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Extremal Problems for Directed Graphs*

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We consider directed graphs without loops and multiple edges, where the exclusion of multiple edges means that two vertices cannot be joined by two edges of the same orientation. Let $L_1, ..., L_q$ be given digraphs.

What is the maximum number of edges a digraph can have if it does not contain any L_i as a subgraph and has given number of vertices?

We shall prove the existence of a sequence of asymptotical extremal graphs having fairly simple structure. More exactly:

There exist a matrix $A = (a_{i,j})_{i,j \leqslant r}$ and a sequence $\{S^n\}$ of graphs such that

(i) the vertices of S^n can be divided into classes $C_1, ..., C_r$ so that, if $i \neq j$, each vertex of C_i is joined to each vertex of C_j by an edge oriented from C_i to C_j if and only if $a_{i,j} = 2$; the vertices of C_i are independent if $a_{i,i} = 0$; and otherwise $a_{i,i} = 1$ and the digraph determined by C_i is a complete acyclic digraph;

(ii) S^n contains no L_i but any graph having $[\epsilon n^2]$ more edges than S^n must contain at least one L_i.

(Here the word graph is an "abbreviation" for "directed graph or digraph.")

NOTATION

The digraphs (= directed graphs) considered in this paper have neither loops nor multiple edges: a vertex cannot be joined to itself and the digraph cannot have two edges joining the vertices x and y and oriented

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from x to y; however, it can contain an edge oriented from x to y and another edge oriented from y to x. The word "digraph" sometimes will be replaced by "graph" where this cannot cause any confusion.

The number of vertices and edges of G will be denoted by $v(G)$ and $e(G)$ respectively. We shall also use upper indices to indicated the numer of vertices: thus G^n always denotes a graph of n vertices. The cardinality of a set E will be denoted by $|E|$.

1. Introduction

The first paper written on extremal digraphs was a joint paper of W. G. Brown and F. Harary [1]. They considered problems which were "digraph analogues of the now classical theorem of P. Turán [2, 3] which determines the maximum number of edges a graph may posses without containing a complete r-graph, K_r." Turán determined the maximum and also characterized the unique extremal graphs. The extremal graphs for K_r had a very simple structure: n vertices were divided into $r-1$ classes each of which contained $[n/(r-1)]$ or $[n/(r-1)]+1$ vertices and two vertices were joined iff they belonged to different classes. Later Erdös and Simonovits [4] proved that these graphs are asymptotic extremal graphs for every family $L_1, \ldots, L_q$ of sample graphs in the following sense:

Let $T(r, n)$ denote the extremal graph for K_r and $L_1, \ldots, L_q$ be given (undirected graphs). Let the chromatic number of each L_i be at least r, the chromatic number of L_1 be exactly r. Then $T(r, n)$ does not contain any L_i but if $n > n_0(\epsilon)$ and

$$e(G^n) > e(T(r, n))(1+\epsilon)$$

then G^n must contain at least one L_i.

P. Erdös and M. Simonovits have also proved independently [5–7] that (using the notations above) for $n > n_0(\epsilon)$ and $\delta > 0$ sufficiently small any G^n which contains no L_i and has at least

$$e(T(r, n))(1-\delta)$$

edges can be obtained from $T(r, n)$ by omitting fewer than ϵn^2 edges and adding at most ϵn^2 new edges.

Brown and Harary considered and solved some special cases of the following general

Problem 1. Let $L_1, \ldots, L_q$ be given digraphs. What is the maximum number of edges a digraph G^n (having n vertices) can have if it does not contain any L_i as a subgraph?

The maximum will be denoted by $f(n; L_1, ..., L_q)$ and the graphs attaining the maximum for a given n will be called extremal digraphs. The graphs $L_1, ..., L_q$ will be called sample digraphs. Problem 1 can also be generalized to infinite families of sample digraphs. If L is an infinite family of sample digraphs, $f(n; \mathsf{L})$ will denote the maximum.

In this paper we shall prove a general existence theorem according to which for every finite or infinite family of sample digraphs there exists a sequence of asymptotical extremal graphs each having a fairly simple structure. To formulate our theorem we need a few definitions.

DEFINITION 1. *Matrix graphs.* If $A = (a_{i,j})_{i,j\leqslant r}$ is a given matrix the elements of which are 0 or 1 in the diagonal and 0 or 2 outside of the diagonal, and $\mathbf{x} = (x_1, ..., x_r)$ is a given vector with non-negative integer coordinates, then the graph $A((\mathbf{x}))$ is defined as follows. We consider r classes $C_1, ..., C_r$ the i-th of which contains x_i vertices, and join each vertex of C_i to each vertex of C_j by an edge oriented from C_i to C_j iff $a_{i,j} = 2 (1 \leqslant i < j \leqslant r)$. Then we enumerate the vertices of C_i by $1, ..., x_i$ and join each pair of vertices by an edge directed from the smaller index to the larger, $i = 1, ..., r$. (In other words: we define a complete acyclic graph on the vertices of C_i.)

One can ask why the elements outside the diagonal are taken to be 2 instead of 1. The advantage of this convention is that, in this case, trivially

$$2e(A(\mathbf{x}))) = \mathbf{x}A\mathbf{x} + O\left(\sum x_i\right). \tag{1}$$

DEFINITION 2. *Optimal matrix graphs.* Let us consider for given n and A all the graph $A((\mathbf{x}))$ such that $x_1 + \cdots + x_r = n$. Of those having the maximum number of edges an arbitrary $A((\mathbf{x}))$ graph will be denoted by $A(n)$ and will be called an optimal matrix graph.

Example 1. Let $T_r = (2 - 2\delta_{i,j})_{i,j\leqslant r}$ where $\delta_{i,j}$ is the Kronecker symbol: 1 if $i = j$ and 0 otherwise. Clearly, $T_r((\mathbf{x}))$ is a complete r-partite digraph with x_i vertices in the i-th class C_i: for each pair (i, j) each vertex of C_i is joined to each vertex of C_j by two edges of opposite directions. It has the maximum number of edges if $|x_i - x_j| \leqslant 1$ for every $1 \leqslant i \leqslant j \leqslant r$.

Example 2. Let $D_r = (2 - \delta_{i,j})_{i,j\leqslant r}$. $D_r((\mathbf{x}))$ can be obtained from the $T_r((\mathbf{x}))$ of the previous example by putting a complete acyclic digraph into each C_i. The maximum is attained under the same condition, e.g., if $n = kr$, then all the classes have k points.

Example 3.

$$A = \begin{pmatrix} 0 & 0 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{pmatrix}.$$

While in the earlier examples the matrices were symmetric and the graphs were easy to visualize, in this case we have a more complicated situation.

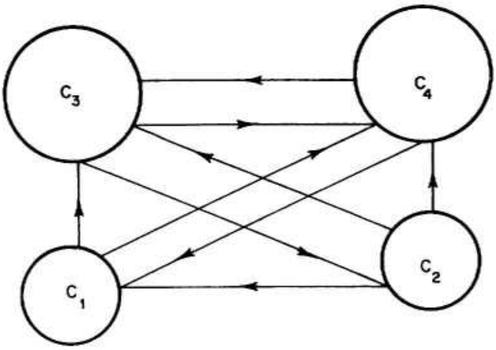

FIGURE 1

The graph itself can be seen in Figure 1. In the case of the optimal graphs the classes C_3 and C_4 are approximately equal and contain asymptotically twice as many vertices as C_1 or C_2 which are also approximately equal. It is also interesting that the graph $A(n)$ is not uniquely determined, e.g., for $n = 6k + 1$ the vectors $(k, k, 2k, 2k + 1)$, $(k, k, 2k + 2k)$, $(k, k + 1, 2k, 2k)$, and $(k + 1, k, 2k, 2k)$ give four different optimal graphs. Let

$$g(A) = \max \left\{ \mathbf{u}A\mathbf{u}\colon u_i \geqslant 0, \sum u_i = 1 \right\}. \tag{2}$$

We shall call $g(A)$ the density of the matrix A. It is trivial from (1) that

$$e(A(n)) = g(A)n^2/2 + o(n^2) \tag{3}$$

and with a little more care one can also prove that

$$e(A(n)) = g(A)n^2/2 + 0(n), \tag{4}$$

but this will not be needed.

DEFINITION 3. The matrix A will be called dense if for every principal proper submatrix A' of A

$$g(A) > g(A').$$

Example 4. The following matrices are *not* dense:

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

(At this stage we have to use direct methods to check that (1) in the cases of A_1 and A_2 and $\left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$ in the case of A_3 are submatrices having the same density as the whole matrix. Later we shall have some simpler methods to check whether or not A is dense.)

DEFINITION 4. The sequence S^n will be called a sequence of asymptotic extremal digraphs for L if S^n does not contain any digraph from L and

$$\lim_{n\to\infty} e(S^n)/f(n; \mathsf{L}) = 1. \tag{5}$$

(To speak about *one* asymptotical extremal graph makes no sense.)

Our main result is

THEOREM 1. *For any finite or infinite family L of sample digraphs there exists a dense matrix A such that $A(n)$ is a sequence of asymptotic extremal graphs for L.*

Example 5. Let L be the graph having 3 vertices a, b, c and 3 edges $(a \to b)$, $(a \to c)$, $(b \to c)$. A trivial modification of Turán's original proof or [1, p. 147] gives that the complete bipartite directed graph $T_2(n)$ (see Example 1) is the only extremal graph for L. Of course, $T_2(n)$ is also a sequence of asymptotical extremal graphs.

Example 6. Let L be the graph having 3 vertices a, b, c and 5 edges $(a \to b)$, $(b \to a)$, $(a \to c)$, $(c \to a)$ and $(b \to c)$. The extremal graphs are completely characterized in [1, p. 147]. Again, $T_2(n)$ is an extremal graph for L but there are very many other extremal graphs as well. The sequence of complete acyclic graphs (corresponding to the dense 1×1 matrix (1) is not a sequence of extremal graphs but it is a sequence of asymptotic extremal graphs.

The main content of Theorem 1 is that one can construct the "almost best" graphs for Problem 1 in a very simple way. However, the word "construct" unfortunately is abused in the sentence above: if we have a matrix A and wish to decide whether or not A yields a sequence of asymptotic extremal graphs, Theorem 1 does not help. We do not even know a finite algorithm which would produce matrix A in the case of finitely many sample graphs. More exactly, we know an algorithm which has in many cases solved the problem but we cannot prove that it will always work.

One can ask whether Theorem 1 can be improved—whether it is possible to obtain some more information on the structure of the sequence of asymptotic extremal graphs. Of course, this can be done by proving theorems on the structure of the dense matrices. Another way of improving Theorem 1 is to prove that only some special types of dense matrices can occur in it. For example the following conjecture would solve the algorithm problem as well:

CONJECTURE 1. Let $A(n)$ be a sequence of asymptotic extremal graphs for $L_1, ..., L_q$, where A is a dense matrix. Then A has less than

$$v(L_1) \cdot v(L_2) \dots v(L_q)$$

rows (and columns).

From another point of view Theorem 1 is the best possible:

THEOREM 2. *There exists for any dense matrix A a finite family of sample graphs for which $A(n)$ is a sequence of extremal graphs. Moreover, any sequence of asymptotic extremal graphs for these sample graphs can be obtained from $B(n)$ by changing $o(n^2)$ edges.*

REMARK 1. The second part of Theorem 2 implies that, if $D(n)$ is a sequence of asymptotic extremal graphs for some D, then $D = A$.

The proof of Theorem 2 is rather complicated; therefore we shall not publish it in this paper.

THE CASE OF UNDIRECTED GRAPHS. Let us omit the directions from a directed graph considered here; then we obtain a non-oriented graph without loops where some pairs of vertices will be joined by 2 edges but never by 3. Let us call these graphs multigraphs. We can associate with a multigraph all the digraphs obtainable by directing the single edges arbitrarily and the multiple edges in opposite directions.

PROBLEM $\hat{1}$. What is the maximum number of edges in a multigraph of n vertices which contains none of the multigraphs $M_1, ..., M_s$?

Let B be an $r \times r$ symmetric matrix, each element of which is 0, 1 or 2. The matrix $\hat{B}((\mathbf{x}))$ can be defined similarly as for digraphs: We join each vertex of C_i to each vertex of C_j by $b_{i,j}$ non-oriented edges of $i \neq j$ and each vertex of C_i to each other vertex of C_i by $b_{i,i}$ edges. An optimal multigraph will be denoted by $\hat{B}(n)$.

THEOREM $\hat{1}$. *For every finite or infinite family of sample multigraphs there exists a dense symmetric matrix B such that $\hat{B}(n)$ is a sequence of asymptotic extremal multigraphs.*

THEOREM $\hat{2}$. *Let B be a dense symmetric matrix. There exists a finite family of sample multigraphs for which $B(n)$ is a sequence of extremal multigraphs. Morever, any sequence of asymptotic extremal graphs for these sample multigraphs can be obtained from $\hat{B}(n)$ by changing $o(n^2)$ edges.*

Theorem $\hat{1}$ is a simple consequence of Theorem 1 while Theorem $\hat{2}$ implies Theorem 2. The proof of Theorem $\hat{2}$ will not be given here. We show how Theorem $\hat{1}$ can be derived from Theorem 1.

Let M be a family of sample multigraphs. By definition, let L be the family of digraphs associated with the multigraphs of M, i.e., obtainable from them by directing the edges in all the permitted ways. According to Theorem 1 there exists a dense A such that $A(n)$ is a sequence of asymptotic extremal graphs for L. Let $B = \frac{1}{2}\,(A + A^*)$, where A^* is the transpose of A. The elements of B are 0, 1, or 2. Since for A and B for every pair of corresponding submatrices of A and B the quadratic form is the same, A is dense if and only if B is dense; i.e., B is also dense. We show that $B(n)$ is a sequence of asymptotic extremal graphs. First, the multigraphs $B(n)$ do not contain any sample multigraph, for otherwise $A(n)$ would contain a directed version of this sample multigraph, i.e. a digraph from L. Further, for ϵ fixed, n sufficiently large, and

$$e(G^n) > e(\hat{B}(n)) + \epsilon n^2 = e(A(n)) + en^2$$

any multigraph must contain a sample multigraph from L; indeed, orienting the edges of G^n in a permitted way we get a graph containing at least one sample digraph from L, the corresponding sample multigraph is trivially contained in G^n. Hence $B(n)$ is really a sequence of asymptotic extremal multigraphs for M.

2. THE STRUCTURE OF MATRIX GRAPHS

(A) First we remark that *every matrix B is either dense or has a proper principal dense submatrix A such that $g(A) = g(B)$.*

Indeed, if A is a minimal submatrix of A such that $g(A) = g(B)$, then for each proper (principal) submatrix A' of A $g(A') < g(A)$, i.e., A is dense.

If B and A are in the relationship described above, we shall write $A = D(B)$. A is not generally uniquely determined by B.

(B) Let A be a dense matrix. For given n we select an optimal vector $(x_1, \dots, x_r) = \mathbf{x}$, i.e., a vector such that

$$A(n) = A((x_1, \dots, x_r)).$$

Let the classes of this $A(n)$ be $C_1, \dots, C_r$. Trivially, if two vertices of $A(n)$ belong to the same C_i, their valence must be the same. We prove that even if two vertices a_1 and a_2 belong to different classes, e.g. to C_1 and C_2, and their valencies are v_1 and v_2, then

$$| v_1 - v_2 | \leqslant 2. \tag{6}$$

Indeed, we can obtain $A(x_1 - 1, x_2 + 1, x_3, \dots, x_r))$ from $A((x_1, \dots, x_r))$ by omitting the v_1 edges incident with a_1 and then joining a_1 to all the vertices in the resulting graph which are joined to a_2, and joining a_1 to a_2 by $a_{2,2}$ edges. The number of edges is increased by at least

$$-v_1 + (v_2 - \tfrac{1}{2}(a_{1,2} + a_{2,1})) \geqslant v_2 - v_1 - 2. \tag{7}$$

On the other hand, the number of edges cannot be increased since $A(n)$ had maximum number of edges. Thus $v_2 - v_1 \leqslant 2$. Q.E.D.

(C) Next we prove that, if $\mathbf{x}_n = (x_{1,n}, \dots, x_{r,n})$ are optimal vectors corresponding to $A(n)$ for a dense A(i.e., $A(n) = A((\mathbf{x}_n)))$, then the vectors $(1/n)\mathbf{x}_n$ converge to a vector $\mathbf{u} = (u_1, \dots, u_r)$ (in the Euclidean norm) where $\mathbf{u}$ is uniquely determined by the system of linear equations

$$(\mathbf{u}; \mathbf{e}) = 1 \qquad \text{and} \qquad (A + A^*)\, \mathbf{u} = 2g(A)\mathbf{e}, \qquad \mathbf{e} = (1, \dots, 1). \tag{8}$$

Indeed, let us suppose that $\mathbf{u}$ is a limit point of the vectors $(1/n)\mathbf{x}_n$. Then, by (3) and (1)

$$2e(A((\mathbf{x}_n))) = \mathbf{x}_n A \mathbf{x}_n + 0(n) = 2e(A(n)) = g(A)n^2 + o(n^2),$$

and therefore

$$\mathbf{u}\, A\, \mathbf{u} = g(A), \qquad \text{while} \quad (\mathbf{u}, \mathbf{e}) = 1.$$

This shows that $\mathbf{u}$ yields the maximum in (2).

If, e.g., $u_r = 0$ were valid, then the submatrix A' obtained from A by omitting the last row and column would have the same maximum: $g(A) = g(A')$ would hold. This contradicts the hypothesis that A is dense. Therefore *each coordinate of* $\mathbf{u}$ *is positive*.

The second equation of (8) can be obtained either by using the Lagrange method for the maximum-problem (2) or from (6). We use the second method. By (3) the average valence in $A(n)$ is $g(A)n + o(n)$. By (6) each vertex has essentially the same valence, i.e. each vertex has the valence $g(A)n + o(n)$. On the other hand, each vertex of C_i has the valence

$$a_{i,i}(x_i - 1) + \sum_{j \neq i} (a_{i,j} + a_{j,i})\, x_j/2. \tag{9}$$

Hence

$$\sum_{j \leqslant r} a_{i,j} u_i + \sum_{j \leqslant r} a_{j,i} u_i = 2g(A). \tag{10}$$

This proves the second equality of (8). Now we prove that $\mathbf{u}$ is the only solution of (8). This will also prove that $(1/n)\mathbf{x}_n$ has only one limit point, i.e., must converge to $\mathbf{u}$.

First we remark that if $\mathbf{v}$ satisfies (8) then

$$2\,\mathbf{v}A\mathbf{v} = \mathbf{v}A\mathbf{v} + \mathbf{v}A^*\mathbf{v} = \mathbf{v}(A + A^*)\mathbf{v} = (\mathbf{v}; 2g(A)\mathbf{e}) = 2g(A)$$

and therefore $\mathbf{v}$ gives the maximum in (3) apart from the fact that $\mathbf{v}$ may have negative coordinates as well. Let now $\mathbf{u}$ be a solution of (8) with positive coordinates, let $\mathbf{v}$ be an arbitrary solution of (8), and let $\mathbf{w} = \omega\mathbf{u} + \mu\mathbf{v}$ where $\omega + \mu = 1$. Then $\mathbf{w}$ is a solution of (8). With suitable (ω, μ) one can get a $\mathbf{w}$ each coordinate of which is non-negative and at least one of which is 0. But, as we have seen, such a $\mathbf{w}$ contradicts the hypothesis that A is dense. Hence (8) has only one solution $\mathbf{u}$ and each coordinate of this $\mathbf{u}$ is positive.

LEMMA 1. *Let* $a_{i,i} = a_{j,j}$ *in a dense matrix* $A = (a_{k,m})$. *Then*

$$a_{i,j} + a_{j,i} > 2a_{i,i}\,.$$

Proof. We may suppose that $i = 1$ and $j = 2$ and that

$$\sum_{3}^{r} (a_{1,j} + a_{j,1} - a_{2,j} - a_{j,2})\, u_j \geqslant 0,$$

where $\mathbf{u}$ is the vector giving the maximum in (2). Let

$$\mathbf{v} = (u_1 + u_2, 0, u_3, \ldots, u_r).$$

Since $\mathbf{u}$ is the only optimum vector,

$$\begin{aligned} 0 &> \mathbf{v}A\mathbf{v} - \mathbf{u}V\mathbf{u} \\ &= u_2 \left(\sum_3^r (a_{1,j} + a_{j,1} - a_{2,j} - a_{j,2})\, u_j \right) + (2a_{1,1} - a_{1,2} - a_{2,1})\, u_1 u_2 \\ &\geqslant (2a_{1,1} - a_{1,2} - a_{2,1})\, u_1 u_2 . \end{aligned}$$

This completes the proof.

3. Augmentation of Matrices

(A) Let A be an $r \times r$ dense matrix and m an integer, $\mathbf{x} = (x_1, \ldots, x_r)$ a vector for which $A(m) = A((\mathbf{x}))$. We construct a new graph by taking x_{r+1} new vertices forming a new class C_{r+1} and joining each vertex of the new class to each vertex of the orlginal class C_j in the same way, $j = 1, \ldots, r$. Then we change the proportions so that the graph obtained should have the maximum number of edges among all graphs of this type with the given number of vertices. (The proportions of the vertices in the original classes may also change.) This construction motivates the following definition.

Definition 5. Let $B = (a_{i,j})$ be an $(r+1) \times (r+1)$ matrix and let A be the submatrix obtained by omitting the last row and column. Let A be dense and u be the vector giving the maximum in (2). Let

$$b = \frac{1}{2} \Big(\sum_{j \leqslant r} a_{r+1,j} u_j + \sum_{j \leqslant r} a_{j,r+1} u_j \Big) > g(A). \tag{11}$$

Then we say that B is obtained from A by *augmentation*.

Remark 2. In the graph construction given to motivate Definition 4 condition (11) means that the new vertices are joined to $A(m)$ by more edges than the valence of the vertices of $A(m)$.

Lemma 2. *If B is obtained from A by augmentation, then $g(B) > g(A)$.*

Proof. Let $g(A) = \gamma$, $\lambda = b/(2b - \gamma)$ (with b defined by (11)) and $\hat{\mathbf{u}} = (\lambda u_1, ..., \lambda u_r, 1 - \lambda)$. Then

$$\begin{aligned} g(B) - \gamma &\geqslant \hat{\mathbf{u}} B \hat{\mathbf{u}} - \gamma = \gamma\lambda^2 + 2\lambda(1 - \lambda)\, b + a_{r+1,r+1}(1 - \lambda)^2 - \gamma \\ &\geqslant \gamma\lambda^2 + 2\lambda b - 2\lambda^2 b - \gamma = 2\lambda b - \gamma - (2b - \gamma)\,\lambda^2 \\ &= 2\lambda b - \gamma - \lambda b = \lambda b - \gamma \\ &= \frac{b^2 - 2\gamma b + \gamma^2}{2b - \gamma} = \frac{(b - \gamma)^2}{2b - \gamma} > 0. \qquad \text{Q.E.D.} \quad (12) \end{aligned}$$

Let us suppose that A_0 is a dense matrix, B_0 is obtained from it by augmentation, $A_1 = D(B_0)$, B_1 is obtained from A_1 by augmentation,... $A_j = D(B_{j-1})$, and B_j is obtained from A_j by augmentation. Since in taking a matrix $A = D(B)$ instead of B we usually have to omit some rows and columns of B, it may happen that in the sequence above A_j does not contain A_0; moreover, A_j need not contain any rows or columns originating from A_0. However, in the process above we never omit the rows and columns of A_0 in which the diagonal element $a_{i,i} = 1$. Here we prove only a slightly weaker lemma.

LEMMA 3. *Let $A_0 = (2 - \delta_{i,j})_{i,j\leqslant r}$ (i.e., the matrix D_r of Example 2). Let B_j be obtained from A_j by augmentation and $A_{j+1} = D(B_j)$, $j = 0, 1, ..., k$. Then each A_j and B_j contain A_0. Further, if one row of B_j originates from A_0 (i.e., is the expansion of a row of A_0) then all its elements are equal to 2; except, of course, the one in the diagonal.*

Proof. (Induction on k). For $k = 0$ the Lemma is trivial. Let us suppose that it is known for $k - 1$. Let

$$B_k = (a_{i,j})_{i,j\leqslant q} \qquad \text{and} \qquad A_{k-1} = (a_{i,j})_{i,j\leqslant q-1},$$

and let the first row of B_k, more exactly $(a_{1,1}, ..., a_{1,r})$, be also a row of A_0. By the hypothesis, $a_{1,2} = a_{1,3} = \cdots = a_{1,q-1} = 2$. Similarly, $a_{2,1} = a_{3,1} = \cdots = a_{q-1,1} = 2$. We prove that $a_{1,q} = a_{q,1} = 2$. We know that, if $\mathbf{u}$ gives the maximum in (2) for A_{k-1}, then (by (10) and (11))

$$\begin{aligned} 2g(A_{k-1}) &= \sum_1^{q-1} (a_{i,1} + a_{1,i})\, u_1 = 4(u_2 + \cdots + u_{q-1}) + 2u_1 \\ &< 2b = \sum_1^{q-1} (a_{i,q} + a_{q,i})\, u_1 \leqslant 4(u_2 + \cdots + u_{q-1}) + (a_{1,q} + a_{q,1})\, u_1 . \end{aligned} \tag{13}$$

Therefore $a_{1,q} + a_{q,1} > 2$. Since, for $i \neq j$, $a_{i,j} = 0$ or 2, $a_{1,q} = a_{q,1} = 2$.

Now we show that in omitting some rows and columns of B_k in order to obtain A_k we cannot omit the first row of B_k. This will complete the proof of Lemma 3. Let us suppose that A_k does not contain the first row (and column) of B_k. Let $\tilde{A}$ be the matrix determined by A_k and the first row and first column of B_k. We can apply Lemma 2 to $\tilde{A}$ and A_k as follows:

The new elements (except the element in the diagonal) are equal to 2; using the notations of (11), $b = 2$ and $g(A)$ is always less than 2. Hence $g(B_k) \geqslant g(\tilde{A}) > g(A_k) = g(B_k)$. This contradiction proves the lemma.

LEMMA 4. *For every dense matrix A, positive integer m, and positive constant ϵ there exists an integer m' such that*

If n is large enough and G^n contains $A(m')$, and if each vertex of G^n has valence $\geqslant (g(A) + \epsilon)n$, then there exists a matrix B obtained from A by augmentation and a maximal dense submatrix $A^* = D(B)$ such that G^n contains $A^*(m)$.

Proof. First we fix a few constants. Let $b = g(A) + \epsilon$, $b' = g(A) + \epsilon/2$ and $b'' = g(A) + \epsilon/4$. Let $c = \epsilon/8r$, where r is the number of rows in A, $m' > c^{-1}$.

We divide G^n into two parts: the vertices of $A(m')$ will be in the first and the other vertices in the second. The number of edges joining the two parts is asymptotically bnm' or more. Therefore the vertices in the second part are joined to $A(m')$ by $\geqslant bm' - 1$ edges in the average. Thus there exists a positive constant c_1 such that at least $c_1 n$ vertices are joined to $A(m')$ by more than $b'm'$ edges. Let us denote the class of these vertices by E_0. Since the vertices of E_0 can be joined to $A(m')$ in only finitely many way, there exists a subclass E_1 of E_0 where $|E_1| \geqslant c_2 n$, whose elements are joined to $A(m')$ in the same way. (We say that x and y are joined to $A(m')$ in the same way if for every $z \in A(m')$ the directed edge $(x \to z)$ belongs to G^n if and only if $(y \to z)$ also belongs to G^n and the directed edge $(z \to x)$ belongs to G^n if and only if $(z \to y)$ belongs to G^n.)

Let $A = (a_{i,j})_{i,j\leqslant r}$ and let $B = (a_{i,j})_{i,j\leqslant r+1}$ be defined as follows:

(i) if there exist cm' vertices in C_j joined to each vertex of E_1 by two edges of different orientation, then $a_{r+1,j} = a_{j,r+1} = 2$;

(ii) if there exist cm' vertices in C_j joined to E_1 by edges directed from C_j to E_1 (from E_1 to C_j) but (i) does not hold then let $a_{j,r+1} = 0$, $a_{r+1,j} = 2$ ($a_{j,r+1} = 2$, $a_{r+1,j} = 0$, except if $a_{j,r+1}$ and $a_{r+1,j}$ are already defined by the first part if (ii)).

(iii) if neither (i) nor (ii) holds, then fewer than $2cm'$ vertices of C_j are joined to E_1. In this case let $a_{r+1,j} = a_{j,r+1} = 0$. Finally, we define $a_{r+1,r+1} = 0$.

We prove that B is obtained from A by augmentation. Clearly, the number of edges between C_j and a point of E_1 is less than

$$\tfrac{1}{2}\,(a_{r+1,j} + a_{j,r+1})|\, C_j \,| + 2cm'.$$

Therefore

$$b'm' \leqslant \tfrac{1}{2} \sum (a_{r+1,j} + a_{j,r+1}) \mid C_j \mid + 2rcm'.$$

Let $\mathbf{u}$ be the vector attaining the maximum in (2) for A. Since $| C_j |/m'$ tends to u_j as m' tends to infinity, we may choose m' so large that

$$\sum (a_{r+1,j} + a_{j,r+1})\, u_j > 2b' - 4cr > 2g(A).$$

This shows that B is obtained from A by augmentation.

It is also clear that, if $k < cm'$, then $B((k\mathbf{e})) = B(((k,..., k)))$ is a subgraph of G^n. Therefore, if $A^* = D(B)$, then $A^*(m)$ is also a subgraph of G^n. Q.E.D.

4. Proof of Theorem 1

For a given family of sample digraphs $L_1 ,..., L_q$ let us first consider a simpler problem instead of Problem 1, which could be called the Zarankiewicz problem corresponding to Problem 1.

Problem 3. For given n, what is the maximum d for which there exists a graph G^n containing no L_i and each vertex of which has valence $\geqslant d$?

We denote by Z^n one of the extremal graphs for Problem 3 and by d_n the minimum valence in Z^n. Let

$$a^* = \limsup_{n\to\infty} d_n/n.$$

There exists a sequence N_1 of integers such that $d_n/n \to a^*$ if $n \in N_1$, $n \to \infty$. Given a dense A and a sequence of graphs G^n, we shall say that A is strongly (weakly) contained by the sequence G^n if the maximum $m = m_n$ for which $A(m) \subset G^n$ tends to infinity (is unbounded) as $n \to \infty$.

Let D_p be the matrix defined in Example 2. If $k = v(L_1)$, $D_k(k)$ contains L_1. (Each class of $D_k(k)$ contains exactly one vertex; therefore each pair of vertices is joined by two edges of opposite directions.) Hence, if $\{Z^n: n \in N_1\}$ contains D_p weakly, then $p < k$. Let r be the maximum of those p for which $\{Z^n: n \in N_1\}$ contains D_p weakly. We may select for every m an $n_m \in N_1$ such that $D_r(m) \subset Z^{n_m}$. This means that a subsequence of N_1 must contain D_r strongly. Now we show that there exists a maximal B such that

(i) B occurs in some sequence described by Lemma 3 and starting with $A_0 = D_r$,

(ii) B is weakly contained by the sequence $\{Z^n : n \in N_2\}$.

Let B satisfy (i) and (ii) and s be an integer such that the number of rows in B is at least $r + 2^s$ but fewer than $r + 2^{s+1}$. A trivial consequence of the Ramsey theorem (or cf. [8]) is that a complete directed graph of 2^s vertices must contain a complete acyclic graph of s vertices. Let us select from each class of $D_r \subset B$ s vertices and from each other class of B just one vertex; let us call them vertices of the first and second type, respectively. Applying Lemma 1 with $a_{i,i} = a_{j,j} = 0$, we obtain that each pair of vertices of the second type are joined by at least one edge. Therefore we can choose s of them spanning a complete acyclic graph of s vertices (perhaps with some additional edges). Applying Lemma 3, we obtain that each vertex of the second type is joined to each vertex of the first type by two edges. Hence B contains a $D_{r+1}(s)$. Since D_{r+1} is not even weakly contained by $\{Z^n : n \in N_2\}$, s must be bounded. Thus the set of matrices B being considered is finite and there exists a B for which $g(B)$ is maximal. We shall prove that $B(n)$ is a sequence of asymptotical extremal graphs.

(It can happen that $\{Z^n : n \in N_1\}$ does not even contain D_1 weakly. In this case $r = 0$ in the argument above and we do not need Lemma 3.)

Each $B(n)$ is contained in some $Z^{n'}$; therefore $B(n)$ cannot contain any sample digraph. The vertices of $B(n)$ have valence $g(B)n + o(n)$ (by (3) and (6)). Therefore

$$a^* \geqslant a_* = \liminf_{n\to\infty} d_n/n \geqslant g(B). \tag{14}$$

But, if in (14) we had $a^* > g(B)$, then Lemma 4 would yield a B' which is obtained from B by augmentation such that a $B^* = D(B')$ is weakly contained in $\{Z^n : n \in N_2\}$. This would contradict the maximality of B. Hence

$$\lim d_n/n = g(B). \tag{15}$$

We prove that $B(n)$ is a sequence of asymptotic extremal digraphs for the considered set of sample digraphs. Since we know that it does not contain any sample digraph, we have only to prove that if

$$e(G^n) > e(B(n)) + \epsilon n^2 = \tfrac{1}{2}(g(B) + 2\epsilon)n^2 + o(n^2)$$

then G^n must contain a sample digraph. Let $g(B) + \epsilon = g'$. We define

G^{n-j} recursively by omitting from G^{n-j+1} one vertex of valence $\leqslant g'(n-j+1)$ if G^{n-j+1} has such a vertex; if not, the recursion stops. Clearly

$$\begin{aligned} e(G^{n-k}) &\geqslant \frac{1}{2}(g(B)+2\epsilon)\, n^2 - g'\,\frac{k(2n-k+1)}{2} + o(n^2) \\ &\geqslant g'\,\frac{(n-k)^2}{2} + \frac{\epsilon n^2}{2} + o(n^2). \end{aligned} \tag{16}$$

From (16) we know that G^{n-k} has at least $\epsilon n^2/4$ edges, i.e., the recursion stops when G^{n-k} has at least $(\epsilon/2)\, n$ vertices. Each vertex of the obtained G^{n-k} has valence $\geqslant g'(n-k)$. If n is large enough, then $n-k$ is also large enough and by (15) G^{n-k} must contain a sample graph. Q.E.D.

5. Final Remarks, Open Problems

(A) Instead of considering graphs without loops and multiple directed edges we could consider for a fixed integer t graphs without loops, where two vertices can be joined by at most t edges of the same direction. We conjecture that Theorem 1 is valid even in this case, if we modify slightly the notion of matrix graphs. Now we consider a matrix A whose diagonal elements are non-negative integers not greater than $t-1$, and whose other elements are even non-negative integers not greater than $2t$. We must also fix a vector $\mathbf{a} = (a_1, \ldots, a_r)$ with non-negative integer coordinates where $2a_i \leqslant a_{i,i}$ will also be assumed. We join each vertex of C_i to each vertex of C_j by $a_{i,j}/2$ edges directed from C_i to C_j if $i \neq j$. The complete acyclic graphs of the original definition now have to be replaced by the following graph: If x_r is the number of vertices in C_r and the vertices are $z_1, \ldots, z_{x_r}$, then for every $1 \leqslant x' < x'' \leqslant x_r$ we join $z_{x'}$ to $z_{x''}$ by $a_{i,i}$ edges, a_i of which are directed from $z_{x'}$ to $z_{x''}$ and the other $a_{i,i} - a_i$ are directed in the opposite way. Most of our results remain true even in this case; in some of them 1 must be replaced by t. But the final part of the proof fails to generalize. Theorem 2 remain valid also in the general case and its proof is not more complicated.

(B) Theorem 1 and 2 imply that for any infinite family of digraphs, L, there exists a finite L^* such that

$$f(n; \mathsf{L}) - f(n; \mathsf{L}^*) = o(n^2). \tag{17}$$

Indeed, let $A(n)$ be a sequence of asymptotic extremal graphs for L. Then there exists a finite L^* such that $A(n)$ is a sequence of asymptotic extremal graphs also for L. This proves (17).

CONJECTURE 2. For any L there exists a finite $L^* \subset L$ such that (17) holds.

Conjecture 2 is trivially valid in the case of undirected graphs without loops or multiple edges. We think that it is always valid, i.e., not only for Problem 1 but also for the generalization of Problem 1 given in (A). An equivalent form of Conjecture 2 is

CONJECTURE 2*. (a) For every constant γ there exist only finitely many dense matrices A such that $g(A) = \gamma$.

(b) Let A be the set of positive constants of form $g(A)$, where A can be any dense matrix. Providing A with the usual ordering of the real numbers yields a well-ordered subset.

(c) In another paper we shall deal with characterizing the structure of dense matrices. Here we mention only one result, the proof of which will be published later.

THEOREM 3. *Let* $A = (a_{i,j})_{i,j \leqslant r}$ *be a dense matrix and* $a_{1,1} = 1$; *then* $a_{i,1} = a_{1,j} = 2$ *for every* $i \geqslant 2$ *and* $j \geqslant 2$.

(d) Some ideas of the proof of Theorem 1 can be found in a paper of Motzkin and Straus [9]; the most important is to associate a quadratic form with a graph and look for the maximum of $\mathbf{x}A\mathbf{x}$ in (2). Another device of [9] is that used to prove Lemma 1, i.e., to consider the vector $\mathbf{v} = (u_1 + u_2, 0, u_3, \ldots, u_r)$. However, we have to mention here that this latter is equivalent to a method used by A. Zykov [10] for the same purpose (i.e., to prove Turán's theorem). On the other hand, the quadratic forms here are much less surprising than in [9], since they express the number of edges of the graphs $A((\mathbf{x}))$.

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