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Preface to

Induced Turán problems and traces of hypergraphs

Let me start with telling what the Turán type problems are. Suppose that F is a given "small" graph, then the maximum number of edges of a "large" graph G with n vertices which containing no copy of F as a subgraph is denoted by $\mathrm{ex}(n, F)$. Pál Turán suggested to study this quantity. Although the classic theorem of Erdős, Stone and Simonovits has asymptotically determined the value of $\mathrm{ex}(n, F)$ for the graphs F having chromatic number > 2, graph theorists are still publishing many papers trying to find, on the one hand, the exact values, on the other hand the asymptotic value for bipartite graphs. In spite of the fact that the original area of Turán type problems has not been completely settled, the curiosity of the mathematicians raised analogous and more general questions. One of these directions is the generalizations for hypergraphs when r-element subsets are the edges on the vertex set in contrast to the traditional graphs when $r = 2$. Already Turán has asked in 1945 what is the maximum number of 3-element subsets on an n element set without having all 4 of them on 4 vertices. It is still unknown. There is no Erdős, Stone and Simonovits type theorem here, only some sporadic results. The other variant is when we do not necessarily forbid the small graph F, only when it is an induced copy of F. Prömel and Steger and recently Loh, Tait, Timmons and Zhou gave strong theorems for this case.

Concerning the hypergraph case, a relatively new development is to forbid the so called Berge hypergraphs. If F if a graph, the corresponding Berge r-hypergraphs are obtained by blowing up the edges of F, that is adding $r - 2$ new vertices to each edge of F in such a way that the new r-edges are different but can overlap each other arbitrarily. The family of all such hypergraphs is denoted by $\mathrm{B}^{\,r}F = \mathrm{B}F$. The corresponding Turán type problem: determine $\mathrm{ex}_r(n, \mathrm{B}F)$, the maximum number of edges in an r-hypergraph with n vertices containing no member of $\mathrm{B}F$ as a subhypergraph. There is a growing interest and there are some nice results in this direction, but no general theory was found.

After all of these it is quite natural to combine the previous problems and ask for $\mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F)$ that is for the maximum number of edges when only the induced copies of the Berge hypergraphs are forbidden in an r-hypergraph on n vertices. But it seems to be hopelessly difficult. This is why it is a big surprise that Zoltán Füredi and Ruth Luo were able to determine the order of magnitude of $\mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F)$ for every graph F reducing this question to an almost classical Turán type problem. Let $\mathrm{ex}(n, K_s, F)$ denote the maximum number of copies of complete s-element subgraphs in a graph with n vertices, containing no copy of F. They proved in their paper in the following pages that the order of magnitude of $\mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F)$ is the same as the largest of the orders of magnitude of $\mathrm{ex}(n, K_s, F)$ $(2 \le s \le r)$. They also found a large class of graphs F for which the winner is $s = 2$ that is the order of magnitude is determined by the solution of a classical Turán type problem.

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Concerning the methods of the paper, let me call the reader's attention to the "α-core" of a hypergraph, I foresee a bright future of this concept.

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Induced Turán problems and traces of hypergraphs

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ABSTRACT

Let F be a graph. We say that a hypergraph $\mathcal{H}$ contains an *induced Berge F* if the vertices of F can be embedded to $\mathcal{H}$ (e.g., $V(F) \subseteq V(\mathcal{H})$) and there exists an injective mapping f from the edges of F to the hyperedges of $\mathcal{H}$ such that $f(xy) \cap V(F) = \{x, y\}$ holds for each edge xy of F. In other words, $\mathcal{H}$ contains F as a trace.

Let $\mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F)$ denote the maximum number of edges in an r-uniform hypergraph with no induced Berge F. Let $\mathrm{ex}(n, K_r, F)$ denote the maximum number of K_r's in an F-free graph on n vertices. We show that these two Turán type functions are strongly related.

1. Definitions, Berge *F* subhypergraphs

A hypergraph $\mathcal{H}$ is *r-uniform* or simply an *r-graph* if it is a family of r-element subsets of a finite set $V(\mathcal{H})$. If the *vertex set* $V(\mathcal{H})$ is clear from the text, then we associate an r-graph $\mathcal{H}$ with its edge set $E(\mathcal{H})$, and hence we use $|\mathcal{H}| = |E(\mathcal{H})|$. Usually we take $V(\mathcal{H}) = [n]$, where $[n]$ is the set of first n integers, $[n] := \{1, 2, 3, \dots, n\}$. We also use the notation $\mathcal{H} \subseteq \binom{[n]}{r}$ to denote that $\mathcal{H}$ is an r-uniform hypergraph on $[n]$. For a set of vertices $S \subseteq V(\mathcal{H})$ define the *codegree* of S, denoted by $\deg(S)$, to be the number of edges of $\mathcal{H}$ containing S. The *s-shadow*, $\partial_s\mathcal{H}$, is the family of s-sets contained in the edges of $\mathcal{H}$. So $\partial_1\mathcal{H}$ is the set of non-isolated vertices, and $\partial_2\mathcal{H}$ is the graph whose edges are the pairs with positive codegree in $\mathcal{H}$.

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Definition 1.1. Let F be a graph with vertex set $\{v_1, \ldots, v_p\}$ and edge set $\{e_1, \ldots, e_q\}$. A hypergraph $\mathcal{H}$ contains a **Berge** F if there exist distinct vertices $\{w_1, \ldots, w_p\} \subseteq V(\mathcal{H})$ and distinct edges $\{f_1, \ldots, f_q\} \subseteq E(\mathcal{H})$, such that if $e_i = v_\alpha v_\beta$, then $\{w_\alpha, w_\beta\} \subseteq f_i$. The vertices $\{w_1, \ldots, w_p\}$ are called the **base vertices** of the Berge F.

Definition 1.2. Let F be a graph with vertex set $\{v_1, \ldots, v_p\}$ and edge set $\{e_1, \ldots, e_q\}$. A hypergraph $\mathcal{H}$ contains an **induced Berge** F if there exists a set of distinct vertices $W := \{w_1, \ldots, w_p\} \subseteq V(\mathcal{H})$ and distinct edges $\{f_1, \ldots, f_q\} \subseteq E(\mathcal{H})$, such that if $e_i = v_\alpha v_\beta$, then $f_i \cap W = \{w_\alpha, w_\beta\}$.

In particular, in the case that $\mathcal{H}$ is a graph (2-uniform), an induced Berge F is just any copy of F in $\mathcal{H}$, not to be confused with the notion of induced subgraphs. If the $\mathcal{H}$ and F have the same number of edges, $e(\mathcal{H}) = e(F)$, then we say that $\mathcal{H}$ itself is a(n induced) Berge F hypergraph. The set of r-uniform (induced) Berge F hypergraphs is denoted by $\{\mathrm{B}(F)\}_r$ ($\{\mathrm{B}_{\mathrm{ind}}(F)\}_r$, resp.). For example, if F is a triangle, $E(F) = \{12, 13, 23\}$, then $\{B(F)\}_3$ contains four triple systems: $\{12a, 13a, 23a\}$, $\{12a, 13a, 23b\}$, $\{12a, 13b, 23c\}$ and $\{123, 13a, 23b\}$. The first three of them contains an induced C_3, the last one does not. Parenthesis and indices are omitted when it does not cause ambiguities.

1.1. Three types of extremal numbers

Given a set of r-graphs $\mathcal{F}$, the hypergraph $\mathcal{H}$ is called *$\mathcal{F}$-free* if it does not have any subgraph isomorphic to any member of $\mathcal{F}$. The *Turán number* of $\mathcal{F}$, denoted by $\mathrm{ex}_r(n, \mathcal{F})$, is the maximum size of an $\mathcal{F}$-free $\mathcal{H} \subseteq \binom{[n]}{r}$. Usually it is assumed that $|\mathcal{F}|$ is finite. So the well-known fact $\mathrm{ex}_2(n, \{C_3, C_4, C_5, \ldots\}) = n - 1$ usually is not considered a Turán type result because the set of forbidden graphs $\mathcal{F}$, the set of all cycles, is infinite. If $r = 2$, then the index is usually omitted. Also, if $\mathcal{F}$ has only one member, $\mathcal{F} = \{F\}$, then we write $\mathrm{ex}_r(n, F)$ instead of $\mathrm{ex}_r(n, \{F\})$.

The *generalized Turán number* for graphs, pioneered by Erdős [1] and recently systematically investigated by Alon and Shikhelman [2], is the following extremal problem. We only formulate the case relevant to this paper. Given a graph F, let $\mathrm{ex}(n, K_r, F)$ denote the maximum possible number of copies of K_r's in an F-free, n-vertex graph, i.e.,

$$\mathrm{ex}(n, K_r, F) := \max\left\{ |\mathcal{N}_r(H)| : H \text{ is } F\text{-free }, H \subseteq \binom{[n]}{2} \right\},$$

where $\mathcal{N}_r(H) \subseteq \binom{[n]}{r}$ is the family of r-element vertex sets that span a K_r in H. In particular $\mathcal{N}_2(H) = E(H)$ and $\mathrm{ex}(n, K_2, F) = \mathrm{ex}(n, F)$ is the classical Turán number of F.

For a graph F and positive integer r, let

$$\mathrm{ex}_r(n, \mathrm{B}F) := \max\{e(\mathcal{H}) : \mathcal{H} \subseteq \binom{[n]}{r} \text{ and } \mathcal{H} \text{ is Berge } F\text{-free}\}.$$

Ever since Győri, G. Y. Katona, and Lemons [3] investigated hypergraphs without long Berge paths there is a renewed interest concerning extremal Berge type problems. Here we define a related function, the *induced Berge Turán number* of F. Special cases were studied earlier, especially the 3-uniform case (e.g., Maherani and Shahsiah [4], Gyárfás [5], Sali and Spiro [6]).

$$\mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F) := \max\{e(\mathcal{H}) : \mathcal{H} \subseteq \binom{[n]}{r} \text{ and } \mathcal{H} \text{ is induced Berge } F\text{-free}\}.$$

We consider the relationship between these three functions. Obviously,

$$\mathrm{ex}(n, K_r, F) \le \mathrm{ex}_r(n, \mathrm{B}F) \le \mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F). \tag{1}$$

To see that, consider a graph G with $|\mathcal{N}_r(G)| = \mathrm{ex}(n, K_r, F)$. Since G is F-free, the r-graph $\mathcal{N}_r(G)$ is Berge F-free, implying $|\mathcal{N}_r(G)| \le \mathrm{ex}_r(n, \mathrm{B}F)$. The second inequality holds because if a hypergraph contains no Berge F then it also contains no induced Berge F.

The induced Berge F problem is motivated by the forbidden configuration problem for matrices (see [7] for a survey). It can also be reformulated as a hypergraph trace problem (see Mubayi and Zhao [8]). Few results are known for the induced Berge Turán problem. In [8], the value of

$ex_r(n, B_{ind}K_t)$ is determined asymptotically for K_3 and K_4, as well as K_t when t is close to the uniformity r.

A special case of induced Berge hypergraphs, so called *expansions* were intensively studied, see, e.g., Pikhurko [9], Kostochka, Mubayi, and Verstraëte [10], and the survey by Mubayi and Verstraëte [11].

There are also other areas of research in extremal graph theory which are called 'induced' Turán type results. E.g., Prömel and Steger [12] investigated the extremal properties of graphs not containing an induced copy of a given graph F. A more recent version is by Loh, Tait, Timmons, and Zhou [13]. But most of these are only distant relatives of our induced Berge question.

2. Main results, bounds for $ex_r(n, B_{ind}F)$

Although there was a enormous expansion of investigating Turán type extremal problems in the last two decades (e.g., Razborov's flag algebra method incorporated almost all earlier elementary tools, and also new algebraic and semi-algebraic constructions were found) the topic is still advancing in small steps, it is in the stage of collecting more and more tools and small results. It is still important to find out problems which are solvable with our state of knowledge. This article is a small contribution of that huge task.

We also introduce a new tool (the α-core of a hypergraph) which can be considered as a direct generalization of the extremely useful simple fact that every graph G with average degree d contains an induced subgraph G' with minimum degree at least $d/2$. It seems that α-cores of hypergraphs might be useful in considering further extremal problems.

2.1. The order of magnitude

Let F be a graph. Our aim is to determine the order of magnitude of the induced Berge Turán number of F as $n \to \infty$, or to reduce it to known problems. In the next subsection we define a large class of 3-chromatic graphs $\mathcal{G}_{tri}$ which contains, e.g., all outerplanar graphs. We then apply our results and methods to determine the induced Berge Turán number of graphs in this class more precisely.

Theorem 2.1. *Let $r \geq 2$, and let F be a graph such that $E(F) \neq \emptyset$. Then, as $n \to \infty$*

$$ex_r(n, B_{ind}F) = \Theta(\max_{2\leq s\leq r}\{ex(n, K_s, F)\}).$$

This theorem shows that the order of magnitudes of the three functions in (1) behave differently as r changes. For small r in the range $r \leq \chi(F) - 1$, the functions $ex_r(n, F)$, $ex_r(n, BF)$, and $ex_r(n, B_{ind}F)$, are all of order $\Theta(n^r)$: the balanced complete $(\chi(F) - 1)$-partite r-graph contains no Berge F (and its 2-shadow, the r-partite Turán graph, is r-chromatic).

If $r \geq |V(F)|$ then $ex(n, K_r, F) = 0$ (since a K_r contains a copy of F). For general graphs F, the behavior of the three functions in the range $\chi(F) \leq r \leq |V(F)| - 1$ is still unknown. Determining the order of $ex(n, K_r, F)$ for r in this range would give an answer for the growth of $ex_r(n, B_{ind}F)$.

Concerning the Berge Turán function Gerbner and Palmer [14] showed that

$$ex_r(n, BF) \leq ex(n, F)$$

for $r \geq |V(F)|$. So in this range $ex_r(n, BF) = O(n^2)$. For the complete graphs the two sides have the same order: $ex_r(n, BK_r) = \Theta(n^2)$ if $r \geq 3$. However this does not hold if r is large compared to $|V(F)|$. Grósz, Methuku, and Tompkins [15] proved that for any non-bipartite F and sufficiently large r, the order of $ex_r(n, F)$ differs from that of $ex(n, F)$: there exists some number $th(F)$ such that if $r \geq th(F)$ then $ex_r(n, F) = o(n^2)$.

In contrast, the order of the induced Berge Turán function $ex_r(n, B_{ind}F)$ is non-decreasing in r. Moreover, it is basically monotone. If $\bigcap E(F) = \emptyset$, i.e., F is not a star, then we will see later by Lemma 3.1 that

$$\left(1 - \frac{r-1}{n}\right) ex_{r-1}(n, B_{ind}F) \leq ex_r(n, B_{ind}F). \quad (2)$$

2.2. Outerplanar graphs and more

We define the class of t-vertex graphs $\mathcal{G}_{\rm tri}^{(t)}$ by induction on t as follows. The class $\mathcal{G}_{\rm tri}^{(2)}$ has only a single member, K_2. For $t > 2$ one obtains each member G of $\mathcal{G}_{\rm tri}^{(t)}$ by taking a $G^{(t-1)} \in \mathcal{G}_{\rm tri}^{(t-1)}$, taking an edge $xy \in G^{(t-1)}$, adding a new vertex $z \notin V(G^{(t-1)})$, and joining z to x and to y. Each $G \in \mathcal{G}_{\rm tri}^{(t)}$ has exactly t vertices and $2t-3$ edges. Finally, let $\mathcal{G}_{\rm tri}$ be the family of all non-empty subgraphs of the members of $\cup_{t\geq 2}\mathcal{G}_{\rm tri}^{(t)}$.

Note that $\mathcal{G}_{\rm tri}$ contains all outerplanar graphs, particularly cycles, C_t, and forests. Each $G \in \mathcal{G}_{\rm tri}$ has chromatic number at most 3 and are obviously planar.

Theorem 2.2. *Let $r \geq 2$ be a positive integer. Fix a graph $F \in \mathcal{G}_{\rm tri}$. As $n \to \infty$ we have $\mathrm{ex}_r(n, \mathrm{B_{ind}}F) = \Theta(\mathrm{ex}(n, F))$.*

This theorem reveals further gaps between $\mathrm{ex}_r(n, \mathrm{B}F)$ and $\mathrm{ex}_r(n, \mathrm{B_{ind}}F)$. Győri and Lemons [16,17] proved that for $r \geq 3$ an r-uniform hypergraph avoiding a Berge cycle C_{2t+1} has at most $O(\mathrm{ex}(n, C_{2t}))$ edges, which is known to be $O(n^{1+(1/t)})$. On the other hand, in the same range, we have $\mathrm{ex}_r(n, \mathrm{B_{ind}}C_{2t+1}) = \Theta(n^2)$.

Together, Theorems 2.1 and 2.2 show that $\mathrm{ex}(n, C_t)$ has the same order as $\max_{2\leq s\leq r}\{\mathrm{ex}(n, K_s, C_t)\}$. We obtain the following (known) corollary. For any $r \geq 2$ and $t \geq 3$

$$\mathrm{ex}(n, K_r, C_t) = O(\mathrm{ex}(n, C_t)).$$

We also state the case of trees.

Corollary 2.3. *Let $r \geq 2$ and T be a forest with at least two edges. Then $\mathrm{ex}_r(n, \mathrm{B_{ind}}T) = \Theta(\mathrm{ex}(n, T)) = \Theta(n)$.*

Finally, we get better bounds for stars, $F = K_{1,t-1}$.

Theorem 2.4. *For any $r \geq 2$, $t \geq 3$, if $n = a(r+t-3)+b$ with $b \leq r+t-4$ then*

$$a\binom{r+t-3}{r} + \binom{b}{r} \leq \mathrm{ex}_r(n, \mathrm{B_{ind}}K_{1,t-1}) \leq \frac{n}{r}\binom{r+t-3}{r-1}.$$

In particular, if n is divisible by $r+t-3$, the lower bound is $\frac{n}{r}\binom{r+t-4}{r-1}$.

3. Constructions and proofs

3.1. Simple constructions and a monotonicity of the induced Berge Turán function

If $E(F)$ has a single edge then for $n \geq |V(F)|+r-2$ we have $\mathrm{ex}(n, F) = \mathrm{ex}(n, K_r, F) = \mathrm{ex}_r(n, \mathrm{B}F) = \mathrm{ex}_r(n, \mathrm{B_{ind}}F) = 0$, so there is nothing to prove, all of our statements trivially hold.

In all other cases we have $\mathrm{ex}_r(n, \mathrm{B_{ind}}F) = \Omega(n)$ as one can see from the following constructions. If F has two non-disjoint edges then a matching of r-sets gives $\mathrm{ex}_r(n, \mathrm{B_{ind}}F) \geq \lfloor n/r \rfloor$. If F has two disjoint edges then the hypergraph consisting of $n-r+1$ sets sharing a common $(r-1)$-set yields $\mathrm{ex}_r(n, \mathrm{B_{ind}}F) \geq n-r+1$.

If $x \in V(F)$ is an isolated vertex then $\mathrm{ex}_r(n, \mathrm{B_{ind}}F) = \mathrm{ex}_r(n, \mathrm{B_{ind}}(F \setminus \{x\}))$ for all $n > (r-2)|E(F)| + |V(F)|$. So we may delete isolated vertices and asymptotically get the same Turán number. From now on, we suppose that F has no isolated vertex and $|E(F)| \geq 2$.

Lemma 3.1. *Fix integers $r, t \geq 2$. If F is a graph on t vertices such that $F \neq K_{1,t-1}$, F has no isolated vertices, and $e(F) \geq 2$, then $\mathrm{ex}_r(n, \mathrm{B_{ind}}F) \geq \mathrm{ex}_{(r-1)}(n-1, \mathrm{B_{ind}}F)$. More precisely, $\mathrm{ex}_r(n, \mathrm{B_{ind}}F) = \Omega(\mathrm{ex}(n, F))$.*

Proof. Let $\mathcal{H}$ be an $(r-1)$-uniform hypergraph on $n-1$ vertices with $\mathrm{ex}_{(r-1)}(n-1, \mathrm{B_{ind}}F)$ edges and no induced Berge F. Construct an r-uniform hypergraph $\mathcal{H}'$ with $V(\mathcal{H}') = V(\mathcal{H}) \cup \{v\}$ such that the edges of $\mathcal{H}'$ are obtained by extending every edge of $\mathcal{H}$ to include the new vertex v. Suppose that $\mathcal{H}'$ contains an induced Berge F. Since $\mathcal{H}$ was induced Berge F-free, v must be a base vertex. Because v is contained in every edge of $\mathcal{H}'$, there is a fixed vertex contained in every edge of F. I.e., $F = K_{1,t-1}$, a contradiction.

Inductively, we obtain $\mathrm{ex}_2(n-r+2, \mathrm{B_{ind}}F) \leq \mathrm{ex}_r(n, \mathrm{B_{ind}}F)$. But $\mathrm{ex}_2(n-r+2, \mathrm{B_{ind}}F) = \mathrm{ex}(n-r+2, F) = \Theta(\mathrm{ex}(n, F))$. □

To show (2) let $\mathcal{H}$ be an induced Berge F-free $(r-1)$-uniform hypergraph on n vertices, $|\mathcal{H}| = \mathrm{ex}_{(r-1)}(n, \mathrm{B_{ind}}F)$. For $x \in V := V(\mathcal{H})$ let $\mathcal{H}_x := \{e \in \mathcal{H} : e \subset V \setminus \{x\}\}$. Since each $\mathcal{H}_x$ is also induced Berge F-free we get

$$(n-r+1)\mathrm{ex}_{(r-1)}(n, \mathrm{B_{ind}}F) = (n-r+1)|\mathcal{H}| = \sum_{x\in V} |\mathcal{H}_x| \leq n \times \mathrm{ex}_{(r-1)}(n-1, \mathrm{B_{ind}}F).$$

By Lemma 3.1 the right hand side is at most $n \times \mathrm{ex}_r(n, \mathrm{B_{ind}}F)$. Rearranging yields (2). □

3.2. The α-core of a hypergraph

Let $\mathcal{H}$ be an r-partite, r-uniform hypergraph with parts $V(\mathcal{H}) = V_1 \cup \cdots \cup V_r$. For some $1 \leq s \leq r$ and edge $e \in \mathcal{H}$, define $e[\bar{s}]$ to be the trace of e onto all parts other than V_s. That is, $e[\bar{s}] = e \setminus V_s$. Let $\mathcal{H}[\bar{s}] = \{e[\bar{s}] : e \in E(\mathcal{H})\}$.

Theorem 3.2. *For positive integers α, r, any r-uniform r-partite hypergraph $\mathcal{H}$ contains edge-disjoint subhypergraphs $\mathcal{A}$ and $\mathcal{B}$ such that*

(a) *For any $S \subseteq V(\mathcal{H})$, with $|S| = r-1$, either $\deg_{\mathcal{A}}(S) = 0$ or $\deg_{\mathcal{A}}(S) \geq \alpha$.*
(b) $|\mathcal{B}| \geq \frac{|\mathcal{H}\setminus\mathcal{A}|}{\alpha-1}$ *and* $|\mathcal{B}| \leq \sum_{s=1}^{r} |\mathcal{B}[\bar{s}]|$.

Proof. We build $\mathcal{A}$ and $\mathcal{B}$ inductively. Initially set $\mathcal{H}_0 := \mathcal{H}$, $\mathcal{B}_0 := \{\emptyset\}$.

At step i, if there exists an $S \subseteq V(\mathcal{H}_{i-1})$ with $|S| = r-1$ and $1 \leq \deg_{\mathcal{H}_{i-1}}(S) \leq \alpha - 1$, then let E_S be the edges of $\mathcal{H}_{i-1}$ containing S. Set $\mathcal{H}_i = \mathcal{H}_{i-1} \setminus E_S$. Pick any edge, say $B_i \in E_S$, and set $\mathcal{B}_i = \mathcal{B}_{i-1} \cup \{B_i\}$.

The process ends after k steps when for every $S \subseteq V(\mathcal{H}_k)$ with $|S| = r-1$, either $\deg_{\mathcal{H}_k}(S) = 0$ or $\deg_{\mathcal{H}_k}(S) \geq \alpha$. Let $\mathcal{A} := \mathcal{H}_k$ and $\mathcal{B} := \mathcal{B}_k = \{B_1, \ldots, B_k\}$. Then $\mathcal{A}$ satisfies (a).

To see that $\mathcal{B}$ satisfies (b), at each step i when we choose $B_i \in E_S$, $|E_S| \leq \alpha - 1$, so we obtain that $|\mathcal{B}|$ is at least a $1/(\alpha-1)$ portion of the deleted edges. Next, at each step, we associated with B_i a distinct set S_i of $r-1$ vertices. If B_i and B_j are associated with sets S_i and S_j respectively such that both sets are contained in $(V_1 \cup \cdots \cup V_r) \setminus V_s$, then in $\mathcal{B}[\bar{s}]$, $B_i[\bar{s}] = S_i$ and $B_j[\bar{s}] = S_j$ are distinct. Hence $\sum_{s=1}^{r} \mathcal{B}[\bar{s}] \geq |\{S_1, \ldots, S_k\}| = |\mathcal{B}|$. □

Let any $\mathcal{A} \subseteq \mathcal{H}$ satisfying (a) be called an α-**core** of $\mathcal{H}$.

Lemma 3.3. *Let α, r be positive integers, and let F be a graph with $|V(F)| - 1 \leq \alpha$. Let $\mathcal{H}$ be an r-uniform, r-partite hypergraph with an α-core $\mathcal{A}$. If the 2-shadow $\partial_2\mathcal{A}$ of $\mathcal{A}$ contains a copy of F, then $\mathcal{A}$ (and therefore $\mathcal{H}$) contains an induced Berge F.*

Proof. We will find an induced Berge F on the same base vertex set $V(F)$. Let xy be an edge in the copy of F, and let e_{xy} be an edge of $\mathcal{A}$ containing $\{x, y\}$ with minimum $|e_{xy} \cap V(F)|$. Such an edge e_{xy} exists by the definition of the 2-shadow. If e_{xy} contains some vertex $z \in V(F) \setminus \{x, y\}$, then the $(r-1)$-set $e_{xy} \setminus \{z\}$ is contained in at least $\alpha - 1$ other edges in $\mathcal{A}$. Since there are $|V(F)| - 3 \leq \alpha - 2$ vertices in $V(F) \setminus \{x, y, z\}$, we may find some $z' \notin V(F) - \{x, y, z\}$ such that $e_{xy} \setminus \{z\} \cup \{z'\} \in E(\mathcal{A})$, contradicting the choice of e_{xy}. Therefore $e_{xy} \cap V(F) = \{x, y\}$. We find such an edge of $\mathcal{A}$ for each edge of F. □

If $\alpha \geq e(F) + |V(F)|$, then with the same method one can find an induced Berge F in $\mathcal{A}$ such that each pair of hyperedges e_{xy} and e_{uv} intersect only at $\{x, y\} \cap \{u, v\}$. This is called an *F-expansion*. But this observation does not seem to help our purposes here.

Claim 3.4. *Suppose that $r \geq 3$ and $\mathcal{A}$ contains an induced Berge F, where $|V(F)| \leq \alpha$ (and $E(F) \neq \emptyset$). Define a new graph $F^+ := F^+_{xy}$ by adding a new vertex $z \notin V(F)$, taking an edge $xy \in E(F)$, and joining z to x and to y. Then $\mathcal{A}$ also contains an induced Berge F^+.*

Proof. By Lemma 3.3, there exists a hyperedge $e_{xy} \in \mathcal{A}$ such that $e_{xy} \cap V(F) = \{x, y\}$. Then for every $z' \in e_{xy} \setminus \{x, y\}$ we have xz' and $yz' \in \partial_2\mathcal{A}$, so F^+ is a subgraph of $\partial_2\mathcal{A}$. Then Lemma 3.3 completes the Claim. □

Lemma 3.5. *Suppose that $G \in \mathcal{G}_{\text{tri}}$ with $t = |V(G)| \geq 3$. Then $G \in \mathcal{G}^{(t)}_{\text{tri}}$.*

Proof. This statement seems to be evident, but still needs a proof. By definition, there exists an $s \geq t$ such that $G \in \mathcal{G}^{(s)}_{\text{tri}}$. Let $s = s(G)$ be the smallest such s. We will show by induction on t that $s(G) = t$. The base case $t = 3$ is obvious. Suppose $t > 3$ and that G is a subgraph of $H \in \mathcal{G}^{(s)}_{\text{tri}}$, where the vertices of H are $\{v_1, \dots, v_s\}$ and each v_i (with $i \geq 3$) has exactly two H-neighbors in $\{v_1, \dots, v_{i-1}\}$. Moreover, these two neighbors (call them $v_{\alpha(i)}$ and $v_{\beta(i)}$) are joined by an edge in H. Let $I \subseteq [s]$, $I := \{i_1, \dots, i_t\}$, $1 \leq i_1 < \cdots < i_t \leq s$, $V_I := \{v_i : i \in I\}$, and suppose that G is a spanning subgraph of $H[V_I]$. Since s is minimal, we have $i_t = s$ and $N_H(v_s) = \{v_{\alpha(s)}, v_{\beta(s)}\}$. $G' := H[V_I] \setminus \{v_s\}$ has $t-1$ vertices, and it belongs to $\mathcal{G}_{\text{tri}}$. By our induction hypothesis there exists a $H' \in \mathcal{G}^{(t-1)}_{\text{tri}}$ such that G' is a subgraph of H' on the same vertex set $V_I \setminus \{v_s\}$. If $\{v_{\alpha(s)}, v_{\beta(s)}\} \subseteq V(H')$ then by adjoining a new vertex z' to H' and connecting it to $v_{\alpha(s)}$ and $v_{\beta(s)}$ we obtain a t-vertex graph H'' from $\mathcal{G}^{(t)}_{\text{tri}}$ containing G. If $|N_H(v_s) \cap V(H')| \leq 1$ then it is even simpler to find such a graph H''. □

3.3. Proofs of the upper bounds for induced berge F problems

We prove a version of Theorem 2.1 with more precise bounds. For positive integers a and b, $(a)_b = (a)(a-1)\cdots(a-b+1)$ denotes the falling factorial.

Theorem 3.6. *Let t, r, n be positive integers, and let F be a graph with $|V(F)| = t$. Let $\mathcal{H}$ be an n-vertex r-uniform hypergraph with no induced Berge F. If $\mathcal{H}$ is r-partite, then*

$$|\mathcal{H}| \leq \sum_{i=2}^{r} (t-2)^{r-i}(r)_{r-i}\text{ex}(n, K_i, F).$$

Proof. We proceed by induction on r. The base case $r = 2$ is trivial since an induced Berge F is just a copy of F. Thus $\text{ex}_2(n, \text{B}_{\text{ind}}F) = \text{ex}(n, K_2, F) = \text{ex}(n, F)$. Now let $r \geq 3$. Let $\mathcal{A}$ and $\mathcal{B}$ be subhypergraphs of $\mathcal{H}$ obtained from Theorem 3.2 with $\alpha = t - 1$. So we have

$$|\mathcal{H}| = |\mathcal{A}| + |\mathcal{H} \setminus \mathcal{A}| \leq |\mathcal{A}| + (t-2)\sum_{s=1}^{r} |\mathcal{B}[\bar{s}]| \leq |\mathcal{A}| + (t-2)(r)\text{ex}_{r-1}(n, \text{B}_{\text{ind}}F),$$

where the last inequality holds because each $\mathcal{B}[\bar{s}]$ is $(r-1)$-uniform, $(r-1)$-partite and does not contain an induced Berge F.

By Lemma 3.3, $\partial_2\mathcal{A}$ contains no copy of F. Furthermore, since each edge in $\mathcal{A}$ creates a K_r in $\partial_2\mathcal{A}$, $|\mathcal{A}| \leq \text{ex}(n, K_r, F)$. Applying the induction hypothesis, we obtain

$$|\mathcal{H}| \leq \text{ex}(n, K_r, F) + (t-2)r\sum_{i=2}^{r-1} (t-2)^{r-1-i}(r-1)_{r-1-i}\text{ex}(n, K_i, F)$$

and we are done. □

Corollary 3.7. *Let t, r, n be positive integers, and let F be a graph with $|V(F)| = t$. Then*

$$\max_{2\le s\le r}\{\mathrm{ex}(n-(r-s), K_s, F)\} \le \mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F) \le \frac{r^r}{r!}\sum_{i=2}^{r}(t-2)^{r-i}(r)_{r-i}\mathrm{ex}(n, K_i, F).$$

In particular, $\mathrm{ex}_r(n, \mathrm{B}_{\mathrm{ind}}F) = \Theta(\max_{2\le s\le r}\{\mathrm{ex}(n, K_s, F)\})$.

Proof. The lower bound follows from Lemma 3.1 and (1). For the upper bound, we use the fact that any r-uniform hypergraph $\mathcal{H}$ has an r-partite subhypergraph with at least $\frac{r!}{r^r}e(\mathcal{H})$ edges. Apply Theorem 3.6 to any such subhypergraph. □

Proof of Theorem 2.2. The lower bound comes from Lemma 3.1. For the upper bound, we proceed by induction on r. First we show that if $\mathcal{H}$ is r-partite with no induced Berge $F \in \mathcal{G}_{\mathrm{tri}}$ then

$$|\mathcal{H}| \le (t-2)^{r-2}\frac{r!}{2}\mathrm{ex}(n, F). \tag{3}$$

The base case $r = 2$ is trivial, so let $r \ge 3$. Let $\mathcal{A}$ and $\mathcal{B}$ be subhypergraphs of $\mathcal{H}$ obtained from Theorem 3.2 with $\alpha = t-1$. We have

$$|\mathcal{H}| \le |\mathcal{A}| + (t-2)\sum_{s=1}^{r}|B[\bar{s}]| \le |\mathcal{A}| + (t-2)(r)\mathrm{ex}_{r-1}(n, \mathrm{B}_{\mathrm{ind}}F). \tag{4}$$

Observe that $\mathcal{A}$ is empty. Indeed, if $\mathcal{A}$ contains at least one edge, then the 2-shadow $\partial_2\mathcal{A}$ contains a K_r. So Claim 3.4 and Lemma 3.5 imply that $\partial_2\mathcal{A}$ contains a copy of F. Then we apply Lemma 3.3 to find an induced Berge F, a contradiction. Hence $|\mathcal{A}| = 0$. Applying induction hypothesis, (4) yields (3).

Finally, if $\mathcal{H}$ is not r-partite, then we apply the previous proof to an r-partite subgraph $\mathcal{H}'$ of $\mathcal{H}$ with at least $\frac{r!}{r^r}|\mathcal{H}|$ edges to obtain $|\mathcal{H}| \le \frac{1}{2}r^r(t-2)^{r-2}\mathrm{ex}(n, F)$. □

Proof of Theorem 2.4. For the lower bound, let each component of $\mathcal{H}$ be a clique such that there are as many cliques of size $r+t-3$ as possible. If $n = a(r+t-3)+b$ where $0 \le b < r+t-3$, then $|\mathcal{H}| = a\binom{r+t-3}{r} + \binom{b}{r}$. Suppose $\mathcal{H}$ contains an induced Berge $K_{1,t-1}$. Then its base vertices, say $\{v_1, \ldots, v_t\}$ must be contained in a single component of $\mathcal{H}$. But each edge in a component contains at least 3 base vertices, a contradiction.

For the upper bound, let $\mathcal{H}$ be an n-vertex, r-uniform hypergraph with no induced Berge $K_{1,t-1}$. We say that a set system $\{f_1, \ldots, f_s\}$ is *strongly representable* if for every $f_i \in \mathcal{F}$, there exists a $v_i \in f_i$ such that $v_i \notin f_j$ for all $j \ne i$. Füredi and Tuza [18] proved that if a set system $\mathcal{F}$ with $|f| \le r$ for all $f \in \mathcal{F}$ does not contain a strongly representable subfamily of size s then $|\mathcal{F}| \le \binom{r+s-1}{r}$. For any vertex $v \in V(\mathcal{H})$, let $E_v := \{e \setminus \{v\} : v \in e \in \mathcal{H}\}$. The $(r-1)$-uniform set system E_v cannot contain a strongly representable subfamily of size $t-1$, otherwise the corresponding edges in $\mathcal{H}$ and their representative vertices would yield an induced Berge $K_{1,t-1}$ in $\mathcal{H}$ with vertex v as the center vertex. Therefore $\deg(v) \le \binom{(r-1)+(t-2)}{r-1}$ so $|\mathcal{H}| \le \frac{n}{r}\binom{r+t-3}{r-1}$. □

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