COMMUTATORS FOR NEAR-RINGS: HUQ $\neq$ SMITH

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Dedicated to the memory of Ervin Fried

Abstract. It is shown that the Huq and the Smith commutators do not coincide in the variety of near-rings.

1. Introduction

As the title shows, the paper is devoted to commutators of ideals (normal subobjects) in the variety (category) of near-rings, and its main purpose is to present a counter-example, due to the third named author, showing that, in the case of near-rings, the Huq and the Smith commutators need not coincide. For readers less familiar with these commutators, let us recall:

What we call the *Huq commutator* is a category-theoretic concept introduced by Huq [10]. In the case of a semi-abelian [12] variety $\mathbf{C}$ of universal algebras, such as the varieties of groups, rings or near-rings, it can be defined as follows: Given X in $\mathbf{C}$ and normal subalgebras A and B of X, the Huq commutator $[A, B]_H$ is the smallest normal subalgebra C of X such that the canonical homomorphism $A * B \to X/C$ factors through the canonical homomorphism $A * B \to A \times B$. Briefly, the existence of such a factorization means that the canonical homomorphism $A \times B \to X/C$ is well defined. Here $A * B$ stands for the free product (in categorical terms, the coproduct or sum) of A and B.

The *Smith commutator* is a concept originally introduced by Smith [15] for congruences in a Mal'tsev (that is, congruence permutable) variety. Together with its various generalizations this notion is well known not only in universal algebra but also in category theory (see e.g. [13] and references therein). In the formulation given in [11], for an algebra X in a Mal'tsev variety with Mal'tsev term $p(x, y, z)$ and

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two congruences α and β on X, the commutator $[\alpha, \beta]_S$ is the smallest congruence on X for which the function

$$p : \{(x, y, z) \mid (x, y) \in \alpha \text{ and } (y, z) \in \beta\} \to X/[\alpha, \beta]_S$$

sending (x, y, z) to the $[\alpha, \beta]_S$-class of $p(x, y, z)$ is a homomorphism.

When X belongs to a semi-abelian variety $\mathbf{C}$ (and in some more general situations), there is a one-to-one correspondence between the normal subalgebras and the congruences on X. Therefore, for normal subalgebras A and B of X and their corresponding congruences α and β on X, one would expect that the congruence corresponding to $[A, B]_H$ coincides with the Smith commutator $[\alpha, \beta]_S$. However, this is not the case in general, which, in a sense, is already suggested by the commutator constructions of Higgins [9]; the first explicit counter-example ('digroups': two independent group structures on the same set with the same identity element) was constructed much later in a joint work of the first named author and Bourn (unpublished, but later mentioned, first in [3], in the form of an observation on change-of-base functors for split extensions). Another counter-example (loops) was given recently by Hartl and van der Linden [8]. The question of when these two commutators coincide, is of sufficient importance to justify a condition "Smith = Huq" in universal algebra around which several theories have been developed, see for example [14].

Let us recall that a near-ring N is a system $N = (N, 0, +, -, \cdot)$ in which $(N, 0, +, -)$ is a group (not necessarily commutative), $(N, \cdot)$ is a semigroup (with $x \cdot y$ written as xy), and the right distributive law $(x+y)z = xy+xz$ holds. Notice that $0x = 0$ is an identity in near-rings but $x0 = 0$ need not be valid. In the semi-abelian variety of near-rings the normal subalgebras are called ideals, and $A \lhd N$ if and only if A is a subgroup of $(N, 0, +, -)$ with an and $n(a + m) - nm$ in A for all $a \in A$ and $n, m \in N$. The next two sections give more information on these two commutators for near-rings, while the last section presents our counter-example.

Throughout this paper N denotes a near-ring, A and B ideals of N, and α and β the corresponding congruences. Furthermore, we shall write $[A, B]_H$ for the Huq commutator of A and B and $[A, B]_S$ for the ideal corresponding to the Smith commutator $[\alpha, \beta]_S$.

2. The Huq commutator for near-rings

Apart from the two commutator operations we are interested in, we introduce two more operations on ideals, namely:

$[A,B]_G$, the ideal of N generated by the usual group-theoretic commutator of A and B considered as subgroups of the additive group of N; that is, $[A,B]_G$ is the ideal of N generated by the set

$$\{a+b-a-b \mid a \in A, b \in B\}.$$

$A \bullet B$, the ideal of N generated by the set

$$\{a(b+a')-aa' \mid a,a' \in A \text{ and } b \in B\}.$$

For our ideals A and B, $[A,B]_H$ is the smallest ideal of N for which the canonical map $\theta_0 : A \times B \to N/[A,B]_H$ is a near-ring homomorphism; the subscript 0 indicates here that we are dealing with ideals, that is, with congruence classes of 0; later we shall deal with congruences themselves. The homomorphism θ_0 must send elements of the form $(a,0)$ and $(0,b)$ to the classes of a and b, respectively, and so

$$\theta_0(a,b) = a+b+[A,B]_H,$$

as follows from $(a,b)=(a,0)+(0,b)$. This formula gives easily:

Theorem 1. $[A,B]_H = [A,B]_G \vee (A \bullet B) \vee (B \bullet A)$ *in the lattice of sub-near-rings of* N *(or, equivalently, in the lattice of ideals of* N*). That is,* $[A,B]_H$ *is the ideal of* N *generated by all elements of the form* $a+b-a-b$, $a(b+a')-aa'$ *and* $b(a+b')-bb'$, *where* a *and* a' *are in* A, *and* b *and* b' *are in* B.

Proof. Just observe that:

– the map θ_0 preserves addition if and only if $[A,B]_G \subseteq [A,B]_H$;

– the map θ_0 preserves multiplication if and only if

$$aa'+bb'-(a+b)(a'+b')$$

is in $[A,B]_H$ for all $a,a' \in A$ and $b,b' \in B$;

– these relations hold since θ_0 is a homomorphism;

– as follows from the right distributive law and the fact that $[A,B]_H$ is an ideal in N containing $[A,B]_G$, for all $a,a' \in A$ and $b,b' \in B$, $aa'+bb'-(a+b)(a'+b')$ is in $[A,B]_H$ if and only if so are all elements of the form $a(b+a')-aa'$ and $b(a+b')-bb'$. ☐

3. The Smith commutator for near-rings

As experience with the Smith commutator theory shows, and as even suggested, in a sense, by classical affine geometry (see e.g. [7]), the suitable congruence counterpart of the map θ_0 is the map

$$\theta : \{(x,y,z) \in N^3 \mid x-y \in A \text{ and } y-z \in B\} \to N/[A,B]_S \qquad (3.1)$$

defined by $\theta(x,y,z) = x-y+z$ where $[A,B]_S$ is the smallest ideal of N for which θ is a near-ring homomorphism. This gives a simple characterization of the Smith commutator, perfectly analogous to the definition of the Huq commutator, and explicitly mentioned in [11] (referring to [13]) in a more general context.

The next theorem will be a counterpart of Theorem 1. In order to formulate it, we introduce two more operations on ideals A, B, C and D of N; this time a ternary and a quaternary operation, respectively:

– $\mathcal{C}(A, B, C)$ is the ideal of N generated by the set

$$\{a(b+c) - ac \mid a \in A, b \in B, c \in C\};$$

note that $\mathcal{C}(A, B, A) = A \bullet B$.

– $\mathcal{C}'(A, B, C, D)$ is the ideal of N generated by the set

$$\{a(b+c+d) - a(c+d) + ad - a(b+d) \mid a \in A, b \in B, c \in C, d \in D\}.$$

Theorem 2. $[A, B]_S = [A, B]_G \vee \mathcal{C}(A, B, N) \vee \mathcal{C}(B, A, N) \vee \mathcal{C}'(N, A, B, N)$ *in the lattice of sub-near-rings of N (or, equivalently, in the lattice of ideals of N). That is, $[A, B]_S$ is the ideal of N generated by all elements of the forms*

$$a+b-a-b,\ a(b+x)-ax,\ b(a+x)-bx,\ x(a+b+y)-x(b+y)+xy-x(a+y) \tag{3.2}$$

where $a \in A$, $b \in B$ and $x, y \in N$.

Proof. We begin as in the proof of Theorem 1. Being a homomorphism, θ preserves addition and multiplication. Preservation of addition is equivalent to $[A, B]_G \subseteq [A, B]_S$ or, in other words, that all elements of the form $a + b - a - b$ with $a \in A$ and $b \in B$ are in $[A, B]_S$. Next, θ preserves multiplication if and only if $[A, B]_S$ contains all elements of the form

$$xx' - yy' + zz' - (x - y + z)(x' - y' + z') \tag{3.3}$$

with $x - y$ and $x' - y'$ in A and $y - z$ and $y' - z'$ in B. Denoting $x - y$, $x' - y'$, $y - z$ and $y' - z'$ by a, a', b and b', respectively, we can rewrite (3.3) as

$$(a + b + z)(a' + b' + z') - (b + z)(b' + z') + zz' - (a + z)(a' + z'), \tag{3.4}$$

and then, using the right distributive law, as

$$\begin{aligned} &a(a' + b' + z') + b(a' + b' + z') + z(a' + b' + z') - z(b' + z') - b(b' + z') \\ &+zz' - z(a' + z') - a(a' + z'). \end{aligned} \tag{3.5}$$

We need to show that given a congruence $\sim$ on N with $a + b \sim b + a$ for all a in A and b in B, all elements of the forms (3.2) are congruent to 0 if and only if so are all elements of the form (3.5).

"*If*": Just note that in the cases $a' = b = z = 0$, $a = b' = z = 0$, and $a = b = 0$, the expression (3.5) reduces to $a(b'+z')-az'$, $b(a'+z')-bz'$, and $z(a' + b' + z') - z(b' + z') + zz' - z(a' + z')$, respectively.

"*Only if*": Assuming that all elements of the forms (3.2) are congruent to 0, we have:

$$\begin{aligned}
&a(a'+b'+z')+b(a'+b'+z')+z(a'+b'+z')-z(b'+z')-b(b'+z')\\
&\quad +zz'-z(a'+z')-a(a'+z')\\
&\sim a(a'+b'+z')+b(a'+b'+z')-b(b'+z')+z(a'+b'+z')\\
&\quad -z(b'+z')+zz'-z(a'+z')-a(a'+z')\\
&\quad \text{(since } z(a'+b'+z')-z(b'+z') \text{ is in } A \text{ and } -b(b'+z') \text{ is in } B,\\
&\quad \text{whence these elements commute up to } [A,B]_G)\\
&\sim a(a'+b'+z')+b(a'+b'+z')-b(b'+z')-a(a'+z')\\
&\quad \text{(since } z(a'+b'+z')-z(b'+z')+zz'-z(a'+z') \sim 0)\\
&\sim a(a'+b'+z')-a(a'+z') \quad \text{(since } b(a'+b'+z')-b(b'+z') \sim 0)\\
&\sim 0.
\end{aligned}$$

□

4. Huq $\neq$ Smith

As mentioned in the Introduction, the purpose of this section is to give an example of a near-ring N with ideals A and B for which $[A,B]_S \neq [A,B]_H$. Since the inclusion $[A,B]_H \subseteq [A,B]_S$ (trivially) holds in general, inequality here means strict inclusion.

Example. We take $N = \Psi$, the near-ring constructed in [16] using an idea of Betsch and Kaarli [1]. Its underlying group is $M^3 = M \times M \times M$ where M is any abelian group with a nonzero proper subgroup K, and its multiplication is defined by

$$(m_1,m_2,m_3)(n_1,n_2,n_3) = \begin{cases} (m_2,0,0) \text{ if } n_2 \neq 0 \neq n_3\\ (0,0,0) \text{ otherwise.}\end{cases}$$

We then take $A = M \times K \times \{0\} = \{(m_1,m_2,m_3) \in M^3 \mid m_2 \in K$ and $m_3 = 0\}$ and $B = M \times \{0\} \times M = \{(m_1,m_2,m_3) \in M^3 \mid m_2 = 0\}$. Then:

– $[A,B]_G = \{0\}$ since M^3 is an abelian group.

– $\mathcal{C}(A,B,N) = K \times \{0\} \times \{0\} = \{(m_1,m_2,m_3) \in M^3 \mid m_1 \in K$ and $m_2 = 0 = m_3\}$. Indeed, on the one hand, $\mathcal{C}(A,B,N) \subseteq K\times\{0\}\times\{0\}$ by the definition of multiplication in N, and, on the other hand, for every non-zero $k \in K$, we have $(k,0,0) = (0,-k,0)[(0,0,k)+(0,k,-k)] - (0,-k,0)(0,k,-k) \in \mathcal{C}(A,B,N)$, and also $\mathcal{C}(A,B,A) = K \times \{0\} \times \{0\}$.

– $\mathcal{C}(B,A,N) = \{0\} \times \{0\} \times \{0\}$, since $bx = 0$ for every $b \in B$ and every $x \in N$, and also $\mathcal{C}(B,A,B) = \{0\} \times \{0\} \times \{0\}$.

– $\mathcal{C}'(N,A,B,N) = M \times \{0\} \times \{0\} = \{(m_1,m_2,m_3) \in M^3 \mid m_2 = m_3 = 0\}$. Indeed, on the one hand $xy \in M \times \{0\} \times \{0\}$ for every

x and y in N, making the inclusion $\mathcal{C}'(N, A, B, N) \subseteq M \times \{0\} \times \{0\}$ obvious; on the other hand, for every non-zero $m \in M$, we choose any non-zero $k \in K$, and we have $(m, 0, 0) = (0, m, 0)[(0, k, 0) + (0, 0, m) + (0, 0, 0)] - (0, m, 0)[(0, 0, m) + (0, 0, 0)] + (0, m, 0)(0, 0, 0) - (0, m, 0)[(0, k, 0) + (0, 0, 0)] \in \mathcal{C}'(N, A, B, N)$.

Therefore $[A, B]_S = M \times \{0\} \times \{0\}$, by Theorem 3. At the same time, using Theorem 1 and the calculation above, we obtain

$[A, B]_H = [A, B]_G \vee (A \bullet B) \vee (B \bullet A) = [A, B]_G \vee \mathcal{C}(A, B, A) \vee \mathcal{C}(B, A, B) = K \times \{0\} \times \{0\}$.

That is, $[A, B]_H \neq [A, B]_S$, as desired.

Remarks. (a) Obviously, the same (counter-)example can be used in any full subcategory $\mathbf{C}$ of the category of near-rings closed under finite products, subobjects and quotient objects, containing the above near-ring N (for at least one M). Moreover, if we allow the ground category to be homological in the sense of [2], then the same applies to, say, all sub-quasi-varieties of the variety of near-rings.

(b) In particular, we can take $\mathbf{C}$ in (a) to be the category of all finite near-rings (using a finite abelian group M); the category of *zero-symmetric near-rings*, that is, those near-rings X in which $x0 = 0$ for every $x \in X$; the variety of near-rings in which the constants form an ideal, cf. [4] or [5]; or we could even require all near-rings to have commutative addition, and/or to satisfy the identity $xyz = 0$.

(c) As mentioned in the example above, we have $xy \in M \times \{0\} \times \{0\}$ for every x and y in N, which implies $[N, N]_S \subseteq M \times \{0\} \times \{0\}$ (which is in fact equality, since we know that $[A, B]_S = M \times \{0\} \times \{0\}$). On the other hand, $xy = (0, 0, 0) = 0$ for every $x \in N$ and $y \in M \times \{0\} \times \{0\}$, which implies $[N, M \times \{0\} \times \{0\}]_S = 0$. This shows that N is a nilpotent object of class 2.

(d) We do not fully understand the role and behaviour of the operations $\bullet$, $\mathcal{C}$ and $\mathcal{C}'$; further investigations, including comparisons with weighted commutators [6], may yield here more information.

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