Lectures

Hypergraph Extremal Problems

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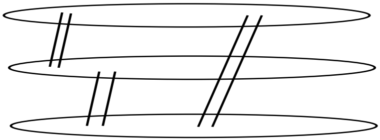
In Moscow, in October and November, 2015 at Moscow Institute of Physics and Technology, I gave five lectures, connected to each other in an involved way. These slides are not the actual lectures but selected/reorganized parts of those lectures with some additional information, sometimes indicated by $*$.

This "lecture" does not try to cover this extremely large theory:
I had to leave out many important parts, results, references!!!

Turán type extremal problems



Theorem (Turán, 1941)

If $e(G_n) > e(T_{n,p})$, then G_n contains a K_{p+1}. [1]

The general question:

- Given a Universe: graphs, hypergraphs, digraphs,. . .
- Given a forbidden family $\mathcal{L}$ of subgraphs, determine

$$\mathbf{ex}(n, \mathcal{L}) := \max\{e(G_n) \ : \ G_n \text{ contains no } L \in \mathcal{L}\}$$

and the graphs attaining this maximum. . . $\mathbf{EX}(n, \mathcal{L})$

Main setting: Universe

- Graphs
- Digraphs
- **Hypergraphs**
- Directed Multihypergraphs

Universe:

We fix some type of *structures*, like graphs, digraphs, or r-uniform hypergraphs, integers, and a family $\mathcal{L}$ of forbidden substructures, e.g. cycles C_{2k} of $2k$ vertices.

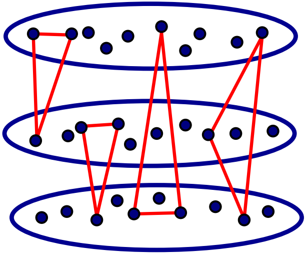
A Turán-type extremal (hyper)graph problem

asks for the maximum number $\mathbf{ex}(n, \mathcal{L})$ of (hyper)edges a (hyper)graph can have under the conditions that it does not contain any *forbidden* substructures.

Hypergraphs

Problem (Turán hypergraph conjecture)

The structure below is the extremal structure for $K_4^{(3)}$.



Problem (Turán conjecture)

The structure above, on the left is the extremal structure for $K_5^{(3)}$.

Hypergraphs

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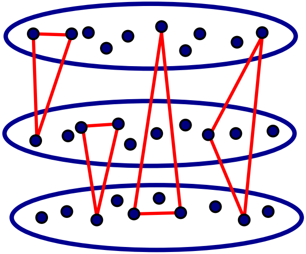
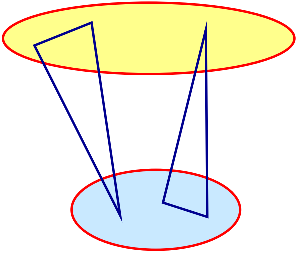




Problem (Turán conjecture)

The structure above, on the left is the extremal structure for $K_5^{(3)}$.

Some remarks *

Originally Turán thought that this hypergraph problem easy but it turned out one of the most difficult problems.
Why?

- Partly because there are many extremal graphs (Katona-Nemetz-Sim., W. G. Brown, Kostochka, ...)
- Partly because the "cutting into two parts" does not work
- and partly because hypergraph problems tend to be much more difficult than ordinary graph problems.

Digraph and multigraph extremal problems seem to behave as a bridge between ordinary and hypergraph problems.

Katona-Nemetz-Sim.: Convergent densities *

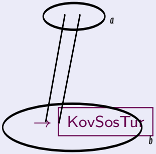
$$\frac{\mathbf{ex}(n, \mathcal{L})}{\binom{n}{r}}$$

is decreasing, therefore it is convergent.

Two important theorems

Kővári-T. Sós-Turán theorem. *Let $2 \leq a \leq b$ be fixed integers. Then*

$$\mathbf{ex}(n, K(a,b)) \leq \frac{1}{2}\sqrt[a]{b-1}\, n^{2-\frac{1}{a}} + \frac{1}{2}an.$$



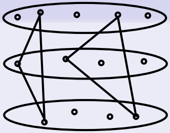
Definition: $K_r^{(r)}(m, \ldots, m) :=$

rm vertices are partitioned into r disjoing m-tuples and we take all the m^r r-tuples intersecting eacs m-tuple.

Theorem (Erdős, hypergaphs thm)

Consider r-uniform hypergaphs. For any fixed m,

$$\mathbf{ex}(n, K_r^{(r)}(m, \ldots, m)) = O(n^{r-(1/m^{r-1})}).$$



$\mathbf{ex}(n, L)$ is degenerate if L is bipartite

Theorem (Corollary of the Erdős-Kővári-T. Sós-Turán theorem)

$$\mathbf{ex}(n, \mathcal{L}) = o(n^2) \quad \textit{if and only if} \quad \min_{L \in \mathcal{L}} \chi(L) = 2.$$

2

This is also a Corollary of Erdős-Stone-Sim. theorem.

Definition

A hypergraph extremal problem (for k-uniform hypergraphs) is degenerate if

$$\mathbf{ex}(n, \mathcal{L}) = o(n^k).$$

For hypergraphs: Erdős hypergraph theorem

ex$(n, \mathcal{L}^r)$ is degenerate iff some $L \in \mathcal{L}$ has a *Strong r-colouring*:

r colours are used and each edge of L has r different colours.

Exercise Prove Erdős' hypergraph theorem. PEH

Exercise Prove that the problem of L is degenerate iff it can be k-colored so at each edge meats each of the k colors. HDeg

A “simple” unsolved problem

Definition

$\mathcal{L}_{r,k,\ell}$ is the family of forbidden r-uniform hypergraphs with k vertices and ℓ r-edges.

Problem (Brown-Erdős-Sós)

Determine

$$\mathbf{ex}(n, \mathcal{L}_{r,k,\ell}).$$

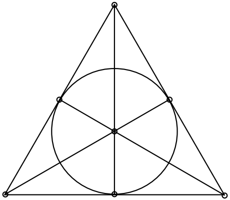
Problem (Erdős: 4-3 problem)

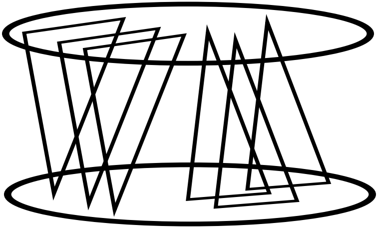
How many triples ensure the existence of $K_4^{(3)} - e$ in an n-vertex three-uniform hypergraph $\mathcal{H}_n^{(3)}$?

The T. Sós conjecture

Conjecture (V. T. Sós)

Partition $n > n_0$ vertices into two classes A and B with $||A| - |B|| \leq 1$ and take all the triples intersecting both A and B. The obtained 3-uniform hypergraph is extremal for $\mathcal{F}_7$.





The conjectured extremal graphs: $\mathcal{B}(X, \overline{X})$

Füredi-Kündgen Theorem

If M_n is an arbitrary multigraph (without restriction on the edge multiplicities, except that they are nonnegative) and all the 4-vertex subgraphs of M_n have at most 20 edges, then

$$e(M_m) \leq 3\binom{n}{2} + O(n).$$

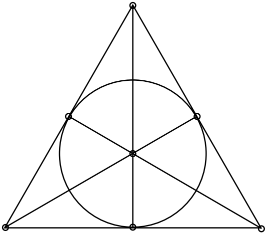
→ FureKund

Theorem (de Caen and Füredi)

→ *FureCaen*

$$\mathbf{ex}(n, \mathcal{F}_7) = \frac{3}{4}\binom{n}{3} + O(n^2).$$

New Results: The Fano-extremal graphs



Main theorem. *If $\mathcal{H}_n^{(3)}$ is a triple system on $n > n_1$ vertices not containing $\mathcal{F}_7$ and of maximum cardinality, then $\chi(\mathcal{H}_n^{(3)}) = 2$.*

$$\Longrightarrow \qquad \mathrm{ex}_3(n, \mathcal{F}_7) = \binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3}.$$

Keevash-Sudakov

Remark

The same result was proved independently, in a fairly similar way, by

Peter Keevash and Benny Sudakov → KeeSud .

Theorem (Stability)

There exist a $\gamma_2 > 0$ and an n_2 such that:
If $\mathcal{F}_7 \not\subseteq \mathcal{H}_n^{(3)}$ and

$$\deg(x) > \left(\frac{3}{4} - \gamma_2\right)\binom{n}{2} \text{ for each } x \in V(\mathcal{H}_n^{(3)}),$$

then $\mathcal{H}_n^{(3)}$ is bipartite, $\mathcal{H}_n^{(3)} \subseteq \mathcal{H}_n^{(3)}(X, \overline{X})$. → *FureSimFano*

Linear cycles?

We consider 3-uniform hypergraphs.

Conjecture (Gyárfás-G.N.Sárközy, [?])

One can partition the vertex set of every 3-uniform hypergraph H into $\alpha(H)$ linear cycles, edges and subsets of hyperedges.

Definition (Strong degree)

$\mathbf{d}^+(x) =$ *Maximum matching in the link of* x.

Theorem (Gyárfás, Győri, Sim.)

If $\mathbf{d}^+(x) \geq 3$, *then* $\mathcal{H}_n^{(3)}$ *contains a linear cycle.*

Theorem (Gyárfás, Győri, Sim.)

If $\mathcal{H}_n^{(3)}$ *does not contain a linear cycle, then* $\alpha(\mathcal{H}_n^{(3)}{}_n) < \frac{2}{5}n$

Hypergraphs, Continuation

Some new results:

Theorem (B. Ergemlidze, Győri, A. Methuku)

If $\mathcal{H}_n^{(3)}$ is linear cycle free 3-uniform hypergraph then max degree (classical definition) is at most $n-2$, what is sharp if $\mathcal{H}_n^{(3)}$ consists of all triples containing a given vertex.

Theorem (B. Ergemlidze, Győri, A. Methuku)

If $\mathcal{H}_n^{(3)}$ is linear cycle free 3-uniform hypergraph not containing K_5^3 then it is 2-colorable (and so $\alpha(\mathcal{H}_n^{(3)})$ is at least $\lceil n/2 \rceil$ what is sharp).

Theorem (Győri, N.Lemons, 2012)

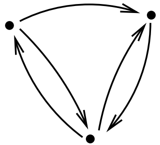
If $\mathcal{H}_n^{(3)}$ is a 3-uniform hypergraph with no Berge-cycle C_{2k+1} then $|E(\mathcal{H}_n^{(3)})|$ is at most $f(k)n^{1+1/k}$.

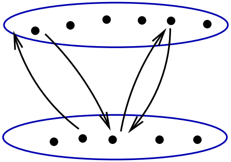
A digraph theorem

Digraph extremal problems are somewhere between hypergraph and ordinary graph extremal problems

They are often tools to solve Hypergraph extremal problems. We have to assume an upper bound s on the multiplicity. (Otherwise we may have arbitrary many edges without having a K_3.) Let $s = 1$.

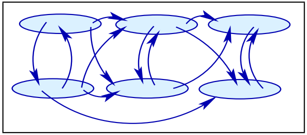
L:





$$\mathbf{ex}(n, L) = 2\mathbf{ex}(n, K_3) \qquad (n > n_0?)$$

Many extremal graphs: We can combine arbitrary many oriented double Turán graph by joining them by single arcs.



Further results?

See also the survey

A. F. Sidorenko: What do we know and what we do not know about Turán Numbers, Graphs Combin. 11 (1995), no. 2, 179–199.

We left out very many things, e.g.:

- results on Hamiltonian hypergraphs
- Codes,
- Covering, ...

Many thanks for your attention.