

FAIR SEMIGROUPS AND MORITA EQUIVALENCE

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ABSTRACT. In analogy to the xst-rings studied by García and Marín, we define fair semigroups and investigate Morita equivalence for a subclass of them. In particular, we present examples for semigroups which are Morita equivalent but not strongly Morita equivalent.

1. INTRODUCTION

In recent years, a satisfactory theory of Morita equivalence has been developed for semigroups with local units, see [6–8]. The efficiency of this theory is largely due to the fact, established by Lawson [8], that in this case Morita equivalence is strong in the sense that it comes from a well-behaved Morita context. Little is known about Morita equivalence under assumptions weaker than having local units, see [6]. In particular, no example has been known for Morita equivalence between two semigroups which are not strongly Morita equivalent.

In the present paper we consider Morita equivalence in a different class of semigroups, which we call fair semigroups. This class corresponds to the class of xst-rings considered by García and Marín [3], based on previous work by Xu, Shum, and Turner-Smith [11], and the tools we use also correspond to those in [3]. Every semigroup S has a largest unitary right ideal $U(S_S)$, and if S is right fair then $U(S_S)$ is a two-sided ideal. We show that if S and T are right fair semigroups such that $U(S_S)$ and $U(T_T)$ have common weak right local units then S and T are Morita equivalent if and only if $U(S_S)$ and $U(T_T)$ are also. This makes it possible to handle cases where one of the semigroups is not factorisable, and we obtain semigroups S, T which are Morita equivalent but not strongly Morita equivalent.

A semigroup S is called **factorisable** if every element of S is a product of two elements. We say that an element s of a semigroup has a **weak right local unit** u if $su = s$. Weak left local units are defined similarly. A semigroup has **weak local units** if each of its elements has both a weak right and a weak left local unit, and **local units** if the elements u above can always be chosen to be idempotent.

We say that a semigroup S has **common weak right local units** if for every $s, t \in S$ there exists $u \in S$ such that $s = su$ and $t = tu$. Semigroups with common weak left local units are defined similarly. A semigroup has **common weak local units** if it has common weak right local units and common weak left local units.

2010 *Mathematics Subject Classification.* 20M10, 20M30.

Key words and phrases. Fair semigroup, unitary act, Morita equivalence.

Research of the first named author was partially supported by the Estonian Institutional Research Project IUT20-57. Research of the second named author was partially supported by the Hungarian National Research Foundation grant no. K101515. Mutual visits of the authors were made possible by the exchange agreement between the Estonian and the Hungarian Academies of Sciences.

Clearly, if S is a semigroup with common weak right local units then for every finite subset $\{s_1, \dots, s_n\} \subseteq S$ there exists $u \in S$ such that $s_k = s_k u$ for all $k \in \{1, \dots, n\}$.

A semigroup S is called **right reductive** if, for every $s, t \in S$, $sz = tz$ for all $z \in S$ implies $s = t$. For a semigroup S we have the following implication:

common weak right local units \Rightarrow right reductive.

Indeed, let $s, t \in S$, where S has common weak right local units. Then $s = su$ and $t = tu$ for some u in S . Suppose that $sz = tz$ for all $z \in S$. In particular, $su = tu$, which means that $s = t$. Hence S is right reductive.

2. FAIR SEMIGROUPS

In this section we introduce and investigate fair semigroups; these are semigroups which correspond to the xst-rings investigated by García and Marín [3].

Definition 1. Let S be a semigroup. A right S -act A_S is called

- (1) **unitary** if $AS = A$;
- (2) **s-unital** if for every $a \in A$ there exists $s \in S$ such that $as = a$.

Clearly, every s-unital right act is unitary.

Definition 2. We say that a semigroup S is a **right fair semigroup** if every subact of a unitary right S -act is unitary. Dually one defines left fair semigroups. By a **fair semigroup** we mean a semigroup which is both left and right fair.

Proposition 1. *A semigroup S is a right fair semigroup if and only if every unitary right S -act is s-unital.*

Proof. Necessity. Let S be a right fair semigroup and let A_S be a unitary act. Then, for every $a \in A$, the subact $aS^1 \subseteq A_S$ is unitary. But this implies that A_S is s-unital.

Sufficiency is evident. □

Proposition 2. *A semigroup S has weak right local units if and only if S is a factorisable right fair semigroup.*

Proof. Necessity. Assume that a semigroup S has weak right local units. Then it is clearly factorisable. Let A_S be unitary and $a \in A$. Then $a = a's$ for some $a' \in A$ and $s \in S$. By assumption, $s = su$ for some $u \in S$. Hence $a = a's = a'su = au$. This proves that S is a right fair semigroup.

Sufficiency. Let S be a factorisable right fair semigroup. Since S is factorisable, the right S -act S_S is unitary, hence for each $s \in S$ there exists $u \in S$ such that $s = su$. □

Example 1. Factorisable semigroups need not have weak right local units. Take the multiplicative semigroup of real numbers between 0 and 1 (0 and 1 excluded). This semigroup is factorisable but none of its elements has a weak right local unit.

There are also fair semigroups which do not have weak right local units (see Example 3(4)).

Next we give a description of fair semigroups which does not involve S -acts.

Theorem 1. *A semigroup S is a right fair semigroup if and only if for every sequence $(s_i)_{i \in \mathbb{N}} \in S^{\mathbb{N}}$ of elements of S there exists $n \in \mathbb{N}$ such that the product $s_n \dots s_1$ has a weak right unit.*

Proof. Necessity. Let S be a right fair semigroup. Consider a sequence $(s_i)_{i \in \mathbb{N}} \in S^{\mathbb{N}}$. We repeat a construction from [5]. Take the right S -act

$$F_S := \prod_{\mathbb{N}} S = \bigcup_{n \in \mathbb{N}} (\{n\} \times S)$$

with the right S -action $(n, s)z := (n, sz)$, and put

$$M_S := F / \sim,$$

where the right S -act congruence \sim on F is defined by

$$(k, s) \sim (l, z) \iff (\exists n \in \mathbb{N})(n \geq k, l \text{ and } s_n \dots s_{k+1}s = s_n \dots s_{l+1}z),$$

$k, l \in \mathbb{N}$, $s, z \in S$ (for $n = k$, $s_n \dots s_{k+1}s := s$). Denote the congruence class of a pair (k, s) by $[k, s]$. Take any $[k, s] \in M$, where $k \in \mathbb{N}$, $s \in S$. Since $(s_{k+2}s_{k+1})s = s_{k+2}(s_{k+1}s)$, we have $(k, s) \sim (k+1, s_{k+1}s)$, and hence $[k, s] = [k+1, s_{k+1}s] = [k+1, s_{k+1}]s \in MS$. Thus M_S is unitary.

By assumption, M_S is s -unital. Take the element $[1, s_1] \in M$. Then $[1, s_1] = [1, s_1]u = [1, s_1u]$ for some $u \in S$. Hence there exists $n \in \mathbb{N}$ such that $s_n \dots s_2s_1 = s_n \dots s_2s_1u$, so u is a weak right unit for $s_n \dots s_2s_1$.

Sufficiency. Let A_S be a unitary S -act and $a_0 \in S$. Repeatedly using unitariness we can find elements $a_1, a_2, \dots \in A$ and $s_1, s_2, \dots \in S$ such that $a_{i-1} = a_i s_i$ for each $i \in \mathbb{N}$. By assumption there exists $n \in \mathbb{N}$ such that $s_n \dots s_1$ has a weak right unit, say u . Hence

$$a_0 = a_1 s_1 = a_2 s_2 s_1 = \dots = a_n s_n \dots s_1 = a_n s_n \dots s_1 u = a_0 u.$$

Thus S is a right fair semigroup. \square

Example 2. Consider any set S which contains at least three elements. Choose distinct elements $0, a, e \in S$ and define a multiplication on S as follows. Put $se := s$ for every $s \in S$ and let all other products be 0. In the smallest case when $S = \{0, a, e\}$, the multiplication table is thus

	0	a	e
0	0	0	0
a	0	0	a
e	0	0	e

If $x, y, z \in S$ and $z \neq e$ then $(xy)z = 0 = x(yz)$. If $z = e$ then $(xy)z = xy = x(yz)$. Hence S is a semigroup. The element e is a right identity of S ; in particular, S is factorisable and right fair. On the other hand, S is not left fair because the subproducts of the sequence a, e, e, e, \dots are all equal to a and $sa = 0$ for all $s \in S$.

Note also that the elements a and e have no common weak left unit, hence a (finite) factorisable semigroup need not have common weak left (or, dually, right) local units.

Let us list some consequences of Theorem 1.

Corollary 1. *Free semigroups and free commutative semigroups are not fair.*

Corollary 2. *If S is a right fair semigroup then for every $s \in S$ there exists $n \in \mathbb{N}$ such that s^n has a weak right unit.*

Let $U(S_S)$ be the union of all right ideals I of a semigroup S which are right unitary, that is, $IS = I$. Then $U(S_S)$ is the largest right ideal of S which is right unitary. Dually one can consider the left ideal $U({}_S S)$. Note that if S is a factorisable semigroup then both S_S and ${}_S S$ are unitary, hence for factorisable semigroups S we have $U(S_S) = U({}_S S) = S$.

Corollary 3. *For every element s of a right fair semigroup S there exists $n \in \mathbb{N}$ such that $s^n \in U(S_S)$.*

Proof. Let $s \in S$. Then there exist $n \in \mathbb{N}$ and $u \in S$ such that $s^n = s^n u$. Now $s^n = s^n u = s^n u u \in (s^n S^1)S$ and, for every v in S , $s^n v = s^n u v \in (s^n S^1)S$, so $s^n S^1$ is a unitary right ideal of S . Hence $s^n S^1 \subseteq U(S_S)$ and $s^n \in U(S_S)$. \square

The next lemma will be used repeatedly.

Lemma 1. *If S is a right fair semigroup then $U(S_S)$ is a two-sided ideal of S .*

Proof. By the construction, $U(S_S)$ is a right ideal of S . Let $s \in S$ and $u \in U(S_S)$. Since $U(S_S)$ is right unitary, it is right s -unital, so $u = ut$ for some $t \in S$. Hence $su = sut$. But then $suS^1 = (suS^1)S$, that is, the principal right ideal suS^1 is a unitary right S -act. By the definition of $U(S_S)$, $su \in suS^1 \subseteq U(S_S)$. This shows that $U(S_S)$ is a left ideal of S . \square

Lemma 2. *Let S be a fair semigroup. For every $s \in S$, the following assertions are equivalent.*

- (1) $s \in U(S_S)$.
- (2) $s = su$ for some $u \in S$.
- (3) $s \in U({}_S S)$.
- (4) $s = us$ for some $u \in S$.

Proof. Since $U(S_S)$ is a right unitary S -act, (1) \Rightarrow (2) because S is a fair semigroup. Similarly, (3) \Rightarrow (4).

(4) \Rightarrow (1). Let $s = us$ for some $u \in S$. By Corollary 3 there exists $n \in \mathbb{N}$ such that $u^n \in U(S_S)$. Hence $s = us = u^n s \in U(S_S)$, because $U(S_S)$ is a right ideal. A similar proof works for (2) \Rightarrow (3). \square

By Lemma 2, $U(S_S) = U({}_S S)$, so we denote this set by $U(S)$ and call it **the unitary part** of the fair semigroup S .

Corollary 4. *If S is a fair semigroup then the set*

$$U(S) = \{s \in S \mid s = su = vs \text{ for some } u, v \in S\}$$

is a two-sided ideal of S . Moreover, $U(S)$ is a semigroup with weak local units (hence also a fair semigroup).

Proof. The first part follows from Lemmas 1 and 2. To prove the second part, take $s \in U(S)$. Then $s = su = vs$ for some $u, v \in S$. By Corollary 3, there exist $m, n \in \mathbb{N}$ such that $u^m, v^n \in U(S)$. Then $s = su^m = v^n s$. \square

Corollary 5. *If S is a right fair semigroup, A_S is a unitary act and $a \in A$ then $a = au$ for some $u \in U(S_S)$.*

Proof. Since A_S is unitary, it is s -unital. Hence $a = as$ for some $s \in S$. By Corollary 3, there exists $n \in \mathbb{N}$ such that $s^n \in U(S_S)$. So $a = as = as^n$. \square

We conclude this section by giving several examples of semigroups which are fair and semigroups which are not.

Example 3. 1. Every semigroup with weak local units (in particular every monoid) is a fair semigroup.

2. If S is a semigroup such that S^n is a right fair semigroup for some $n \in \mathbb{N}$ then S itself is a right fair semigroup. In particular, every nilpotent semigroup (a semigroup with zero in which every product of a given length is zero) and every inflation of a fair semigroup is a fair semigroup.

3. The multiplicative semigroup of a right xst-ring is a right fair semigroup. To see this, recall first that a module M_R over an arbitrary ring R is said to be unitary if $MR = M$ and that every module M_R has a largest submodule which is unitary. Let S be a right xst-ring and consider its multiplicative semigroup. Take any sequence $(s_i)_{i \in \mathbb{N}} \in S^{\mathbb{N}}$. In the proof of Proposition 6 of [3] it is shown that there exists $n \in \mathbb{N}$ such that $s_n s_{n-1} \dots s_1 \in \overline{U(S_S)}$, where $\overline{U(S_S)}$ denotes the largest unitary right ideal of the ring S (we have to distinguish between the largest unitary right ideal of S in the ring sense and in the semigroup sense). By Proposition 1 of [3] this means that the product $s_n s_{n-1} \dots s_1$ has a weak right unit. By our Theorem 1, this shows that S is a right fair semigroup.

4. Every finite monogenic semigroup $\langle s \rangle$ is a fair semigroup. If the index of s is at least two then this semigroup is non-factorisable.

5. A homomorphic image of a fair semigroup is a fair semigroup.

6. A direct product of finitely many fair semigroups is a fair semigroup.

Example 4. 1. A subsemigroup of a fair semigroup need not be a fair semigroup: for any semigroup T , T^1 is a fair semigroup.

2. A direct product of infinitely many fair semigroups is not necessarily a fair semigroup. Let $T_n = \langle t_n \rangle$, $n \in \mathbb{N}$, be an n -element monogenic semigroup, where $t_n^n = t_n^{n+1}$. Put $S := \prod_{n \in \mathbb{N}} T_n$ and consider the element $s = (t_1, t_2, t_3, \dots) \in S$. If we suppose that S is a fair semigroup then, by Corollary 2, there exists $k \in \mathbb{N}$ and $u = (u_1, u_2, u_3, \dots) \in S$ such that $s^k u = s^k$. Then $t_i^k u_i = t_i^k$ for each $i \in \mathbb{N}$. In particular, $t_{k+1}^k = t_{k+1}^k u_{k+1} = t_{k+1}^{k+1}$, which cannot happen.

3. An ideal extension of a fair semigroup by a fair semigroup need not be itself fair: in Example 2, $I = \{0, a\}$ is an ideal of $S = \{0, a, e\}$, S/I is a two-element semilattice, hence both of them are fair semigroups by items 1 and 2 in Example 3, but S is not fair.

4. A finite semigroup generated by two elements need not be fair: $\{a, e\}$ is a generating set for the non-fair semigroup $S = \{0, a, e\}$ in Example 2.

To give further examples for fair semigroups, we need the following observation.

Lemma 3. *If a semigroup S has the descending chain condition (DCC) for principal left ideals then for each sequence $(s_i)_{i \in \mathbb{N}} \in S^{\mathbb{N}}$ there exist $n \in \mathbb{N}$ and $u \in S^1$ such that*

$$s_n \dots s_1 = u s_{n+1} s_n \dots s_1.$$

Proof. Consider a sequence $(s_i)_{i \in \mathbb{N}} \in S^{\mathbb{N}}$ and the descending chain

$$S^1 s_1 \supseteq S^1 s_2 s_1 \supseteq S^1 s_3 s_2 s_1 \supseteq \dots$$

By assumption, there exists $n \in \mathbb{N}$ such that $S^1 s_n \dots s_1 = S^1 s_{n+1} s_n \dots s_1$. Hence $s_n \dots s_1 = u s_{n+1} s_n \dots s_1$ for some $u \in S^1$. \square

Proposition 3. *Every commutative semigroup with DCC for principal ideals is a fair semigroup.*

Proof. Let S be a commutative semigroup with DCC for principal ideals. Consider a sequence $(s_i)_{i \in \mathbb{N}} \in S^{\mathbb{N}}$. By Lemma 3, there exist $n \in \mathbb{N}$ and $u \in S^1$ such that $s_n \dots s_1 = u s_{n+1} s_n \dots s_1$. By commutativity, $s_n \dots s_1 = (s_n \dots s_1)(s_{n+1}u)$. Hence S is a fair semigroup by Theorem 1. \square

Corollary 6. *Every finite commutative semigroup is a fair semigroup.*

The converse of Proposition 3 is not true: a commutative monoid need not have DCC for principal ideals.

3. MORITA EQUIVALENCE

In this section we study Morita equivalence of right fair semigroups whose unitary part has common weak right local units. We denote the category of all unitary right S -acts by \mathbf{UAct}_S . First we define a notion which is analogous to that of a torsionfree module.

Definition 3. A right S -act A_S is called **nonsingular** if $a = a'$ ($a, a' \in A$) whenever $as = a's$ for all $s \in S$.

Proposition 4 (cf. [3], Proposition 2). *Let S be a fair semigroup such that $U(S)$ has common weak right local units. Then every unitary right S -act is nonsingular.*

Proof. Let A_S be unitary and let $a, a' \in A$ be such that $as = a's$ for each $s \in S$. By Corollary 5, there exist $u, u' \in U(S)$ such that $a = au$ and $a' = a'u'$. Let $uv = u$ and $u'v = u'$ where $v \in U(S)$. Then $av = a'v$ and

$$a = au = auv = av = a'v = a'u'v = a'u' = a'.$$

\square

Hence, under the assumptions of Proposition 4, \mathbf{UAct}_S is the category of unitary nonsingular right S -acts.

Definition 4. A right S -act A_S is called **closed** if the mapping $\mu_A : A \otimes_S S \rightarrow A, a \otimes s \mapsto as$ is bijective.

The category of all closed right S -acts is denoted by \mathbf{Fact}_S .

Obviously, A_S is unitary if and only if μ_A is surjective. Hence, for any semigroup S , \mathbf{Fact}_S is a subcategory of \mathbf{UAct}_S .

Lemma 4. *Let S be a semigroup with common weak right local units. Then $\mathbf{UAct}_S = \mathbf{Fact}_S$.*

Proof. We only need to show that if A_S is a unitary act then μ_A is injective. Suppose that $as = a's'$, $a, a' \in A$, $s, s' \in S$. Then there exists $u \in S$ such that $s = su$ and $s' = s'u$. Hence $a \otimes s = a \otimes su = as \otimes u = a's' \otimes u = a' \otimes s'u = a' \otimes s'$ in $A \otimes_S S$. \square

Proposition 5 (cf. [3], Proposition 7). *Let S be a right fair semigroup such that $I := U(S_S)$ has common weak right local units. Then*

- (1) *the categories \mathbf{UAct}_S and \mathbf{UAct}_I are isomorphic;*
- (2) *the categories \mathbf{Fact}_S and \mathbf{Fact}_I are isomorphic.*

Proof. (1) In a natural way, every right S -act is also a right I -act. By Corollary 5, every unitary right S -act is a unitary right I -act. Also, every morphism of right S -acts is a morphism of right I -acts. In this way we obtain a functor $F : \mathbf{UAct}_S \rightarrow \mathbf{UAct}_I$.

Let now $X_I \in \mathbf{UAct}_I$. We endow the set X with a right S -action as follows. Let $x \in X$ and $s \in S$. Since X_I is unitary, it is s -unital and hence $x = xu$ for some $u \in I$. Then $us \in I$, because I is an ideal. We put

$$x \cdot s := x(us).$$

Let us check that the definition does not depend on the choice of u . Suppose that also $x = xv$ where $v \in I$. By assumption, $us = usw$ and $vs = vsw$ for some $w \in I$. Hence $sw \in I$ and

$$x(us) = x(usw) = (xu)(sw) = x(sw) = (xv)(sw) = x(vsw) = x(vs).$$

To prove that we obtain an S -act, take $x \in X$ and $s, t \in S$. Let $x = xu$ for some $u \in I$. Then $us = (us)v$ for some $v \in I$. So $x(us) = x((us)v)$ and

$$\begin{aligned} (x \cdot s) \cdot t &= x(us) \cdot t = (x(us))(vt) = x((us)(vt)) = x(((us)v)t) = x((us)t) \\ &= x(u(st)) = x \cdot (st). \end{aligned}$$

Suppose now that $g : X_I \rightarrow Y_I$ is a morphism of unitary right I -acts. Let $x \in X$, $s \in S$, and let $u \in I$ be such that $x = xu$. Then $g(x) = g(x)u$ and

$$g(x \cdot s) = g(x(us)) = g(x)(us) = g(x) \cdot s,$$

which means that g is a morphism of right S -acts. So putting $G(X_I) := X_S$ and $G(g) := g$ we obtain a functor $G : \mathbf{UAct}_I \rightarrow \mathbf{UAct}_S$.

Let us prove that $G(F(A_S)) = A_S$, that is, that the S -action of $G(F(A_S))$ coincides with the S -action of A_S . Take $a \in A$, $s \in S$ and let $a = au$ where $u \in I$. Then the S -action of $G(F(A_S))$ is $(a, s) \mapsto a(us) = (au)s = as$, as needed.

Finally we show that $F(G(X_I)) = X_I$. Consider a unitary I -act X_I with the I -action $(x, u) \mapsto x * u$. Take $x \in X$ and $v \in I$. Let $u \in I$ be such that $x = x * u$. Then the I -action of $F(G(X_I))$ is $(x, v) \mapsto x * (uv) = (x * u) * v = x * v$, that is, the same as the I -action of X_I . This completes the proof of the first part.

(2) a) First we show that F takes closed right S -acts to closed right I -acts. Because of the remark after Definition 4, we only need to consider injectivity of the corresponding μ . Assume that A_S is closed and consider the mapping $\mu : A \otimes_I I \rightarrow A$, $a \otimes u \mapsto au$. Suppose that $au = a'u'$ where $a, a' \in A$ and $u, u' \in I$. We need to prove that $a \otimes u = a' \otimes u'$ in $A \otimes_I I$. By assumption, $a \otimes u = a' \otimes u'$ in $A \otimes_S S$. Hence

$$\begin{array}{rcl} a & = & a_1 s_1 & s_1 u & = & t_1 z_1 \\ a_1 t_1 & = & a_2 s_2 & s_2 z_1 & = & t_2 z_2 \\ a_2 t_2 & = & a_3 s_3 & s_3 z_2 & = & t_3 z_3 \\ & & \dots & & & \dots \\ a_{n-1} t_{n-1} & = & a_n s_n & s_n z_{n-1} & = & t_n u' \\ a_n t_n & = & a' & & & \end{array}$$

for some $a_k \in A$, $s_k, t_k \in S^1$, $z_l \in S$, where $k \in \{1, \dots, n\}$ and $l \in \{1, \dots, n-1\}$. For each k there exists $u_k \in I$ such that $a_k = a_k u_k$. Now $u_k t_k z_k \in I$, and since every finite subset of I has a common weak right local unit, there exists $v \in I$ such

that $u_k t_k z_k = u_k t_k z_k v$ for each k . Therefore

$$\begin{aligned} a \otimes u &= a_1 u_1 s_1 \otimes u = a_1 \otimes u_1 s_1 u = a_1 \otimes u_1 t_1 z_1 = a_1 \otimes u_1 t_1 z_1 v \\ &= a_1 u_1 t_1 \otimes z_1 v = a_2 u_2 s_2 \otimes z_1 v = a_2 \otimes u_2 s_2 z_1 v \\ &= a_2 \otimes u_2 t_2 z_2 v = \dots = a' \otimes u' \end{aligned}$$

in $A \otimes_I I$, as needed.

b) We prove that G takes closed right I -acts to closed right S -acts. Consider a closed I -act X_I . Take $x, x' \in X$ and $s, s' \in S$. Let u, u' be such that $x = xu$ and $x' = x'u'$. Suppose that $x \cdot s = x' \cdot s'$, that is, $x(us) = x'(u's')$. By assumption, $x \otimes us = x' \otimes u's'$ in $X \otimes_I I$. But then obviously $x \otimes us = x' \otimes u's'$ in $X \otimes_S S$ and hence $x \otimes s = x' \otimes s'$ in $X \otimes_S S$. So the mapping $\mu : X \otimes_S S \rightarrow X$, $x \otimes s \mapsto x \cdot s$ is injective and X_S is closed. \square

In the ring case (see [3], Proposition 7) no assumption is needed on $U(S_S)$, but here we cannot do without assuming the existence of common weak right units in it. Namely, in the ring case this condition follows from the s -unital property by a simple orthogonalisation (see [10], Theorem 1), but in semigroups it is obviously not true.

Definition 5. Semigroups S and T are called **right Morita equivalent** if the categories \mathbf{FAct}_S and \mathbf{FAct}_T are equivalent, **left Morita equivalent** if the categories ${}_S\mathbf{FAct}$ and ${}_T\mathbf{FAct}$ are equivalent, **Morita equivalent** if they are left and right Morita equivalent, and **strongly Morita equivalent** (cf. [9]) if there exists a unitary Morita context $(S, T, {}_S P_T, {}_T Q_S, \theta, \phi)$ with θ and ϕ surjective.

By Proposition 1 of [6] we know that if two semigroups are strongly Morita equivalent then both of them must be factorisable.

In the papers [6], [8], and [9] one can find many examples of (strongly) Morita equivalent semigroups with local units. Next we give some examples in the case of semigroups which do not have local units.

Example 5. Consider a non-trivial semigroup S with zero multiplication. Then S is a fair semigroup where $U(S) = \{0\}$ has common weak local units. By Proposition 5 and its dual, S is right and left Morita equivalent to a one-element semigroup. But a non-trivial semigroup with zero multiplication is not factorisable, hence it cannot be strongly Morita equivalent to any semigroup.

Consider now Example 2 again: let Z be a semigroup with zero multiplication, and extend Z with a right identity e such that $ez = 0$ for all $z \in Z$. As we have seen in Example 2, S is factorisable and right fair but not left fair. We claim that S is strongly Morita equivalent to the two-element semilattice. Indeed, by Theorem 9 of [8], a semigroup S is strongly Morita equivalent to a monoid if and only if $S = SeS$ for some idempotent $e \in S$. In our case we have $s = see$ for every $s \in S$, hence $S = SeS$. From the proof of Theorem 9 in [8] it follows that S is strongly Morita equivalent to the two-element monoid $eSe = \{e, 0\}$.

Proposition 6. *Every finite monogenic semigroup is Morita equivalent to its group part. Consequently, two such semigroups are Morita equivalent if and only if their periods are equal.*

Proof. If S is a finite monogenic semigroup then $U(S)$ is the group part of S , and the first claim is true by Proposition 5. By [1] and [4], two groups are Morita equivalent if and only if they are isomorphic, which gives the second claim. \square

Example 6. Suppose that $S = \langle s \rangle$ is a finite monogenic semigroup where the index of s is ≥ 2 and the group part is T . Then S and T are left and right Morita equivalent but they are not strongly Morita equivalent since S is not factorisable (the element s has no factorisation).

Let us also mention that Banaschewski [1] and Knauer [4] showed that if two monoids S and T are Morita equivalent and S is a group or is commutative then S and T are isomorphic. The proposition above shows that the situation is different in the case of semigroups: a (finite) abelian group can be Morita equivalent to a semigroup which is not even factorisable.

Theorem 2. *Let S, T be right fair semigroups such that $U(S_S), U(T_T)$ have common weak right local units. Then the following are equivalent.*

- (1) *The categories \mathbf{UAct}_S and \mathbf{UAct}_T are equivalent.*
- (2) *The categories \mathbf{Fact}_S and \mathbf{Fact}_T are equivalent.*
- (3) *The semigroups S and T are Morita equivalent.*
- (4) *The categories $\mathbf{UAct}_{U(S_S)}$ and $\mathbf{UAct}_{U(T_T)}$ are equivalent.*
- (5) *The categories $\mathbf{Fact}_{U(S_S)}$ and $\mathbf{Fact}_{U(T_T)}$ are equivalent.*
- (6) *The semigroups $U(S_S)$ and $U(T_T)$ are Morita equivalent.*

Proof. (1) \Leftrightarrow (4) and (2) \Leftrightarrow (5) by Proposition 5. (4) \Leftrightarrow (5) by Lemma 4. (2) \Leftrightarrow (3) and (5) \Leftrightarrow (6) by Definition 5. \square

Theorem 2 reduces the study of Morita equivalence of fair semigroups S, T whose unitary parts $U(S), U(T)$ have common weak right local units to the study of Morita equivalence of their unitary parts. Since $U(S), U(T)$ are factorisable semigroups (actually semigroups with weak local units), results from [2] concerning unitary acts can be applied. If it happens that $U(S), U(T)$ are semigroups with local units then also the theory developed in [8] and [6] is applicable.

For a semigroup S , define the congruence

$$\zeta_S = \{(s_1, s_2) \in S \times S \mid ss_1 = ss_2 \text{ for all } s \in S\}$$

and denote by S' the quotient semigroup S/ζ_S . From [2], Theorem 3 we have the following result.

Proposition 7. *Let S and T be factorisable semigroups. The category of unitary nonsingular right S -acts is equivalent to the category of unitary nonsingular right T -acts if and only if the semigroups S' and T' are strongly Morita equivalent.*

For left reductive semigroups, ζ_S is the equality relation and $S' = S$. Now, if $U(S), U(T)$ have common weak local units then they are left reductive, hence $U(S)' = U(S)$ and $U(T)' = U(T)$.

Proposition 8. *Let S, T be fair semigroups such that $U(S), U(T)$ have common weak local units. Then the following are equivalent.*

- (1) *The semigroups S and T are Morita equivalent.*
- (2) *The semigroups $U(S)$ and $U(T)$ are Morita equivalent.*
- (3) *The semigroups $U(S)$ and $U(T)$ are strongly Morita equivalent.*

Proof. The equivalence of (1) and (2) follows immediately from Theorem 2. By the same theorem, $U(S)$ and $U(T)$ are Morita equivalent if and only if $\mathbf{UAct}_{U(S)}$ and $\mathbf{UAct}_{U(T)}$ are equivalent. By Proposition 4, $\mathbf{UAct}_{U(S)}$ ($\mathbf{UAct}_{U(T)}$) is the category of unitary nonsingular right $U(S)$ -acts ($U(T)$ -acts). It follows from Proposition 7 that the last two categories are equivalent if and only if $U(S)$ and $U(T)$ are strongly Morita equivalent. This completes the proof of the equivalence of (2) and (3). \square

Corollary 7. *Two semigroups with common weak local units are Morita equivalent if and only if they are strongly Morita equivalent.*

Proof. By Proposition 2, a semigroup has weak local units if and only if it is factorisable and fair, and for a factorisable semigroup S we have $U(S) = S$, and now our claim follows from the equivalence of the last two conditions of the foregoing proposition. \square

Concluding remarks. 1. For semigroups with local units, left Morita equivalence, right Morita equivalence, and strong Morita equivalence are the same relation by [8]. It is not known how much of this remains true for semigroups which do not have local units. We do not have an answer even for semigroups with weak local units or for factorisable semigroups. All we know is that Morita equivalence and strong Morita equivalence do not coincide in the class of all semigroups.

2. In [7], several properties are shown to be invariant under strong Morita equivalence in the presence of factorisability or various kinds of local units. Example 7 shows that the properties of being a group, being a monoid, having local units or weak local units, and factorisability are not invariant under Morita equivalence in the class of all semigroups. In Example 5 we have seen that every semigroup with zero multiplication is Morita equivalent to the one-element semigroup, and by Theorem 16 in [6] we know that a semigroup is strongly Morita equivalent to a one-element semigroup if and only if it is a rectangular band. Hence being a rectangular band is also not invariant under Morita equivalence. These examples show also that the ideal lattice of a semigroup need not be preserved under Morita equivalence.

ACKNOWLEDGEMENT

We are grateful to Iris Lüsi for finding Example 2. We would also like to thank Victoria Gould for discussions that led us to Proposition 3.

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