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Minimal abundant packings and choosability with separation

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Abstract

A (v, k, t) packing of size b is a system of b subsets (blocks) of a v-element underlying set such that each block has k elements and every t-set is contained in at most one block. $P(v, k, t)$ stands for the maximum possible b. A packing is called *abundant* if $b > v$. We give new estimates for $P(v, k, t)$ around the critical range, slightly improving the Johnson bound and asymptotically determine the minimum $v = v_0(k, t)$ when *abundant* packings exist. For a graph G and a positive integer c, let $\chi_\ell(G, c)$ be the minimum value of k such that one can properly color the vertices of G from any assignment of lists $L(v)$ such that $|L(v)| = k$ for all $v \in V(G)$ and $|L(u) \cap L(v)| \leq c$ for all $uv \in E(G)$. Kratochvíl, Tuza and Voigt in 1998 asked to determine $\lim_{n\to\infty} \chi_\ell(K_n, c)/\sqrt{cn}$ (if it exists). Using our bound on $v_0(k, t)$, we prove that the limit exists and equals 1. Given c, we find the exact value of $\chi_\ell(K_n, c)$ for infinitely many n.

Keywords Packing of sets · t-designs · Choosability · Complete graph · Graph colorings

Mathematics Subject Classification 05C15 · 05B40

1 Preliminaries on hypergraphs

A *hypergraph* $\mathcal{H} = (V, \mathcal{E})$ consists of a set of vertices $V = V(\mathcal{H})$ and a collection $\mathcal{E}$ of subsets of V called edges or blocks, i.e., multiple copies of edges are allowed. Often we take $V(\mathcal{H}) = [v]$, where $[v] := \{1, 2, 3, \ldots, v\}$. The *degree* of a vertex $x \in V$, denoted by $d_{\mathcal{H}}(x)$ or just by d_x, is the number of edges containing the vertex x.

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A set of distinct vertices $\{x_1, , \dots, x_m\}$ is called a *system of distinct representatives* (SDR, for short) of the (multi)family $\mathcal{E} = \{E_1, \dots, E_m\}$ if $x_i \in E_i$ for each $i \in [m]$. By classic Hall's Theorem [1], $\mathcal{E}$ has an SDR if and only if it satisfies Hall's condition

$$\left| \bigcup_{E \in \mathcal{E}'} E \right| \geq |\mathcal{E}'| \quad \text{for all } \mathcal{E}' \subseteq \mathcal{E}. \tag{1}$$

A hypergraph is k-*uniform* if each of its edges has k elements. It is a t-*packing* if $|E \cap E'| < t$ for any two distinct edges $E, E' \in \mathcal{E}$. The following theorem is usually attributed to Johnson [2], who used it to get upper bounds for error-correcting codes. It was rediscovered several times, e.g., Bassalygo [3], Corrádi [4]. Let $\mathcal{E} := \{E_1, \dots, E_m\}$ be a family of k-sets such that $|E_i \cap E_j| < t$ for all $1 \leq i < j \leq m$. Then

$$v := \left| \bigcup_{i=1}^{m} E_i \right| \geq \frac{mk^2}{(m-1)(t-1)+k}. \tag{2}$$

A (v, k, t) packing of size b is a system of b subsets (blocks) of a v-element underlying set such that each block has k elements and every t-set is contained in at most one block. $P(v, k, t)$ stands for the maximum possible b. A packing is called *abundant* if $b > v$. For example, the finite affine plane $AG(2, q)$ of order q is a (perfect) $(q^2, q, 2)$ packing with $q^2 + q$ blocks, so it is abundant.

Let $v_0(k, t)$ stand for the minimum v that $P(v, k, t) > v$. For example, we have $v_0(q, 2) \leq q^2$ if an $AG(2, q)$ exists. Applying (2) to $v + 1$ blocks of an abundant packing one gets $v \geq (v+1)k^2/(v(t-1)+k)$. Rearranging we get $v\left(v(t-1) - k^2 + k\right) \geq k^2$ and thus

$$v_0(k, t) > (k^2 - k)/(t-1). \tag{3}$$

2 Main result and an application

Our main aim here is to show that (3) gives the true order of magnitude of v_0.

Theorem 1 *Let $t \geq 2$ and suppose that $k \to \infty$. Then $v_0(k, t) = (1 + o(1))\dfrac{k^2}{t-1}$.*

Leaving out an arbitrary element from each block of a (v, k, t) packing one obtains a $(v, k-1, t)$ packing of the same size (for $k > t \geq 2$). In this way one can see that the sequence $\{v_0(k, t) : k = t, t+1, t+2, \dots\}$ is strictly increasing. So Theorem 1 follows from (3) and an explicit construction of an infinite series of abundant packings for a dense sequence of k's giving us an asymptotically matching upper bound. From now on, it will be more convenient to use c for $t - 1$. The following construction is presented in Sect. 4.

Construction 2 *Let $c \geq 1$ and suppose that q is a prime power, $c < q - 1$ and c divides $q - 1$. Then*

$$P\left(\frac{q^2-1}{c} + 1, q, c+1\right) \geq \frac{q^2-1}{c} + \left\lfloor \frac{q+1}{c} \right\rfloor.$$

We obtain for these values that

$$v_0(q, c+1) \leq \frac{q^2-1}{c} + 1. \tag{4}$$

It is known (see [5]) that for every sufficiently large real k there exists a prime $q \in [k, k+k^{0.6}]$ such that c divides $q-1$. Then the monotonicity of v_0 and (4) yield

$$v_0(k, c+1) \le v_0(q, c+1) \le \frac{q^2-1}{c} + 1 < \frac{k^2}{c} + O(k^{1.6}).$$

This together with (3), completes the proof of Theorem 1.

Theorem 1 can answer a question on list colorings of graphs. Recall that a *list* L for a graph G is an assignment to every $v \in V(G)$ a set $L(v)$ of colors that may be used for coloring v. Graph G is *L-colorable* if there exists a proper coloring f of the vertices of G from L, i.e., if $f(v) \in L(v)$ for all $v \in V(G)$ and $f(u) \neq f(v)$ for all $uv \in E$. The *list chromatic number* of G, $\chi_\ell(G)$, is the least k such that G is L-colorable, whenever $|L(v)| = k$ for all $v \in V(G)$.

A list L for a graph G is a *(k, c)-list* if $|L(v)| = k$ for all $v \in V(G)$ and $|L(u) \cap L(v)| \le c$ for all $uv \in E(G)$. Kratochvíl, Tuza and Voigt [6] introduced $\chi_\ell(G, c)$, the least k such that G is L-colorable from each (k, c)-list L. They showed that $\sqrt{cn/2} \le \chi_\ell(K_n, c) \le \sqrt{2ecn}$, where K_n is the complete graph on n vertices. They asked whether the limit $\lim_{n\to\infty} \chi_\ell(K_n, c)/\sqrt{cn}$ exists. In Sect. 5 we use Theorem 1 to prove that the limit exists and is 1. We also find the exact value of $\chi_\ell(K_n, c)$ for infinitely many values of n.

3 Explicit gaps

There are many results concerning packings when equality holds in (2). These packings have $1+(k^2-k)/(t-1)$ blocks and are called *symmetric* $(t-1)$-designs, see [1]. By improving (3) we establish large explicit gaps between $v_0(k, t)$ and $v_0(k+1, t)$ for many cases. These gaps will be used in our second topic concerning (k, c)-list colorings of graphs (see Sect. 5).

Claim 3 *Let $q > c \ge 1$. Then* $v_0(q+1, c+1) \ge \frac{1}{c}\left(q^2 + q + \frac{2(q-c+1)}{c+1}\right) + 1.$

Note that using (2), i.e., the inequality $v \ge (v+1)(q+1)^2/(vc+q+1)$, leads to the bound $v_0 > (q^2+q)/c + O(1)$. So we have a slight improvement on the Johnson bound in this critical range.

One can summarize (4) and Claim 3 in one formula:

$$v_0(q, c+1) \le \frac{q^2-1}{c} + 1 < \frac{1}{c}\left(q^2 + q + \frac{2(q-c+1)}{c+1}\right) + 1 \le v_0(q+1, c+1) \quad (5)$$

whenever q is a prime power, $1 \le c < q-1$, and c divides $q-1$.

Lemma 4 *Let $c \ge 1$ and suppose that $\mathcal{E}$ is a q-uniform hypergraph on vertex set Y such that $|\mathcal{E}| = q+2$ and $|E \cap E'| \le c-1$ for any two distinct edges. Then $|Y| \ge \frac{1}{c}(q^2+q+\frac{2(q-c+1)}{c+1})$.*

Proof Let d_y be the degree of the vertex y. We have

$$\sum_{y\in Y} \binom{d_y}{2} = \sum_{E, E' \in \mathcal{E}: E \neq E'} |E \cap E'| \le (c-1)\binom{q+2}{2}, \quad (6)$$

$$\sum_{y\in Y} d_y = \sum_{E\in\mathcal{E}} |E| = (q+2)q. \quad (7)$$

Multiply (6) by -2, (7) by $2c$, add them up and rearrange. We get

$$c(c+1)|Y| + \sum_y -(d_y - c)(d_y - c - 1) \geq (c+1)q(q+1) + 2(q-c+1).$$

Discarding the summation and rearranging we get the desired lower bound for $|Y|$. □

Proof of Claim 3 Let $\mathcal{P}$ be an abundant $(v, q+1, c+1)$ packing on the vertex set V. Since

$$\sum_{x \in V} d_x = \sum_{P \in \mathcal{P}} |P| = |\mathcal{P}|(q+1) > v(q+1),$$

there exists an $x \in V$ with $d_x > q+1$. So one can find $q+2$ edges of $\mathcal{P}$ of the form $\{x\} \cup E_i$ where the family $\{E_1, \ldots, E_{q+2}\}$ is a $(v-1, q, c)$ packing on the vertex set $Y = V \setminus \{x\}$. One can now apply Lemma 4 to complete the proof. □

4 Construction of a packing

In this section we present Construction 2, a $\left(\frac{q^2-1}{c} + 1, q, c+1\right)$ packing $\mathcal{P}$ of size $\frac{q^2-1}{c} + \left\lfloor \frac{q+1}{c} \right\rfloor$ whenever $c \geq 1$, q is a prime power, $c < q-1$, and c divides $q-1$.

Let $\mathbf{F}$ be the q-element finite field and let g be an element of order c in the multiplicative group $\mathbf{F}\backslash\{0\}$. Set $H = \{1, g, g^2, ..., g^{c-1}\}$. It is a c-element subgroup of $\mathbf{F}\backslash\{0\}$. For $(a, b), (a', b') \in (\mathbf{F} \times \mathbf{F})$ we say that $(a, b) \sim (a', b')$ if there exists an $h \in H$ such that $(a', b') = (ha, hb)$. This is an equivalence relation with $\{(0, 0)\}$ being a 1-element class. Each other equivalence class is a collection of c elements in $(\mathbf{F} \times \mathbf{F})\backslash\{(0, 0)\}$. So there are $1 + (q^2 - 1)/c$ equivalence classes. The equivalence class containing (a, b) is denoted by $\langle a, b\rangle$. These equivalence classes form the vertex set V of the packing $\mathcal{P}$.

For $(a, b) \neq (0, 0)$, define the set $L\langle a, b\rangle = \{\langle x, y\rangle : ax + by \in H\}$. Since H is a group, $ax + by \in H$ implies $(h'a)x + (h'b)y \in H$, for all $h' \in H$. Hence $L\langle a, b\rangle$ is a well-defined subset of V. The next statement is a consequence of basic linear algebra.

Claim 5 (Furedi [7]) *Let $(V, \mathcal{L})$ be the hypergraph with vertex set $V = \{\langle a, b\rangle : a, b \in \boldsymbol{F}\}$ and edge set $\mathcal{L} = \{L\langle a, b\rangle : a, b \in \boldsymbol{F}, (a, b) \neq (0, 0)\}$. Then*

- (i) *$\mathcal{L}$ is a q-uniform hypergraph, $|L\langle a, b\rangle| = q$,*
- (ii) *V has $1 + (q^2 - 1)/c$ vertices,*
- (iii) *$\mathcal{L}$ has $(q^2 - 1)/c$ edges.*
- (iv) *Suppose that $(a, b) \nsim (a', b')$. Then $|L\langle a, b\rangle \cap L\langle a', b'\rangle| = c$ whenever $\det\begin{pmatrix} a & b \\ a' & b' \end{pmatrix} \neq 0$ and $|L\langle a, b\rangle \cap L\langle a', b'\rangle| = 0$ whenever this determinant is 0.*

Define the sets $V_m := \{\langle x, y\rangle : y = mx, (x, y) \neq (0, 0)\}$ for $m \in \mathbf{F}$ and let $V_\infty := \{\langle x, y\rangle : x = 0, (x, y) \neq (0, 0)\}$. Then $|V_\alpha| = (q-1)/c$ and these sets form a partition of $V \setminus \langle 0, 0\rangle$. Moreover, $|V_\alpha \cap L\langle a, b\rangle| \leq 1$ for each $\langle a, b\rangle \in V$.

Select $\lfloor (q+1)/c \rfloor$ disjoint c-sets $C_1, C_2, \ldots$ from $\mathbf{F} \cup \{\infty\}$ and define $L(i) := \cup\{V_\alpha : \alpha \in C_i\} \cup \{\langle 0, 0\rangle\}$. Then these are q-element sets pairwise meeting in $\langle 0, 0\rangle$. Moreover, $|L(i) \cap L\langle a, b\rangle| \leq c$. Finally, $\mathcal{P} := \mathcal{L} \cup \{L(1), L(2), \ldots\}$ is a packing we were looking for.

5 List colorings

In this section we answer the question of Kratochvíl, Tuza and Voigt [6] on colorings of complete graphs from (k, c)-lists.

Theorem 6 *Let $c \geq 1$. Then*

(i) $\lim_{n\to\infty} \chi_\ell(K_n, c)/\sqrt{cn} = 1$.
(ii) *If q is a prime power, $c < q - 1$ and c divides $q - 1$, then $\chi_\ell(K_n, c) = q + 1$ for all*

$$n \in \left[\frac{q^2-1}{c} + 2, \frac{1}{c}\left(q^2 + q + \frac{2(q-c+1)}{c+1}\right) + 1\right].$$

Proof The complete graph K_n is L-colorable if and only if the set of lists $\{L(v) : v \in [n]\}$ satisfy Hall's condition (1). (This observation is due to Vizing [8].) A (k, c)-list corresponds to a $(c+1)$-packing of k-sets. So $\chi_\ell(K_n, c) > k$ if and only if there is an abundant $(v, k, c+1)$ packing with $v \leq n$. Hence

$$\chi_\ell(K_n, c) = q + 1 \iff v_0(q, c+1) < n \leq v_0(q+1, c+1). \tag{8}$$

To make (8) more clear, let us explain. If $v_0(q, c+1) < n$, then there exists a $(v, q, c+1)$ packing $\mathcal{P}$ of size $v + 1 \leq n$. Assign the members of $\mathcal{P}$ to the first $v + 1$ vertices of K_n and assign completely disjoint q-sets to the rest of the vertices. This assignment does not satisfy Hall's condition, so we obtain $\chi_\ell(K_n, c) > q$. On the other hand, if $n \leq v_0(q + 1, c + 1)$ then any $(q + 1, c)$-list assignment L of K_n is a $(c + 1)$-packing of $(q + 1)$-sets of size at most $v_0(q+1, c+1)$. So neither $\{L(v) : v \in [n]\}$ is abundant, nor any part of it is abundant. Therefore, it satisfies Hall's condition and thus implying K_n is L-colorable.

The proof now follows from Theorem 1, (5), and (8). □

For a fixed $c \geq 1$, one might be interested in knowing what is the maximum value of $\chi_\ell(G, c)$ over all n-vertex graphs G. Note that if H is an induced subgraph of G, then $\chi_\ell(H, c) \leq \chi_\ell(G, c)$, but this may not hold true for non-induced subgraphs. We have the following conjecture.

Conjecture 7 *If $c, n \geq 1$ and G is an n-vertex graph, then $\chi_\ell(G, c) \leq \chi_\ell(K_n, c)$.*

Work [6] generated lots of further research, especially concerning planar graphs, e.g., [9]. For further recent results concerning separated list colorings see [10, 11].

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