Choosability with Separation of Complete Multipartite Graphs and Hypergraphs

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Abstract: For a hypergraph G and a positive integer s, let $\chi_\ell(G, s)$ be the minimum value of l such that G is L-colorable from every list L with $|L(v)| = l$ for each $v \in V(G)$ and $|L(u) \cap L(v)| \leq s$ for all $u, v \in e \in E(G)$. This parameter was studied by Kratochvíl, Tuza, and Voigt for various kinds of graphs. Using randomized constructions we find the asymptotics of $\chi_\ell(G, s)$ for balanced complete multipartite graphs and for complete k-partite k-uniform hypergraphs. © 2013 Wiley Periodicals, Inc. J. Graph Theory 76: 129–137, 2014

1. INTRODUCTION

Given a hypergraph G, a *list* L for G is an assignment to every $v \in V(G)$ of a set $L(v)$ of colors that may be used for the coloring of v. We say that G is *L-colorable*, if there exists a proper coloring f of the vertices of G from L, i.e. if $f(v) \in L(v)$ for all $v \in V(G)$ and no edge of G is monochromatic in f. A list L for a hypergraph G is a *k-list* if $|L(v)| = k$ for all $v \in V(G)$. An extensively studied parameter is the *list chromatic number* of G, $\chi_\ell(G)$, introduced by Vizing [9] and Erdős et al. [3]. For a hypergraph G, $\chi_\ell(G)$ is the least k such that G is L-colorable for every k-list L. This parameter is also sometimes called *choice number*, or *choosability* of G.

By definition, $\chi_\ell(G) \geq \chi(G)$ for any hypergraph G. Moreover, $\chi_\ell(G)$ may be much larger than $\chi(G)$. For example, $\chi_\ell(K_{n,n})$ has the order of $\log n$ (see, e.g. [1]), while $\chi(K_{n,n}) = 2$. It is natural to ask what happens when the lists of adjacent vertices in (hyper)graphs do not intersect too much.

For a positive integer s, a list L for a hypergraph G is *s-separated* if $|L(u) \cap L(v)| \leq s$ for all pairs $\{u, v\}$ such that some edge of G contains both, u and v. If G is a graph, this means that for each $uv \in E(G)$, $L(u)$, and $L(v)$ share at most s colors. Let $\chi_\ell(G, s)$ denote the minimum k such that G is L-colorable from each s-separated k-list L. By definition, for every $1 \leq s_1 \leq s_2$,

$$\chi_\ell(G, s_1) \leq \chi_\ell(G, s_2) \leq \chi_\ell(G). \tag{1}$$

Kratochvíl et al. [7] studied $\chi_\ell(G, s)$ for various G and s. They showed the following.

Theorem 1 ([7]). *For positive integers s, n with $s \leq n$, $\sqrt{\frac{1}{2}sn} \leq \chi_\ell(K_n, s) \leq \sqrt{2esn}$.*

So the ratio of the upper and lower bounds is $2\sqrt{e} \sim 3.29$. In [4], the asymptotics of $\chi_\ell(K_n, s)$ for every fixed s was found.

Theorem 2 ([4]). *For every fixed s, $\lim_{n \to \infty} \frac{\chi_\ell(K_n, s)}{\sqrt{sn}} = 1$.*

Since $\chi_\ell(K_n) = \chi(K_n) = n$, Theorems 1 and 2 show that for fixed s and large n, $\chi_\ell(K_n, s)$ is much less than $\chi_\ell(K_n)$. In this article, we study list colorings from s-separated lists of balanced complete multipartite graphs and uniform hypergraphs. It turns out that even for small s, $\chi_\ell(K_{n,n}, s)$ and $\chi_\ell(K_{n,n})$ are asymptotically the same. Let

$K(k, m) = K_{m,m,\ldots,m}$ denote the complete multipartite graph with k partite sets of size m. One of our main results is

Theorem 3. *For every fixed k,*

$$\chi_\ell(K(k, m), 1) = (1 + o(1))\chi_\ell(K(k, m)) = (1 + o(1))\log_{k/(k-1)} m.$$

In view of (1), this means that for any s and any fixed k,

$$\lim_{m \to \infty} \frac{\chi_\ell(K(k, m), s)}{\log_{k/(k-1)} m} = 1.$$

We also prove a result of similar nature for balanced complete k-uniform k-partite hypergraphs. Recall that a hypergraph is *k-partite* if its vertex set can be partitioned into k sets $V_1, \ldots, V_k$ so that each edge contains at most one vertex from each set. A k-partite (hyper)graph is *balanced* if all parts have equal sizes.

Let $K^k(k, m) = K^k_{m,m,\ldots,m}$ denote the complete k-uniform k-partite hypergraph with partite sets of size m. Our second main result is:

Theorem 4. *For every fixed k,*

$$\chi_\ell(K^k(k, m), 1) = (1 + o(1))\chi_\ell(K^k(k, m)) = (1 + o(1))\log_k m.$$

The upper bounds in Theorems 3 and 4 were known. To prove the lower bounds, we need constructions of several uniform **nearly disjoint** hypergraphs on the same vertex set each of which has small independence number. We show that such probabilistic constructions are possible and present them in the next section. We think that these constructions are of interest by themselves. Using these construction and the approach to list colorings used in [3] and later in [6], we prove Theorems 3 and 4 in Sections 3 and 4, respectively. Although we use only basic probabilistic tools, namely the first moment method, our asymptotics for the nearly disjoint case are very close to the classical ones.

2. NEARLY DISJOINT HYPERGRAPHS WITH SMALL INDEPENDENCE NUMBER

Hypergraphs H_1 and H_2 are *nearly disjoint* if every edge of H_1 meets every edge of H_2 in at most one vertex. Hypergraphs $H_1, H_2, \ldots, H_k$ are nearly disjoint if they are pairwise nearly disjoint.

For a hypergraph H, $\Delta(H)$ denotes the *maximum degree* of the vertices in H, $\alpha(H)$ denotes the *independence number of H*, i.e. the size of a largest subset of vertices of H not containing edges of H, and $\tau(H) = |V(H)| - \alpha(H)$ denotes the *transversal number* of H. We are interested in constructing k nearly disjoint hypergraphs each of size m with small independence numbers. We first cite two results and prove a lemma which we then use to construct these nearly disjoint hypergraphs.

The following theorem is due to Erdős [2]. He proved it for the case when $k = 2$, but his proof can easily be extended to any fixed k.

Theorem 5 ([2])**.** *Let $k \geq 2$ be fixed. For r sufficiently large, there exists r-uniform hypergraphs H on $n = \lceil \frac{k-1}{2}r^2 \rceil$ vertices with at most $\frac{e}{2}r^2 k^{r-1}(k-1)\ln k$ edges such that $\alpha(H) < n/k$.*

The following theorem is a partial case of a more general result by Lovász (Corollary 2 in [8]).

Theorem 6 ([8])**.** *Let u be the minimum size of an edge in a hypergraph G with maximum degree Δ. Then*

$$\tau(G) \leq (1 + 1/2 + \cdots + 1/\Delta)|V(G)|/u.$$

For a positive integer r and a real number b, the binomial coefficient $\binom{b}{r}$ is defined as $\frac{1}{r!}b(b-1)\cdots(b-r+1)$.

Let $a_1, \ldots, a_n$ be nonnegative integers and $a = \max_i a_i$. If $a > 0$, then

$$\sum_{i=1}^{n} \binom{a_i}{2} \leq \sum_{i=1}^{n} \frac{a_i(a-1)}{2} = \binom{a}{2}\frac{\sum_{i=1}^{n} a_i}{a}. \tag{2}$$

We now prove our main lemma.

Lemma 1.   *For each $0 < a < 1$ and integers $t \geq 1$, $r \geq 2$ and $q \geq \frac{r^2}{a}$, there exists an r-uniform hypergraph H with tq vertices such that*

   (i) *$V(H) = T_1 \cup T_2 \cup, \ldots, \cup T_q$, where $|T_1| = \cdots = |T_q| = t$ and disjoint,*
   (ii) *every edge in H meets every T_i in at most one vertex,*
   (iii) *$\alpha(H) < atq$,*
   (iv) *$|E(H)| \leq (1/a)^r \cdot 4tq$.*

   Proof.   Let $S = T_1 \cup T_2 \cup, \ldots, \cup T_q$. To prove the lemma, we will construct an auxiliary hypergraph $\mathcal{H}$ whose vertex set is

$$V(\mathcal{H}) = \bigcup_{1 \leq i_1 < i_2 \cdots < i_r \leq q} T_{i_1} \times T_{i_2} \times \cdots \times T_{i_r}.$$

For every set $X \subset S$, we consider the set E_X of r-sets that are contained in X and members of $V(\mathcal{H})$. The sets E_X for every set X with $|X| = \lceil a|S| \rceil$ will form the edges of $\mathcal{H}$. A vertex cover of this hypergraph $\mathcal{H}$ gives us a collection of r-subsets of S with the property that if we take any set $X \subset S$ with $|X| \geq a|S|$, then we get an r-set that is entirely contained in X. A minimum vertex cover of $\mathcal{H}$ gives us our required hypergraph H with vertex set $V(H) = S$ and $|E(H)| = \tau(\mathcal{H})$.

We first estimate the size of E_X. Let A_X denote the number of r-subsets of X that meet some T_i in at least two vertices. (Note that we may assume that $t \geq 2$, since $A_X = 0$ if $t = 1$.) Then

$$A_X \leq \sum_{i=1}^{q} \binom{|X \cap T_i|}{2}\binom{|X| - 2}{r - 2} \leq \frac{|X|}{t}\binom{t}{2}\binom{|X| - 2}{r - 2} = \frac{1}{2}\binom{|X|}{r}\frac{(t-1)r(r-1)}{|X| - 1}$$

$$\leq \frac{1}{2}\binom{|X|}{r},$$

where the second inequality is due to (2) and the last inequality is by the choice of q, since

$$(t-1)r(r-1) \leq (t-1)r^2 \leq a(t-1)q \leq atq - 1 \leq |X| - 1.$$

Hence

$$|E_X| = \binom{|X|}{r} - A_X \ge \frac{1}{2}\binom{|X|}{r}.$$

Applying Theorem 6 for $G = \mathcal{H}$ and $u = \min |E_X|$ we get,

$$\tau(\mathcal{H}) \le \frac{|V(\mathcal{H})|}{\min |E_X|}(1 + \ln \Delta) \le \frac{\binom{|S|}{r}}{\frac{1}{2}\binom{|X|}{r}}(1 + \ln 2^{|S|}) \le \frac{\binom{|S|}{r}}{\frac{1}{2}\binom{a|S|}{r}}(|S|)$$

$$= 2\left(\frac{1}{a}\right)^r \frac{(|S|)(|S|-1)\cdots(|S|-(r-1))}{(|S|)(|S|-\frac{1}{a})\cdots(|S|-(\frac{r-1}{a}))}|S| \le 2\left(\frac{1}{a}\right)^r |S| \prod_{0 \le i \le r-1} \frac{|S|-i}{|S|-\frac{1}{a}i}.$$

Here the last product is less than 2, since

$$\prod_{0 \le i \le r-1} \frac{|S|-i}{|S|-\frac{1}{a}i} = \prod_{0 \le i \le r-1}\left(1 + \frac{(1-a)i}{a|S|-i}\right) \le \exp \frac{(1-a)\sum_{i<r} i}{r^2 - r + 1} < \exp \frac{1-a}{2}.$$

Hence

$$\tau(\mathcal{H}) \le (1/a)^r \cdot 4tq. \qquad \blacksquare$$

Construction: Iterative Method for constructing nearly disjoint hypergraphs:

Let an integer $q \ge \frac{r^2}{a}$ be fixed. We start with a q-vertex empty hypergraph. We use Lemma 1 and obtain a hypergraph G_1^1 such that $\alpha(G_1^1) < a|V(G_1^1)|$. After $i-1$ more iterations, we have hypergraphs $G_1^i, G_2^i, \ldots, G_i^i$, where G_j^i is just q vertex disjoint copies of G_j^{i-1} (where $j < i$) and G_i^i is obtained by taking q copies of $V(G_{i-1}^{i-1})$ and using Lemma 1.

Note that we have the following:

1. G_α^i, G_β^i are nearly disjoint for all $\alpha \ne \beta$;
2. $|V(G_j^i)| = q^i$ for all $j \le i$;
3. $|E(G_j^i)| \le (1/a)^r \cdot 4 \cdot q^{i-1}q = (1/a)^r \cdot 4q^i$ for all $j \le i$;
4. $\alpha(G_j^i) < a|V(G_j^i)|$ for all $j \le i$.

Remark.   Note that it the above construction we took a q-vertex empty hypergraph and applied Lemma 1 to it to get G_1^1. One can instead (for r sufficiently large) start with the hypergraph G_1^1 given in Theorem 5 and slightly improve the result.

We have the following corollary from the above construction.

Corollary 1.   *Let $k \ge 2$, $r \ge 2$, $0 < a < 1$ and $q = \lceil \frac{r^2}{a} \rceil$. There exist k nearly disjoint r-uniform hypergraphs $H_1, H_2, \ldots H_k$ on the same vertex set with q^k vertices each with $\lfloor 4q^k(\frac{1}{a})^r \rfloor$ edges such that $\alpha(H_i) < a|V(H_i)|$, for all $1 \le i \le k$.*

Proof.   From the construction mentioned above we see that we have nearly disjoint hypergraphs $H_j = G_j^k$ such that $\alpha(H_j) < a|V(H_j)|$ and $|E(H_j)| \le 4q^k(\frac{1}{a})^r$, for all $1 \le j \le k$. We just need to show we can add edges in H_j such that $|E(H_j)| = \lfloor 4q^k(\frac{1}{a})^r \rfloor$. It is in fact true that at every iteration step i, one can have $|E(G_j^i)| \le 4q^i(\frac{1}{a})^r$, for all

$1 \leq j \leq i$, since at step i, we make q copies of the vertex set with q^{i-1} from the previous step and we have at least $\binom{q}{r}(q^{(i-1)})^r$ possibilities, which is much greater than $4q^i(\frac{1}{a})^r$. ∎

Remark.   We required the sizes of the hypergraphs in the above corollary to be equal since the edges of these hypergraphs will form the list assignment for the vertices of a balanced multipartite graph. But it is not necessary. We shall use Corollary 2 to generalize the result to unbalanced multipartite graphs.

Corollary 2.   *Let $k \geq 2$, $r \geq 2$, $0 < a < 1$ and $q = \lceil \frac{r^2}{a} \rceil$. There exist k nearly disjoint r-uniform hypergraphs $H_1, H_2, \ldots H_k$ on the same vertex set with q^k vertices with $|E(H_i)|$ obtaining any value in $[4q^k(\frac{1}{a})^r, \binom{q}{r}q^{(i-1)r+(k-i)}]$ such that $\alpha(H_i) < a|V(H_i)|$, for all $1 \leq i \leq k$.*

Proof.   Consider the hypergraph G_i^i obtained at Step i in the construction. As we saw in Corollary 1, for every i, G_i^i can have at most $4q^i(\frac{1}{a})^r$ edges. In fact, we can add all the possible $\binom{q}{r}(q^{(i-1)})^r$ edges. It is easy to see that we still maintain that the hypergraphs obtained so far in the construction are nearly disjoint. Moreover, we also do not increase the independence number of the hypergraphs by adding more edges. In the next $k - i$ steps of the construction we just take q^{k-i} copies of G_i^i to obtain H_i that has at most $\binom{q}{r}q^{(i-1)r+(k-i)}$ edges. ∎

3.   COLORING COMPLETE MULTIPARTITE GRAPHS WITH *s*-SEPARATED LISTS

Recall that $K(k, m) = K_{m,m,\ldots,m}$ denotes the complete multipartite graph with k partite sets of size m. We will use the ideas of [3] and [6] and the results of the previous section to prove Theorem 3. For convenience, we restate it here.

Theorem 7.   *For every fixed k,*

$$\chi_\ell(K(k, m), 1) = (1 + o(1))\chi_\ell(K(k, m)) = (1 + o(1)) \log_{k/(k-1)} m.$$

Proof.   Let G be a copy of $K(k, m)$ with partite sets $V_1, \ldots, V_k$. Let L be an r-list for G. Let $C := \bigcup_{v \in V(G)} L(v)$. Then, since G is complete k-partite, G is L-colorable if and only if we can partition C into sets $C_1, \ldots, C_k$ so that for each $i = 1, \ldots, k$ and each $v \in V_i, C_i \cap L(v) \neq \emptyset$. Let $H = H(G, L)$ be the r-uniform hypergraph with the vertex set C whose edges are the lists of the vertices of G. Since lists of some vertices in G may coincide, H may have multiple edges. For $i = 1, \ldots, k$, let E_i be the set of edges of H that correspond to the lists of the vertices in V_i. So, G is L-colorable if and only if

(*) *we can color $V(H)$ with k colors $1, \ldots, k$ so that for every i and every edge $A \in E_i$, A contains a vertex of color i.*

To get the upper bound we show the following statement.

$$\text{Let } km < \left(\frac{k}{k-1}\right)^r. \text{ Then } \chi_\ell(K(k, m), 1) \leq \chi_\ell(K(k, m)) \leq r. \tag{3}$$

By the above, it is enough to prove that for $km < \left(\frac{k}{k-1}\right)^r$, every r-uniform hypergraph H with $E(H) = E_1 \cup \ldots \cup E_k$ where $|E_i| = m$ for $i = 1, \ldots, m$ has a k-coloring satisfying

(*). We color each $v \in V(H)$ randomly: v gets color i with probability $1/k$ independently from all other vertices. An edge $A \in E_i$ is *happy* if some vertex of A gets color i, and *unhappy* otherwise. For each $A \in E(H)$, the probability that A is unhappy is $(1 - 1/k)^r$. Thus the expectation of the number of unhappy edges is at most $km\left(\frac{k-1}{k}\right)^r < 1$. So, there exists a coloring f such that every edge is happy. This proves (3).

To prove the lower bound, observe that L is 1-separated if and only if the corresponding hypergraphs $H_1 = (V(H), E_1), \ldots, H_k = (V(H), E_k)$ are nearly disjoint. Let $q := \lceil \frac{kr^2}{(k-1)} \rceil$. By Corollary 1 for $a = (k-1)/k$, there exist nearly disjoint r-uniform hypergraphs $H_1, \ldots, H_k$ on the same vertex set, say V, such that for every $i = 1, \ldots, k$,

(a) $|E(H_i)| \leq 4(\frac{k}{k-1})^r q^k \leq 4(\frac{k}{k-1})^r 2^k r^{2k}$;
(b) $\alpha(H_i) < \frac{k-1}{k}|V|$.

We claim that the hypergraph $H := \bigcup_{i=1}^k H_i$ does not satisfy (*). Indeed, suppose that there is a k-coloring f such that for every i and every edge $A \in E_i$, $f^{-1}(i) \cap A \neq \emptyset$. We may assume that $|f^{-1}(1)| \geq \cdots \geq |f^{-1}(k)|$. Let $B = V - f^{-1}(k)$. By our ordering, $|B| \geq \frac{k-1}{k}|V|$. So by (b), some edge of H_k is contained in B, a contradiction to the choice of f. Thus if k is fixed and positive integers r and m satisfy

$$m \geq 4\left(\frac{k}{k-1}\right)^r (2r^2)^k,$$

then $\chi_\ell(K(k, m), 1) \geq 1 + r$. Since for fixed k,

$$\ln\left(4\left(\frac{k}{k-1}\right)^r (2r^2)^k\right) = r\ln\left(\frac{k}{k-1}\right) + 2k\ln r + (k+2)\ln 2$$

$$= r\ln\left(\frac{k}{k-1}\right)(1 + o(1)),$$

the theorem is proved.   ∎

We now state an easy generalization of Theorem 7 for unbalanced multipartite graphs. It follows easily from Corollary 2 and the proof of Theorem 7.

Theorem 8.   *Given positive integers $k, m_1, m_2, \ldots m_k$ such that $m_1 \leq m_2 \leq \cdots \leq m_k$, let r be the largest integer such that $m_1 \geq 4q^k(\frac{k}{k-1})^r$, where $q = \lceil \frac{k}{k-1}r^2 \rceil$.*

If $m_i \in [4q^k(\frac{k}{k-1})^r, \binom{q}{r}q^{(i-1)r+(k-i)}]$, for all $1 \leq i \leq k$, then $\chi_\ell(K_{m_1, m_2, \ldots m_k}, 1) > r$.
In other words $\chi_\ell(K_{m_1, m_2, \ldots m_k}, 1) \geq (1 - o(1))\log_{k/(k-1)} m_1$

Remark.   One might want to show a matching upper bound or improve the lower bound and give a new bound in terms of m_k, when m_k is not too large compared to m_1. A similar result was shown about the the choice number χ_ℓ of unbalanced multipartite graphs in [5].

4. COLORING COMPLETE *K*-UNIFORM *K*-PARTITE HYPERGRAPHS WITH *s*-SEPARATED LISTS

Recall that $K^k(k, m) = K^k_{m,m,\ldots,m}$ denotes the complete k-uniform k-partite hypergraph with k partite sets of size m. In this section, we prove Theorem 4. For convenience, we restate it here.

Theorem 9. *For every fixed k,*

$$\chi_\ell(K^k(k, m), 1) = (1 + o(1))\chi_\ell(K^k(k, m)) = (1 + o(1)) \log_k m.$$

Proof. Let G be a copy of $K^k(k, m)$ with partite sets $V_1, \ldots, V_k$. Let L be an r-list for G. Let $C := \bigcup_{v \in V(G)} L(v)$. Since G is complete k-uniform k-partite, a coloring of $V(G)$ is proper if and only if no color is present on each V_i. Thus, G is L-colorable if and only if we can partition C into sets $C_1, \ldots, C_k$ so that for each $i = 1, \ldots, k$ and each $v \in V_i$, C_i does not contain $L(v)$. As in the proof of Theorem 3, let $H = H(G, L)$ be the r-uniform hypergraph with the vertex set C whose edges are the lists of the vertices of G. For $i = 1, \ldots, k$, let E_i be the set of edges of H that correspond to the lists of the vertices in V_i. So, G is L-colorable if and only if

(**) *we can color $V(H)$ with k colors $1, \ldots, k$ so that for every i and every edge $A \in E_i$, A is not monochromatic of color i.*

First we prove:

$$\text{Let } m < k^{r-1}. \text{ Then} \chi_\ell(K^k(k, m), 1) \le \chi_\ell(K^k(k, m)) \le r. \tag{4}$$

It is enough to prove that for $m < k^{r-1}$, every r-uniform hypergraph H with $E(H) = E_1 \cup \ldots \cup E_k$ where $|E_i| = m$ for $i = 1, \ldots, m$ has a k-coloring satisfying (**). We color each $v \in V(H)$ randomly: v gets color i with probability $1/k$ independently from all other vertices. An edge $A \in E_i$ is *happy* if some vertex of A gets color distinct from i, and *unhappy* otherwise. For each $A \in E(H)$, the probability that A is unhappy is k^{-r}. Thus the expectation of the number of unhappy edges is at most $kmk^{-r} < 1$. So, there exists a coloring c such that every edge is happy. This proves (4).

Now we prove the lower bound. Recall that L is 1-separated if and only if the corresponding hypergraphs $H_1 = (V(H), E_1), \ldots, H_k = (V(H), E_k)$ are nearly disjoint. Let $q := kr^2$. By Corollary 1 for $a = 1/k$, there exist nearly disjoint r-uniform hypergraphs $H_1, \ldots, H_k$ on the same vertex set, say V such that for every $i = 1, \ldots, k$,

(a) $|E(H_i)| \le 4k^r q^k$;
(b) $\alpha(H_i) < \frac{1}{k}|V|$.

We claim that the hypergraph $H := \bigcup_{i=1}^k H_i$ does not satisfy (**). Indeed, suppose that there is a k-coloring f such that for every i and every edge $A \in E_i$, $A \nsubseteq f^{-1}(i)$. We may assume that $|f^{-1}(1)| \ge |V|/k$. Then by (b), some edge of H_1 is contained in $f^{-1}(1)$, a contradiction to the choice of f. Thus if k is fixed and positive integers r and m satisfy

$$m \ge 4k^r(kr^2)^k,$$

then $\chi_\ell(K^k(k, m), 1) \ge 1 + r$. Since for fixed k,

$$\ln\left(4k^r(kr^2)^k\right) = (r + k)\ln k + 2k\ln r + 2\ln 2 = r\ln k(1 + o(1)),$$

the theorem is proved. ∎

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