcal{L}$ of EXCLUDED GRAPHS,

$$\mathbf{ex}(n, \mathcal{L}) := \max\{e(G_n) \ : \ L \not\subseteq G_n \text{ if } L \in \mathcal{L}\}.$$



Notation: G_n, K_p, ... $T_{n,p}$

$T_{n,p}$: Turán graph of n vertices and p classes.
In case of graphs the subscript: mostly the number of vertices
$\mathbf{ex}(n, \mathcal{L})$,

The family of extremal graphs: $\mathbf{EX}(n, \mathcal{L})$.

How had extremal graph theory started?

1. Mantel, (1907) $\mathbf{ex}(n, K_3) = \left[\frac{n^2}{4}\right]$
2. Erdős (1938), Multiplicative Sidon problem, [0]
   Eszter Klein construction
   = first finite geometry construction in Extremal Graph Theory
3. TURÁN THEOREM, Turán problems (1941): when we exclude a not necessarily complete L. BREAKTHROUGH
4. TURÁN'S QUESTIONS: What if we exclude a path P_k?
   Answer: Erdős-Gallai theorem. What if we exclude a graph of a Platonic body?
5. Erdős-Stone (1946)
6.

Kővári-Sós-Turán / Erdős (1954)

$$\mathbf{ex}(n, K_{a,b}) \le \frac{1}{2}\sqrt[a]{b-1}\cdot n^{2-(1/a)} + O(n).$$

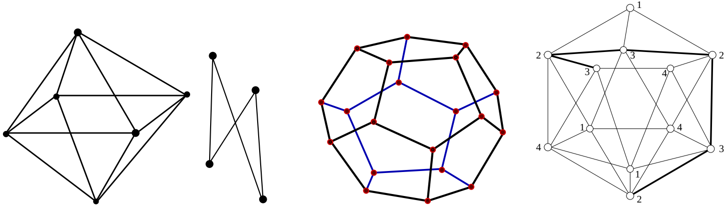
Other directions in extremal graph theory:

1. Hypergraph versions: Mostly very difficult,
2. Hamiltonicity, or other spanning subgraphs
   (Dirac theorem, Pósa theorem, . . .)
3. Further Universes, e.g. digraphs,. . .
   Brown-Erdős-Sim.
4. Ramsey-Turán (survey: Sim.-Sós [0])
5. Anti-Ramsey

Turán's problems: Platonic bodies

What happens if we exclude the graphs of the Platonic bodies?

1. Tetrahedron, (Turán)
2. Octahedron: Erdős-Simonovits:



3. Cube: Erdős-Simonovits, this is perhaps the most difficult Turán problem, because of the missing lower bound.
4. Dodecahedron: Simonovits
5. Icosahedron: This lead to the results primarily discussed in this lecture.

\+ Paths,. . . : Erdős-Gallai [0]

More explicitly

Cube theorem, Erdős-Sim.:

$$\mathbf{ex}(n, Q_8) = O(n^{8/5}).$$

Erdős-Sim.: Is this sharp? For some $c_Q > 0$, is

$$\mathbf{ex}(n, Q_8) \geq c_Q(n^{8/5}).$$

We do not even know e.g. that $\frac{\mathbf{ex}(n,Q_8)}{n^{3/2}} \to \infty$.

Dodecahedron theorem, Sim.:

For $n > n_0$

$$\mathbf{EX}(n, D_{20}) = \{H(n, 2, 6)\}$$

where $H(n, p, s) = K_{s-1} \otimes T_{n-s+1,p}$.

Large part of the theory asserts that
the general case is very similar to the case of Turán's theorem.

The Multiplicative Sidon problem

Erdős, 1938:

Problem: Multiplicative Sidon [0]

Assume that $a_1, \ldots, a_m \in [1, n]$ are integers satisfying the MULTIPLICATIVE SIDON condition: all the pairwise products are different, in the sense that

$$\text{if } a_i a_j = a_k a_\ell \text{ then } \{i, j\} = \{k, \ell\}. \tag{1}$$

How large can m be?

Connected to

Lemma:

If $G_n \subseteq K(n, n)$ and $C_4 \not\subseteq G_n$ then $e(G_n) \leq 3n\sqrt{n}$.

Generalizations, Erdős, A. Sárközy, Sós / Győri:

They connect certain number theory questions to $\mathbf{ex}(n, C_6)$ . . .

Stability results

Given a class of excluded graphs, $\mathcal{L}$, with

$$p := \min_{L \in \mathcal{L}} \chi(L) - 1.$$

Erdős-Sim. Theorem [0]

$$\mathbf{ex}(n, \mathcal{L}) = e(T_{n,p}) + o(n^2).$$

$\rho(G, H)$= Hamming distance of G and H:

then minimum number of edges one has to add to or delete from G to get a G_n^* isomorphic to H.

Erdős-Sim. Theorem [0],[0],[0]

For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if G_n is $\mathcal{L}$-free and

$$e(G_n) > \mathbf{ex}(n, \mathcal{L}) - \delta n^2$$

then $\rho(G_n, T_{n,p}) < \varepsilon n^2$.

This applies to the extremal and almost extremal graphs.

Extremal graphs

If $S_n \in \mathbf{EX}(n, \mathcal{L})$ then

$$d_{\min}(S_n) \geq \left(1 - \frac{1}{p}\right) n - o(n).$$

This does not apply to the almost extremal graphs, (since deleting edges the min degree can be pushed down).

Decomposition classes

Definition: Decomposition

Given $\mathcal{L}$, $p = p(\mathcal{L})$, we define the decomposition class $\mathcal{M}$ of $\mathcal{L}$ as the family of graphs M for which $M \otimes K_{p-1}(t, \ldots, t)$ contains some forbidden graph.

Examples:

The octahedron graph $K(2,2,2)$ has C_4 in its decomposition class.

Octahedron Theorem

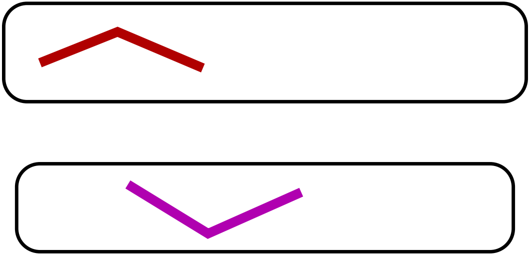
We start with an illustration. Let $O_6 = K(2,2,2)$ be the octahedron graph.

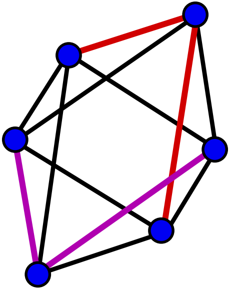
Octahedron Theorem, Erdős and Sim. [0]

If S_n is an extremal graph for the octahedron O_6 for n sufficiently large, then there exist extremal graphs G_1 and G_2 for the circuit C_4 and the path P_3 such that $S_n = G_1 \otimes G_2$ and $|V(G_i)| = \frac{1}{2}n + o(n)$, $i = 1,2$.
If G_1 does not contain C_4 and G_2 does not contain P_3, then $G_1 \otimes G_2$ does not contain O_6.
Thus, if we replace G_1 by any $H_1 \in \mathbf{EX}(v(G_1), C_4)$ and G_2 by any $H_2 \in \mathbf{EX}(v(G_2), P_3)$, then $H_1 \otimes H_2$ is also extremal for O_6.

Path-Path excluded:

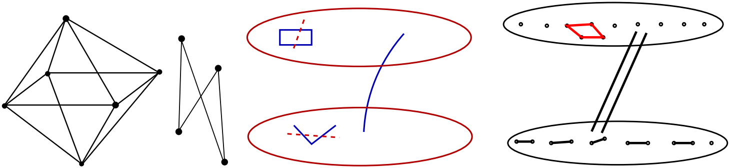
In proving the Octaherdon theorem, it is important, that putting a P_3 into both classes of a complete bipartite graph we get an Octahedron, though P_3 is not in the Decomposition class.
So the Decomposition class does not determine the extremal structure completely.





Generalized Octahedron theorem [0]

If S_n ix extremal for $K(a, b, c, \dots, r_p)$ then the vertices of S_n can be partitioned into p classes, $\mathcal{C}_1, \dots, \mathcal{C}_p$ so that $\mathcal{C}_1$ contains no $K(a, b)$, $\mathcal{C}_2, \dots, \mathcal{C}_p)$ contains no $K(1, c)$.



Examples: $K_3(a, b, c)$, $a \leq b \leq c$

$K(a, b)$ is the important graph in $\mathcal{M}$.

Erdős-Gallai, Erdős, Moon, Sim.

Examples: Moon theorem, s vertex-disjoint K_{p+1}

s independent edges form the important graph in $\mathcal{M}$.

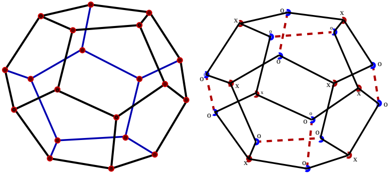
More examples on the decomposition

Dodecahedron Theorem: $\mathcal{L} = \{D_{20}\}$

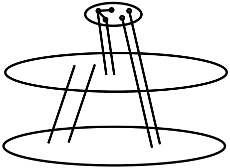
For $n > n_0$ $H(n, 2, 6)$ is the (only) extremal graph for D_{20}

Lemma: Deleting 6 independent edges we can obtain a bipartite graph.

So the Decomposition class $\mathcal{M} = 6K_2$ =6 independent edges



Dodecahedron graph



The extremal structure

We need

Lemma:

Deleting 5 vertices of D_{20} we cannot obtain a bipartite graph.

Error terms in Erdős-Sim. Stability theorem

The error terms primarily depend on the decomposition class, i.e. on $\mathbf{ex}(n/p, \mathcal{M})$:

(a) Since we can put an $\mathcal{M}$-extremal graph into the first class of a $T_{n,p}$, and the resulting graph contains no $L \in \mathcal{L}$, therefore

$$\mathbf{ex}(n, \mathcal{L}) \geq e(T_{n,p}) + \mathbf{ex}(n/p, \mathcal{M}).$$

(b) The upper bound is

$$\mathbf{ex}(n, \mathcal{L}) \leq e(T_{n,p}) + p \cdot \mathbf{ex}(n/p, \mathcal{M}) + O(n).$$

Octahedron graph theorem

Octahedron Theorem: [0]

If S_n is an extremal graph for the octahedron O_6, for n sufficiently large, then there exist extremal graphs G_1 and G_2 for the circuit C_4 and the path P_3 such that $S_n = G_1 \times G_2$ and $|V(G_i)| = \frac{1}{2}n + O(n)$, $i = 1, 2$. ∎

The (k, ℓ)-problem: G_n has fewer than ℓ edges on any k-vertex subgraph.

The (6,12)-theorem: Griggs, Sim. Thomas

If S_n is an extremal graph for the (6,12)-problem, for n sufficiently large, then there exist extremal graphs G_1 and G_2 for the circuits C_3, C_4 and the path P_2 (!) such that $S_n = G_1 \times G_2$ and $|V(G_i)| = \frac{1}{2}n + O(n)$, $i = 1, 2$. ∎

Füredi-Sim. . .

Griggs-Sim.-Thomas thm

There are many similar results where the family $\mathcal{L}_{k,\ell}$ of excluded graphs are the graphs of k vertices and ℓ edges, (earlier e.g. Griggs-Sim.-Thomas thm, see [0] and recenty e.g. Füredi and Simonovits (manuscript))

Griggs-Sim.-Thomas

If S_n is an extremal graph for $\mathcal{L}_{6,12}$ for n sufficiently large, then there exist extremal graphs G_1 and G_2 for the circuits $\{K_3, C_4\}$ and the path P_2 such that $S_n = G_1 \otimes G_2$ and $|V(G_i)| = \frac{1}{2}n + o(n)$, $i = 1, 2$.

The product conjecture

Let $\mathcal{L}$ be finite. If

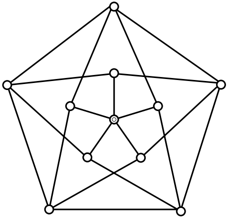
$$\mathbf{ex}(n, \mathcal{L}) > e(T_{n,p}) + cn^{1+\gamma}$$

for some $c > 0$ and $\gamma > 0$ then for each $n > n_0$ the extremal graphs are of product forms: the vertices of $S_n \in \mathbf{EX}(n, \mathcal{L})$ can be partitioned into p classes $\mathcal{C}_i$ of roughly the same size, so that any two vertices x, y from distinct classes are connected to each other.

Corollary, Reduction

The general extremal graph problems can be reduced to degenerate extremal problems.

Critical edge theorem

Odd cycles, 

Critical edge theorem

The following assertions are equivalent:

(a) For $n > n_0$ $T_{n,p}$ is extremal graph for $\mathcal{L}$.

(b) For $n > n_1$ $T_{n,p}$ is the only extremal graph for $\mathcal{L}$.

(c) There exist an $L \in \mathcal{L}$ with $\chi(L) = p + 1$, and an edge e in it, such that $\chi(L - e) = p - 1$.

The critical edge principle

If in ordinary extremal graph problems we can prove a theorem for $\mathcal{L} = \{K_p\}$ then probably we can prove it for any case where in a finite family $\mathcal{L}$ of excluded graphs there is a $p+1$-edge-colour-critical L.

Examples

1. Andrásfai-Erdős-Sós [1]
   $\rightarrow$ Erdős–Simonovits [0]
2. Erdős-Kleitman-Rothschild thm [0]
   $\rightarrow$ Ph. G. Kolaitis, H. J. Prömel, and B. L. Rothschild [0]
3. Babai-Simonovits-Spencer [0]

The Smolenice paper

. . . It discusses among others extremal graph problems where the class $\mathcal{L}_{k,\ell}$ of excluded graphs consists of the k-vertex graphs L of at least ℓ edges.

. . .

In some sense this lead to the two important hypergraph papers of Brown, Erdős, and Sós [0] and [0].

The Ruzsa-Szemerédi theorem [0] answering a question of [0] led to the Removal Lemma.

Chromatic conditions

Erdős, Andrásfai, Gallai [0].

If G_n is K_3-free and is not bipartite, then

$$e(G_n) < \left[\frac{n^2}{4}\right] - \frac{n}{2} + O(1).$$

Andrásfai-Erdős-Sós generalizations [1]. Erdős-Sim. generalization. The simple CHROMATIC CONDITIONS are like

For given p, q the chromatic condition $\mathbf{Chrom_{p,q}}$ is that one cannot delete $\le q$ vertices from G to obtain a $\le p$-chromatic graph.

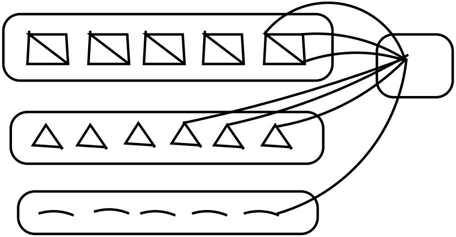
Extremal problems with chromatic conditions

We have a family $\mathcal{L}$ of forbidden graphs, and q, then we can optimize over the $\mathcal{L}$-free graphs $G_n \in \mathbf{Chrom_{p,q}}$.

Our results hold for these extremal problems, too.

Symmetric graph sequences

We have a class $\mathcal{L}$ of excluded graphs, the corresponding minimum chromatic number p and a parameter r.
We have p classes of vertices, $\mathcal{C}_i$ of roughly the same sizes, and a class R^* of exceptional vertices, where $|R^*| \le r$.
Into each class $\mathcal{C}_i$ of a $T_{n,p}$ we put some connected isomorphic graphs B_i of $\le r$ vertices, completely covering $\mathcal{C}_i$, and join each vertex of an "Exceptional class" R^*
to each block B_j **in the same way**.



The meaning of "in the same way"

More precisely:

(a) The blocks for different classes mostly are different, however, for the same class they are the same.

(b) in each class $\mathcal{C}_i$ the vertices of the blocks are labeled in the same way, by $1, \dots, |B_i|$

(c) If a $v \in R^*$ is joined to the j^{th} vertex of a block in B_i then it is joined the j^{th} vertex of each block $B_{j'} \subset \mathcal{C}_i$.

The simplest case

If all the blocks are K_1 (=one vertex graph), consider
$H(n,p,s) := K_{s-1} \otimes T_{n-s+1,p}$

The Turán graph $T_{n,p}$ can be characterized as

an n-vertex $\leq p$-chromatic graph with maximum number of edges.

Characterization: $H(n,p,s)$ is the n-vertex graph with the "chromatic property" that one can delete $< s$ vertices to get a $\leq p$-chromatic graph and maximum number of edges

Dodecahedron theorem

For $n > n_0$ $H(n,2,6)$ is the only extremal graph for D_{12}.

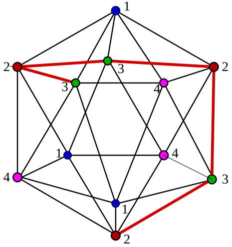
Ambiguity?

There is no problem with speaking of “corresponding vertices” if the blocks have no automorphisms, however, we have to be slightly more careful when some blocks have symmetries.
Therefore we mostly fix some automorphisms

$$\psi_{i,j} : B_{i,1} \to B_{i,j},$$

or the fixed “labeling”.

Path decomposition theorem

The icosahedron graph = 

Main theorem (when a path is in the decomposition class)

If $\mathcal{L}$ contains an L which can be $p+1$-coloured so that the first two classes form a graph contained by a path P_τ, then for $n > n_0(\mathcal{L})$ there exist extremal graphs $S_n \in \mathbb{G}(n, p, r)$.

Is it true that all for some r all the extremal graphs belong to $\mathbb{G}(n, p, r)$?

Characterization of all the extremal graphs

There is a theorem on this but here we skip it.

The tree decomposition conjecture

Can one extend the the path decomposition theorem to all the cases when the decomposition class $\mathcal{M}(\mathcal{L})$ contains some tree?

How to solve an extremal graph theorem with linear remainder term?

We consider here only finite $\mathcal{L}$.
The remainder term $\mathbf{ex}(n, \mathcal{L}) - \mathbf{ex}(n, K_{p+1})$ is $O(n)$ iff $\mathcal{M}(\mathcal{L})$ contains a tree.

1. Check if s independent edges belong to $\mathcal{M}$?
2. Next check if $P_\tau \in \mathcal{M}(\mathcal{L})$ or not?
3. If YES, then check if apply the general theorem.

The excluded Petersen graph

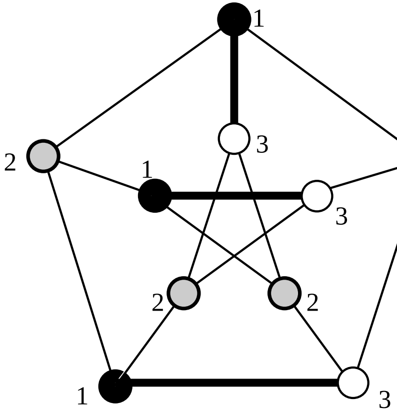


FIGURE 3:
Petersen graph

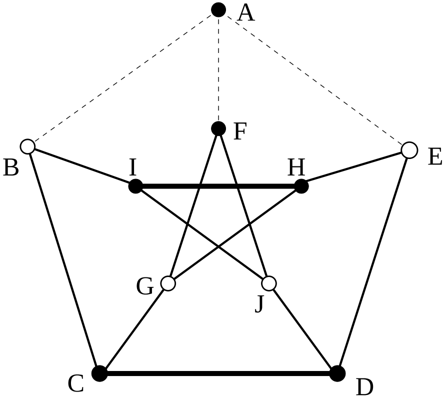


FIGURE 4:
Petersen Truncated

$\chi(\mathbb{P}_{10}) = 3$, the first and last colours span 3 independent edges (see Fig 3) so the decomposition class contains P_6.

Theorem: Petersen-Extremal graphs

For $n > n_0$ $H_{n,2,3}$ is the (only) extremal graph for the Petersen graph $\mathbb{P}_{10}$.

Follows from Theorem 2.2 of [0]:

Theorem: $H_{n,p,t}$-theorem

(i) Let $L_1, \dots, L_\lambda$ be given graphs with $\min \chi(L_i) = p+1$. Assume that omitting any $t-1$ vertices of any L_i we obtain a graph of chromatic number $\geq p+1$, but L_1 can be colored in $p+1$ colors so that the subgraph of L_1 spanned by the first two colors is the union of t independent edges and (perhaps) of some isolated vertices. Then, for $n > n_0(L_1, \dots, L_\lambda)$, $H_{n,p,t}$ is the (only) extremal graph.

(ii) Further, there exists a constant $C > 0$ such that if G_n contains no $L_i \in \mathcal{L}$ and

$$e(G_n) > e(H_{n,p,t}) - \frac{n}{p} + C,$$

then one can delete $t-1$ vertices of G_n so that the remaining G_{n-t+1} is p–colorable.

This theorem is strongly connected with the CRITICAL EDGE THEOREM. It is natural to ask if the uniqueness holds here as well or not:

Open problem

Is there a family $\mathcal{L}$ of forbidden graphs for which for $n > n_0$ $H(n,p,t)$ is extremal but it is not the unique extremal graph?

Remarks

The condition on L_1 is equivalent to that $L_1 \subseteq T_{m,p,t}$ for some m. One could also formulate this by saying that the decomposition class contains the graph consisting of t independent edges.

The meaning of (ii) is that the extremal structure is stable in some sense. To understand this stability better, we introduce the notion of chromatic properties, first only in its simplest form.

Definition, $\mathbb{B}_{p,t}$–property

We shall say that a graph has property $\mathbb{B}_{p,t}$ if one cannot delete $t-1$ vertices from it to get a p–colorable graph.

We shall not distinguish a property of graphs from the set of graphs having this property. If a graph $G \notin \mathbb{B}_{p,t}$ and $H \subseteq G$, then $H \notin \mathbb{B}_{p,t}$ either. To have such a property means that the graph is "big" in some sense, to not have means, that it is "small".

The case of the Nesetril graph

Theorem, Sim.

There exists an $n_0(N_{12})$ such that for $n > n_0$, $H_{n,2,2}$ is the (only) extremal graph.

Remark

Theorem 37 does not follow from Path Decomposition Theorem. Below we shall use the labeling of Figures 3-4.

(a) One can delete 3 independent edges, e.g. BC, EF, and IH to get a bipartite graph from N_{12} and the omission of 2 edges is obviously not enough.

(b) We could apply Path Decomposition Theorem if we could show that the omission of any 2 vertices leaves us with a 3-chromatic graph. The extremal graph would be $H_{n,3,2}$. However, this is not the case: the omission of A and L results a tree.

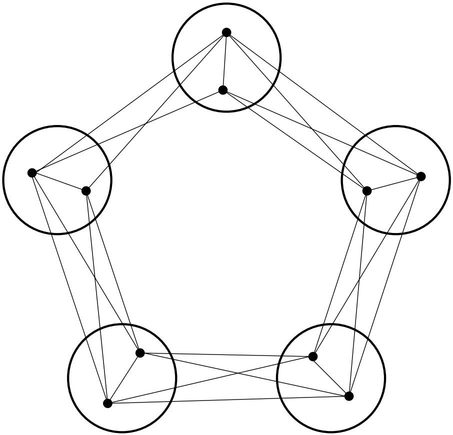


FIGURE 13:
The Łuczak graph

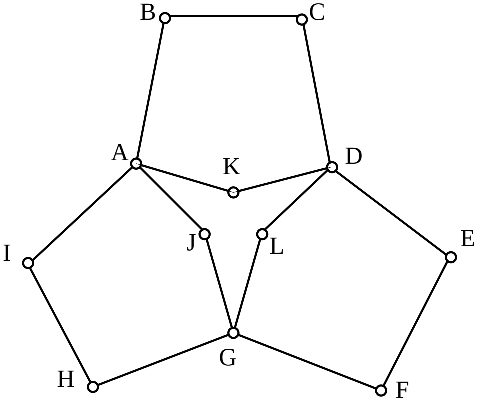


FIGURE 14:
Nešetřil Graph

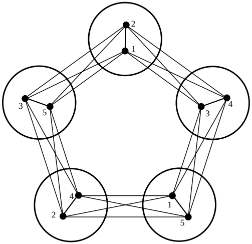
Luczak graph excluded

Problem: the Luczak graph is excluded

Determine $\mathbf{ex}(n, L_{10})$. What are the extremal graphs?

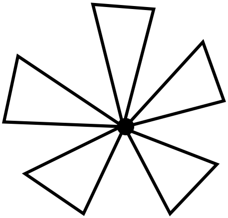
Theorem on Luczak graph

For L_{10}, $H_{n,4,2}$ is the (only) extremal graph, for $n > n_0(L_{10})$.



This is really easy: One can see that L_{10} is 5-chromatic and that removing any vertex of L_{10} it remains 5-chromatic, but one can color it in 5 colors so that the first two colors span 2 independent edges (see Figure 13). Hence we can apply our Main Theorem.

Erdős-Füredi-Gould-Gunderson [0]



The excluded graph is the Friendship graph $F^k = L_{2k+1}$ where k triangles have one common vertex.

The decomposition contains $kK_2 = k$ independent edges and also $K(1,k)$.

Earlier the exact value of **ex**(n, L) was known only for a few graphs L. In [0] the exact value of **ex**(n, F^k) is determined for any $n \geq 50k^2$.

The extremal graph is for n even obtained from a $T_{n,2}$ by putting two vertex-disjoint copies of K_k, for n=odd it is slightly more complicated and the result holds for any $n > 50k^2$.

MathSciNet: "The proof is quite technical and complicated, and essentially gives the uniqueness of the maximum F^k-free graph."

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