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NEARLY SUBADDITIVE SEQUENCES

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Dedicated to the 80th birthday of Endre Szemerédi

Abstract. We show that the de Bruijn–Erdős condition for the error term in their improvement of Fekete's Lemma is not only sufficient but also necessary in the following strong sense. Suppose that given a sequence $0 \le f(1) \le f(2) \le f(3) \le \cdots$ such that

$$\sum_{n=1}^{\infty} f(n)/n^2 = \infty. \tag{1}$$

Then, there exists a sequence $\{b(n)\}_{n=1,2,\dots}$ satisfying

$$b(n+m) \le b(n) + b(m) + f(n+m) \tag{2}$$

such that the sequence of slopes $\{b(n)/n\}_{n=1,2,\dots}$ takes every rational number.

When the series (1) is bounded we improve their result as follows. If there exist an N and a real $\mu > 1$ such that (2) holds for all pairs (n,m) with $N \le n \le m \le \mu n$, then $\lim_n b(n)/n$ exists.

1. Fekete's lemma on subadditive sequences

An infinite sequence of reals $a(1), a(2), \dots, a(n), \dots$ is called *subadditive* if $a(n+m) \le a(n) + a(m)$ holds for all integers $n,m \ge 1$. Every calculus textbook contains Fekete's [7] Lemma as a theorem or as an exercise, see, e.g., [11]. It says that if the sequence $\{a(n)\}$ is subadditive, then $\{a(n)/n\}$

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has a limit (possible negative infinity). Moreover, that limit equals to the infimum

$$\lim_{n\to\infty} \frac{a(n)}{n} = \inf_{k\geq 1} \frac{a(k)}{k}.$$

The aim of this manuscript is to explore what enhancements of Fekete's lemma are possible.

2. Sub-2 sequences and an error term by de Bruijn and Erdős

A sequence $\{a(n)\}$ is called *μ-subadditive* with a *threshold* N ((μ, N)-subadditive, for short) if

$$a(n+m) \leq a(n) + a(m) \tag{3}$$

holds for all integers n, m such that $N \leq n \leq m \leq \mu n$.

Theorem 1 (de Bruijn and Erdős, [4, Theorem 22]). *Suppose that the sequence $\{a(n)\}$ satisfies (3) for $N \leq n \leq m \leq 2n$. Then the sequence of slopes $\{a(n)/n\}$ has a limit (possible negative infinity). Moreover, that limit equals to the infimum,*

$$\lim_{n\to\infty} \frac{a(n)}{n} = \inf_{k\geq N} \frac{a(k)}{k}.$$

Actually, in [4] they considered the case $N = 1$ only. For self-containedness we present a greatly simplified proof for Theorem 1 in Section 5.

Nearly subadditive sequences. Let $f(n)$ be a non-negative, non-decreasing sequence. deBruijn and Erdős [4] called the sequence $\{a(n)\}$ subadditive with an *error term* f (or *nearly f-subadditive*, or *f-subadditive* for short) if

$$a(n+m) \leq a(n) + a(m) + f(n+m) \tag{4}$$

holds for all positive integers $n, m \geq 1$. The case $f(x) = 0$ corresponds to the cases discussed above.

They showed that if the error term f is small,

$$\sum_{n=1}^{\infty} f(n)/n^2 \text{ is finite,} \tag{5}$$

and (4) holds for all $n \leq m \leq 2n$, then the limit of $\{a(n)/n\}$ still exists.

Nearly subadditivity is *really* important. Subadditivity is important, it appears in all parts of mathematics. We all have our favorite examples and applications. But nearly subadditivity is even more applicable, here we mention a few areas.

In the beginning of the Bollobás–Riordan book [2] the de Bruijn–Erdős theorem is listed (as Lemma 2.1 on p. 37) among the important useful tools in Percolation Theory. The de Bruijn–Erdős theorem is widely used in investigating sparse random structures, e.g., Bayati, Gamarnik, and Tetali [1] (Proposition 5 on p. 4011), Turova [13], or Kulczycki, Kwietniak, and Jian Li [10] concerning entropy of shift spaces.

Also, recurrence relations of type (4) are often encountered in the analysis of divide and conquer algorithms,

$$a(n+m) \le a(n) + a(m) + \text{cost of cutting}.$$

see, e.g., Hsien-Kuei Hwang and Tsung-Hsi Tsai [9]. In economics it is an essential property of some cost functions that $\text{COST}(X+Y) \le \text{COST}(X) + \text{COST}(Y)$. Similar relations appear in physics and in combinatorial optimization (see, e.g., Steele [12]).

Also see, e.g., Capobianco [5] concerning cellular automatas, Ceccherini-Silberstein, Coornaert, and Krieger[6] for an analogue on cancellative *amenable semigroups*.

3. Sub-μ sequences with $\mu < 2$

De Bruijn and Erdős [4] stated that '...the inequality in (7.1) [i.e., the condition $n/2 \le m \le 2n$] cannot be replaced by $\mu^{-1}n \le m \le \mu n$ for any $\mu < 2$'. In their papers [3,4] they deal with many conditions and sequences, we could not really know what was in their minds, but our first new result is a strengthening of Theorem 1 for all $\mu > 1$.

THEOREM 2. *Suppose $\mu > 1$ and $N \ge 1$ are given. If $\{a(1), a(2), \dots\}$ is (μ, N)-subadditive, i.e.,*

$$a(n+m) \le a(n) + a(m) \quad \forall n \le m \le \mu n,\ n, m \ge N,$$

then $\lim_{n\to\infty} \frac{a(n)}{n}$ exists and equals $\inf_{k \ge N} \frac{a(k)}{k}$. (It may be $-\infty$.)

Let us call a sequence $\{a(n)\}$ (μ, N, f)-*subadditive* if (4) holds for all $N \le n \le m \le \mu n$. Our Theorem 2 yields the following corollary.

THEOREM 3. *Suppose $\mu > 1$ and $N \ge 1$ are given and f is a non-negative monotone increasing real function satisfying* (5). *If the sequence $\{a(1), a(2), \dots\}$ is (μ, N, f)-subadditive, i.e.,*

$$a(n+m) \le a(n) + a(m) + f(n+m) \quad \forall m \le n \le \mu m,\ m, n \ge N,$$

then $\lim_{n\to\infty} \frac{a(n)}{n}$ *exists.* (*It may be* $-\infty$.)

The proofs are presented in Section 5.

4. How large the error term $f(x)$ could be?

It is very natural to ask how more one can extend the de Bruijn–Erdős theorem concerning f-nearly subadditive sequences (the case $\mu = 2$, $N = 1$). Especially, how large the error term could be?

$f(x) = o(x)$ is necessary. Suppose that $f(n)$ is non-negative and $\limsup f(n)/n > L > 0$. We can easily construct a sequence $\{a(n)\}$ satisfying (4) for all pairs $m, n \geq 1$ such that $\lim a(n)/n$ does not exist. We do not even use that f is monotone or not.

Given such an f one can find a sequence of integers $1 \leq n_1 < n_2 < n_3 < \dots$ such that $f(n_i)/n_i > L/2$, and $n_{i+1} \geq n_i + 2$ for all $i \geq 1$. Define $a(n) = f(n_i)$ if $n = n_i$ and 0 otherwise. □

$f(x) = o(x)$ is not sufficient. Condition (5) allows $f(x) = O(x^{1-c})$ ($c > 0$ fixed) or even $f(x) = O(x/(\log x)^{1+c})$. The first author observed that $f(x)$ could not be $\Omega(x/\log x)$. In 2016 he [8] proposed the following problem for Schweitzer competition for university students. "Prove that there exists a sequence $a(1), a(2), \dots$ of real numbers such that

$$a(n+m) \leq a(n) + a(m) + \frac{n+m}{\log(n+m)}$$

for all integers $m, n \geq 1$, and the set $\{a(n)/n : n \geq 1\}$ is everywhere dense on the real line." There were two correct solutions: by Nóra Frankl, and Kada Williams and two partial solutions by Balázs Maga, and János Nagy.

de Bruijn and Erdős got the best result. We show that the de Bruijn–Erdős condition (5) for the error term is not only sufficient but also necessary in the following strong sense.

THEOREM 4. *Let* $f(n)$ *be a non-negative, non-decreasing sequence and suppose*

$$\sum_{1 \leq n < \infty} f(n)/n^2 = \infty. \tag{6}$$

Then there exists a nearly f*-subadditive sequence* $b(1), b(2), b(3), \dots$ *of rational numbers, i.e., for all integers* $m, n \geq 1$

$$b(n+m) \leq b(n) + b(m) + f(n+m)$$

such that the set of slopes takes all rationals exactly once.

In particular we have $\{b(n)/n : n \geq 1\} = \mathbf{Q}$. The proof is constructive and presented in Section 6.

5. Proofs of the improvements

Here we give the proofs of Theorems 1–3.

A new proof for Theorem 1. Fix a k, $k \geq N$. Write n as $n = (\lfloor n/k \rfloor - 1)k + \beta$ where $k \leq \beta \leq 2k-1$. We will show that

$$a(n) \leq (\lfloor n/k \rfloor - 1)a(k) + a(\beta). \tag{7}$$

To prove (7) we need a definition. A sequence of positive integers $X := \{x_1, x_2, \dots, x_t\}$ is called 2-*good* if $1/2 \leq x_i/x_j \leq 2$ holds for all $1 \leq i, j \leq t$. Take the two smallest members $x_i, x_j \in X$, delete them from X and join a new member $x_{\text{new}} := x_i + x_j$. The new sequence $X' := X \setminus \{x_i, x_j\} \cup \{x_{\text{new}}\}$ is 2-good as well. Note that the sum of the members of X is the same as in X'. If the sequence $\{a(x)\}$ is $(2,N)$-subadditive then $a(x_{\text{new}}) \leq a(x_i) + a(x_j)$, which implies that

$$\sum_{x \in X'} a(x) \leq \sum_{x \in X} a(x). \tag{8}$$

Define the set $X_{\lfloor n/k \rfloor}$ of length $\lfloor n/k \rfloor$ as $\{k, k, k, \dots, k, \beta\}$. It is obviously a 2-good sequence with sum n. Define the sets X_t of length t for $\lfloor n/k \rfloor \geq t \geq 1$ by the above rule, $X_{t-1} := X'_t$. We obtain $X_{\lfloor n/k \rfloor} \longrightarrow \cdots \longrightarrow X_t \longrightarrow X_{t-1} \longrightarrow \cdots \longrightarrow X_1 = \{n\}$. Then (8) gives

$$\begin{aligned} a(n) = \sum_{x \in X_1} a(x) \leq \cdots \leq \sum_{x \in X_t} a(x) \\ \leq \cdots \leq \sum_{x \in X_{\lfloor n/k \rfloor}} a(x) = \big(\lfloor n/k \rfloor - 1\big)\, a(k) + a(\beta). \end{aligned}$$

To complete the proof observe that (7) implies

$$\frac{a(n)}{n} \leq a(k)\frac{\lfloor n/k \rfloor - 1}{n} + \frac{1}{n}\big(\max_{k \leq \beta \leq 2k-1} a(\beta)\big)$$

for all $n \geq k \geq N$. Therefore

$$\limsup_{n \to \infty} \frac{a(n)}{n} \leq \frac{a(k)}{k}$$

holds for every k. So the limit superior of the sequence $\{a(n)/n\}$ does not exceed its infimum, these two values must be equal, so the sequence is convergent. □

Sub-1^+ sequences. For the proof of Theorem 2 we investigate sequences where the subadditivity holds only for a very sparse set of pairs (n,m).

A sequence $\{a(n)\}$ is called $(1^+,N)$ *subadditive* if the following two inequalities hold for all $n \ge N$:

$$a(2n) \le a(n) + a(n), \quad a(2n+1) \le a(n) + a(n+1).$$

Define $q(n) := \max\{ \frac{a(n)}{n}, \dots, \frac{a(2n-1)}{2n-1}, \frac{a(2n)}{2n} \}$.

LEMMA 5. *Suppose that the sequence* $\{a(n)\}$ *is* $(1^+,N)$ *subadditive. Then for* $n \ge N$ *the sequence* $\{q(n)\}$ *is non-increasing,* $q(n) \ge q(n+1)$.

We only have to show that $q(n)$ is at least as large as $a(2n+1)/(2n+1)$ and $a(2n+2)/(2n+2)$. The $(1^+,N)$ subadditivity implies

$$q(n) \ge \begin{cases} \frac{a(n+1)}{n+1} \ge \frac{a(2n+2)}{2n+2}, \\ \max\left\{\frac{a(n)}{n}, \frac{a(n+1)}{n+1}\right\} \ge \frac{n}{2n+1}\frac{a(n)}{n} + \frac{n+1}{2n+1}\frac{a(n+1)}{n+1} \ge \frac{a(2n+1)}{2n+1}. \end{cases}$$

Proof of Theorem 2. Since the case $\mu \ge 2$ is covered by Theorem 1, we may suppose that $1 < \mu < 2$. We can fix a positive integer k such that

$$(1+\mu)^{k-1} \le 2^{k+1} < (1+\mu)^k.$$

Given any n define the sequences $u_0, u_1, \dots, u_k$ and $v_0, v_1, \dots, v_k$ as follows.

$$u_0 = v_0 := n, \quad u_{i+1} := 2u_i, \quad v_{i+1} := v_i + \lfloor \mu v_i \rfloor, \quad (i = 0, 1, \dots, k-1).$$

We have $u_k = 2^k n$ and $v_k > (1+\mu)^k n - (1+\mu)^k/\mu$. So there exists an N_1 (depending only on μ and k) such that $2u_k \le v_k$ holds in the above process for every integer $n \ge N_1$.

Let $N_2 := \max\{N, 1/(\mu-1)\}$. Then the sequence $\{a(n)\}$ is $(1^+, N_2)$ subadditive. Lemma 5 implies that $L = \lim_{n\to\infty} q(n)$ exists. If $L = -\infty$ then $\lim_{n\to\infty} a(n)/n = -\infty$ as well, and we are done. Since $L < \infty$, from now on, we may suppose that L is a real number.

Choose an (arbitrarily small) $\varepsilon > 0$. There exists an N_3 (depending on ε, μ, N, and $\{a(n)\}$) such that $q(n) < L + \varepsilon$ for every $n \ge N_3$. By the definition of q we get

$$a(n)/n < L + \varepsilon \tag{9}$$

for every $n \ge N_3$.

We are going to show that for $n \ge \max\{N_1, N_2, N_3\}$

$$a(n)/n > L + \varepsilon - \varepsilon\,(1+\mu)^k. \tag{10}$$

Since this holds for every $\varepsilon > 0$ the limit $a(n)/n$ exists and equals to L.

To prove (10) we need the following claim which holds for each $i \in \{0, 1, \dots, k-1\}$.

Claim 6. *If* $a(w)/w \le L + \varepsilon - \eta$ *for every* $w \in [u_i, v_i]$*, then* $a(z)/z < L + \varepsilon - \frac{\eta}{1+\mu}$ *for every* $z \in [u_{i+1}, v_{i+1}]$.

Indeed, every $z \in [u_{i+1}, v_{i+1}]$ can be written in the form $z = x + y$ where $x \in [u_i, v_i]$, $x \le y \le \mu x$. Apply subadditivity for (x, y) and the upper bound $L + \varepsilon - \eta$ for $a(x)/x$ and the upper bound $L + \varepsilon$ for $a(y)/y$. We obtain

$$\frac{a(z)}{z} = \frac{a(x+y)}{x+y} \le \frac{a(x)+a(y)}{x+y}$$
$$= \frac{a(x)}{x}\frac{x}{x+y} + \frac{a(y)}{y}\frac{y}{x+y} < (L+\varepsilon-\eta)\frac{x}{x+y} + (L+\varepsilon)\frac{y}{x+y}$$
$$= L + \varepsilon - \eta\frac{x}{x+y} \le L + \varepsilon - \eta\frac{1}{1+\mu}.$$

The end of the proof of Theorem 2. Consider any n with $n \ge \max\{N_1, N_2, N_3\}$. By (9) we have $a(n)/n = L + \varepsilon - h$ for some $h > 0$. Consider the intervals $[u_i, v_i]$ for $i = 0, 1, \dots, k$, where $[u_0, v_0]$ consists of a single element, namely n. Using Claim 6 we get that $a(x) < L + \varepsilon - h/(1+\mu)^i$ for each $x \in [u_i, v_i]$ for $1 \le i \le k$. Especially, $a(x)/x < L + \varepsilon - h/(1+\mu)^k$ for each $x \in [u_k, v_k]$. Since $2u_k \le v_k$ we obtain $q(u_k) < L + \varepsilon - h/(1+\mu)^k$. But $q(u_k) \ge L$, since $q(n)$ is non-increasing. This implies $h < \varepsilon(1+\mu)^k$. We obtained that $a(n)/n = L + \varepsilon - h > L + \varepsilon - \varepsilon(1+\mu)^k$ as claimed in (10). $\square$

Proof of Theorem 3 using Theorem 2. We utilize the proof from [4] (bottom of p. 163). For $n \ge N$ define

$$G(n) := a(n) + 3n\Bigl(\sum_{x \ge n} f(x)/x^2\Bigr).$$

Then the monotonicity of f, the relation $n \le m \le \mu n$, and an easy calculation imply that

$$G(n+m) \le G(n) + G(m)$$

whenever (4) holds for (n, m).

Theorem 2 can be applied to $\{G(n)\}$, so we have that the limit

$$\lim_{n\to\infty} \frac{G(n)}{n} = \lim_{n\to\infty}\left(\frac{a(n)}{n} + 3\Bigl(\sum_{x\ge n}\frac{f(x)}{x^2}\Bigr)\right)$$

exists. Here the last term tends to 0 as $n \to \infty$ by (5) and we are done. $\square$

6. Proof of Theorem 4, a construction

A typical subadditive function is concave like, e.g., for $a(x)=\sqrt{x}$ we have $\sqrt{x+y}\le\sqrt{x}+\sqrt{y}$ (for $x,y\ge 0$). The main idea of the construction for Theorem 4 is that a nearly f-subadditive sequence $\{a(n)\}$ could be (strictly) convex with $\lim_{n\to\infty} a(n)/n=\infty$.

A convex *f*-subadditive function.

CLAIM 7. *Suppose that $f(n)$ is a non-negative, non-decreasing sequence, $0\le f(2)\le f(3)\le\dots$. Define $f(1)=a(1)=0$ and in general let*

$$a(n):=n\biggl(\sum_{i=1}^{n}\frac{f(i)}{i^2}\biggr). \tag{11}$$

Then the sequence $\{a(n)\}$ is nearly f-subadditive, it satisfies (4).

PROOF. Write down the definition of $a(n)$, simplify, use the monotonicity of f, finally the estimate $\left(\sum_{u<i\le v}1/i^2\right)<(1/u)-(1/v)$ (for integers $1\le u<v$). We obtain

$$\begin{aligned}
&a(n+m)-a(n)-a(m)\\
&=n\biggl(\sum_{i\le n+m}\frac{f(i)}{i^2}\biggr)+m\biggl(\sum_{i\le n+m}\frac{f(i)}{i^2}\biggr)-n\biggl(\sum_{i\le n}\frac{f(i)}{i^2}\biggr)-m\biggl(\sum_{i\le m}\frac{f(i)}{i^2}\biggr)\\
&=n\biggl(\sum_{n<i\le n+m}\frac{f(i)}{i^2}\biggr)+m\biggl(\sum_{m<i\le n+m}\frac{f(i)}{i^2}\biggr)\\
&\le nf(n+m)\Bigl(\frac{1}{n}-\frac{1}{n+m}\Bigr)+mf(n+m)\Bigl(\frac{1}{m}-\frac{1}{n+m}\Bigr)=f(n+m). \qquad \square
\end{aligned}$$

CLAIM 8. *The above sequence $\{a(n)\}$ defined by* (11) *is non-negative and convex, i.e., for $n\ge 2$ we have*

$$a(n)\le\frac{a(n-1)+a(n+1)}{2}.$$

PROOF. We have

$$\begin{aligned}
&a(n-1)+a(n+1)-2a(n)\\
&=(n-1)\biggl(\sum_{i\le n-1}\frac{f(i)}{i^2}\biggr)+(n+1)\biggl(\sum_{i\le n+1}\frac{f(i)}{i^2}\biggr)-2n\biggl(\sum_{i\le n}\frac{f(i)}{i^2}\biggr)\\
&=\frac{f(n+1)}{(n+1)}-(n-1)\frac{f(n)}{n^2}\ge\frac{f(n+1)}{(n+1)n^2}\ge 0. \qquad \square
\end{aligned}$$

The end of the proof of Theorem 4. In this section $\{f(n)\}$ is given by Theorem 4, and $\{a(n)\}$ is the well-defined nearly f-subadditive, convex sequence obtained by (11) in Claim 8. Then (6) implies $\lim_{n\to\infty} a(n)/n = \infty$.

For the rest of the proof the main observation is the following: If $c(1) \le c(2) \le c(3) \le \cdots$ is a monotone sequence, and $\{a(n)\}$ is f-subadditive, then

$$b(n) := a(n) - c(n)n \ \text{ is } f \text{ subadditive as well.}$$

Indeed,

$$\begin{aligned} b(n+m) - b(n) - b(m) - f(n+m) &= \big[a(n+m) - c(n+m)(n+m)\big] \\ - \big[a(n) - c(n)n\big] &- \big[a(m) - c(m)m\big] - f(n+m) \\ &= \big[a(n+m) - a(n) - a(m) - f(n+m)\big] \\ + \big(c(n) - c(n+m)\big)n &+ \big(c(m) - c(n+m)\big)m \le 0. \end{aligned}$$

Let $r_1, r_2, r_3, \ldots$ be an enumeration of $\mathbf{Q}$. We will define a sequence $1 \le n_0 \le n_1 \le n_2 \le \cdots$ and simultaneously $\{c(n)\}$ (and thus $\{b(n)\}$ as well) such that

(D) the slopes $\{b(n)/n : 1 \le n \le n_i\}$ are all distinct and rational, and

(R) $r_i \in \{b(n)/n : 1 \le n \le n_i\}$ $(i \ge 1)$.

We proceed by induction on i. Let n_0 be the smallest $x \ge 1$ such that $f(x) > 0$. Equation (6) implies that $1 \le n_0 < \infty$. Choose $c(1) \le \ldots \le c(n_0)$ such that for all $1 \le x \le n_0$, $x \in \mathbf{N}$ the fractions $b(x)/x = (a(x) - c(x)x)/x$ are all rationals and they are all distinct. Since these are finitely many constraints of the form

$$\frac{a(x)}{x} - c(x) \ne \frac{a(y)}{y} - c(y) \quad 1 \le x \ne y \le n_0$$

and the set $\mathbf{Q}$ is everywhere dense on $\mathbf{R}$, one can easily choose appropriate $c(x)$'s.

If $n_0, n_1, \ldots, n_i$ has been already defined (satisfying properties (D) and (R)) then proceed as follows.

If $r_{i+1} \in \{b(x)/x : 1 \le x \le n_i\}$, then let $n_{i+1} := n_i$.

If $r_{i+1} \notin \{b(x)/x : 1 \le x \le n_i\}$ then define n_{i+1} as the smallest integer x satisfying

$$x > n_i, \quad \frac{a(x)}{x} - c(n_i) > r_{i+1}.$$

Such x exists. Let $c(n_{i+1}) := \frac{a(n_{i+1})}{n_{i+1}} - r_{i+1}$. It follows that $c(n_i) < c(n_{i+1})$. Then select $c(x)$ for integers x with $n_i < x < n_{i+1}$ such that the values of $a(x)/x - c(x)$ are all rationals, distinct from each other, have no common values with $\{b(n)/n : 1 \le n \le n_i\} \cup \{r_{i+1}\}$ and also $c(n_i) \le c(n_i+1) \le \cdots \le c(n_{i+1})$. These are finitely many conditions but $c(n_i) < c(n_{i+1})$ and $\mathbf{Q}$ is everywhere dense, so the induction step can be done. This completes the construction. □

7. Conclusion, problems

Let $X \subseteq \mathbf{N} \times \mathbf{N}$, $f\colon \mathbf{N} \to \mathbf{R}$. The sequence $\{a(n)\}$ is (X,f)-subadditive if $a(m+n) \le a(n) + a(m) + f(n+m)$ holds for $(n,m) \in X$. We have found conditions for X and f, strengthening the original Fekete's lemma and its de Bruijn–Erdős generalization, which ensure that $\lim a(n)/n$ exists. Concerning further thinning of X we propose two problems.

Is it possible to replace the constraint $n \le m \le \mu n$ in Theorem 2 by some condition like $n \le m \le n + r(n)$ where $r(n) = o(n)$ is some slow growing function? (Probably not).

What is the structure of 1^+ subadditive sequences? Can we tell more than Lemma 5?

Having the threshold N is a genuine extension. Indeed, consider the following sequence. Suppose that $2 \le N \le n_1 < n_2 < n_3 < \ldots$ are integers such that $n_{i+1} - N \ge n_i$ and $\limsup n_{i+1}/n_i = \infty$. Define for all $i \ge 1$ and positive integer n

$$a(n) := \begin{cases} 1 & \text{if } n \le n_1 \\ 1 & \text{if } \exists\, i \text{ such that } n_{i+1} - N \le n \le n_{i+1} - 2, \\ \lceil n/n_i \rceil & \text{if } \exists\, i \text{ such that } n_i \le n < n_{i+1} \text{ (but } |n - n_{i+1}| \notin [2,N]). \end{cases}$$

This sequence is subadditive for $m \ge n \ge N$. Indeed, if $f(n+m) \le 2$ then $f(n) + f(m) \ge f(n+m)$ since $f(x) \ge 1$ for all x. Otherwise, $n_i \le (n+m) < n_{i+1}$ for some i and $f(n+m) = \lceil (n+m)/n_i \rceil > 2$. We obtain $(n+m)/n_i > 2$ so $m > n_i$ and $n_i < m < n_{i+1}$. If $n \ge n_i$ then $n_i \le n \le m < n_{i+1} - N$ so $f(n) = \lceil n/n_i \rceil$ and $f(m) = \lceil m/n_i \rceil$ and we are done. So we may suppose that $n < n_i$. Then

$$f(n+m) = \lceil (n+m)/n_i \rceil \le 1 + \lceil m/n_i \rceil \le f(n) + f(m)$$

completing the proof.

However it does not seem to be easily transformed to a true subadditive one, because there are infinitely many (x,y) pairs, namely $1 \le x < N$ and $x + y = n_{i+1} - 1$, such that $a(x+y) - a(y) - a(x) = \lceil (n_{i+1}-1)/n_i \rceil - 2$ is arbitrarily large.

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