Discrete Mathematics 345 (2022) 112667

Contents lists available at ScienceDirect

Discrete Mathematics

www.elsevier.com/locate/disc

Turán number of special four cycles in triple systems

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ARTICLE INFO

Article history:
Received 18 March 2021
Received in revised form 28 September 2021
Accepted 29 September 2021
Available online 7 October 2021

Keywords:
Triple system
Turán number
Extremal combinatorics

ABSTRACT

A *special four-cycle* F in a triple system consists of four triples *inducing* a C_4. This means that F has four special vertices v_1, v_2, v_3, v_4 and four triples in the form $w_i v_i v_{i+1}$ (indices are understood $\pmod 4$) where the w_js are not necessarily distinct but disjoint from $\{v_1, v_2, v_3, v_4\}$. There are seven non-isomorphic special four-cycles, their family is denoted by $\mathcal{F}$. Our main result implies that the Turán number $\mathrm{ex}(n, \mathcal{F}) = \Theta(n^{3/2})$. In fact, we prove more, $\mathrm{ex}(n, \{F_1, F_2, F_3\}) = \Theta(n^{3/2})$, where the F_i-s are specific members of $\mathcal{F}$. This extends previous bounds for the Turán number of triple systems containing no Berge four cycles.

We also study $\mathrm{ex}(n, \mathcal{A})$ for all $\mathcal{A} \subseteq \mathcal{F}$. For 16 choices of $\mathcal{A}$ we show that $ex(n, \mathcal{A}) = \Theta(n^{3/2})$, for 92 choices of $\mathcal{A}$ we find that $\mathrm{ex}(n, \mathcal{A}) = \Theta(n^2)$ and the other 18 cases remain unsolved.

1. Introduction

A *triple system* $H = (V, E)$ has vertex set V and E consists of some triples of V (repeated triples are excluded). For any fixed family $\mathcal{H}$ of triple systems, the Turán number $\mathrm{ex}(n, \mathcal{H})$ is the maximum number of triples in a triple system of n vertices that is $\mathcal{H}$-free, i.e., does not contain any member of $\mathcal{H}$ as a subsystem.

Our interest here is the family $\mathcal{F}$ of *special four cycles*: they have four distinct base vertices v_1, v_2, v_3, v_4 and four triples $w_i v_i v_{i+1}$ (indices are understood $\pmod 4$) where the w_js are not necessarily distinct but $w_i \neq v_j$ for any pair of indices $1 \leq i, j \leq 4$.

There are seven non-isomorphic special four cycles. The linear (loose) four cycle F_1 is obtained when all w_j-s are different and in F_2 all w_js coincide. When two pairs coincide we get either F_3 ($w_1 = w_2, w_3 = w_4$) or F_4 ($w_1 = w_3, w_2 = w_4$). The F_4 is the Pasch configuration. We define F_5 with $w_1 = w_2 = w_3$ (but w_4 is different). In F_6 we have $w_1 = w_3$ (and w_2, w_4 are different from w_1 and from each other). When only w_1, w_2 coincide we get F_7. Set $\mathcal{F} = \{F_1, \dots, F_7\}$. For the convenience of the reader, the special four cycles are shown on Fig. 1 and Fig. 2.

Turán numbers of various members of $\mathcal{F}$ have been investigated before. Füredi [4] proved that $\mathrm{ex}(n, F_3) \leq \frac{7}{2}\binom{n}{2}$. Mubayi [8] showed that $\mathrm{ex}(n, F_2) = \Theta(n^{5/2})$. Rödl and Phelps [7] gave the bounds $c_1 n^{5/2} \leq \mathrm{ex}(n, F_4) \leq c_2 n^{11/4}$. In fact, the upper bound is Erdős' upper bound [2] for $\mathrm{ex}(n, K^3_{2,2,2})$. The lower bound comes from a balanced 3-partite triple system where every vertex of the third partite class form a triple with the edges of a bipartite C_4-free graph between the first two partite classes.

We prove that $\mathrm{ex}(n, \mathcal{F}) = \Theta(n^{3/2})$, thus has the same order of magnitude as $\mathrm{ex}(n, C_4)$ for graphs. In fact, it is enough to exclude three of the special four cycles.

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https://doi.org/10.1016/j.disc.2021.112667

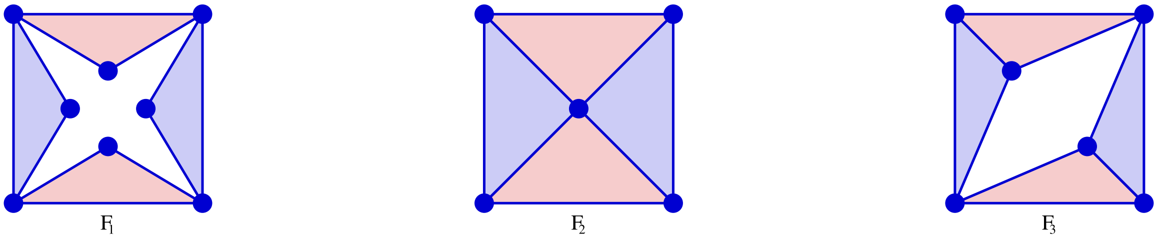

Fig. 1. The family of special four cycles F_1, F_2, F_3 of Theorem 1.

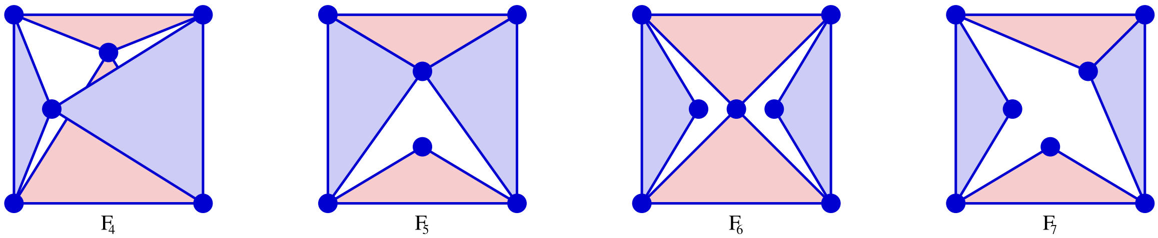

Fig. 2. The family of the other four special four cycles $F_4, \dots, F_7$.

Theorem 1. $\mathrm{ex}(n, \{F_1, F_2, F_3\}) = \Theta(n^{3/2})$.

The family $\mathcal{F}$ of special four cycles is a subfamily of a wider class, the class of *Berge four cycles*, where the vertices w_i can be selected from the base vertices as well, requiring only that the four triples $w_i v_i v_{i+1}$ are different. Theorem 1 extends previous similar upper bounds (Füredi and Özkahya [5], Gerbner, Methuku, Vizer [6]) where the family of Berge four cycles were forbidden.

The appearance of the set $\{F_1, F_2, F_3\}$ is not accidental. If any of F_1, F_2, F_3 is missing from $\mathcal{A} \subset \mathcal{F}$ then $\mathrm{ex}(n, \mathcal{A})$ is essentially larger than $n^{3/2}$.

- (C1) Ruzsa and Szemerédi [9] constructed triple systems on n vertices that do not carry three triples on six vertices and have more than $n^{2-\varepsilon}$ triples for any fixed ε. This provides an example which contains only F_1 from $\mathcal{F}$,
- (C2) The $\binom{n-1}{2}$ triples containing a fixed vertex from n vertices contains only F_2 from $\mathcal{F}$,
- (C3) Partition n vertices evenly into three parts, take a pairing between two equal parts and extend each pair with all vertices of the third class to a triple. This gives a triple system with approximately $n^2/9$ triples and contains only F_3 from $\mathcal{F}$.

In Section 3 we discuss $\mathrm{ex}(n, \mathcal{A})$ for all $\mathcal{A} \subseteq \mathcal{F}$. It turns out that in 92 cases $\mathrm{ex}(n, \mathcal{A}) = \Theta(n^2)$ and 18 cases remain unsolved.

2. Proof of Theorem 1

Assume H is a triple system with n vertices containing no subsystem from the set F_1, F_2, F_3. Applying the standard approach (based on [3]), we may assume that H is 3-partite with vertex partition $[A_1, A_2, A_3]$ where $|A_i| \in \{\lfloor n/3 \rfloor, \lceil n/3 \rceil\}$ and contains at least $2/9$ of the triples of the original triple system.

The triples of H define a bipartite graph $B = [A_1, A_2]$ as follows. If (a_1, a_2, a_3) is a triple of H with $a_i \in A_i$ then $a_1 a_2$ is considered as an edge of B. Define the label $L(a_1, a_2)$ of $a_1 a_2 \in E(B)$ as the set $\{z \in A_3 : a_1 a_2 z \in E(H)\}$. Then

$$|E(H)| = \sum_{a_1 a_2 \in E(B)} |L(a_1, a_2)|. \tag{1}$$

Lemma 1. *The bipartite graph B has at most $O(n^{3/2})$ edges.*

Proof of Lemma 1. We denote by $N(x, y)$ the set of common neighbors (in B) of $x, y \in A_2$ in A_1. Similarly, let $N(u, v)$ be the set common neighbors of $u, v \in A_1$ in A_2.

For distinct vertices $x, y \in A_2$, define the digraph $D = D(x, y)$ with vertex set A_3. For every $u \in A_1$ such that $u \in N(x, y)$ and $a_i \in L(u, x), a_j \in L(u, y)$, a directed edge $a_i a_j$ is defined in $D(x, y)$. We claim that $D(x, y)$ is a very special digraph.

Claim 1.

- *(1.1) There are no multiple loops or parallel directed edges in $D(x, y)$,*
- *(1.2) There is at most one loop in $D(x, y)$,*
- *(1.3) Two non-loop edges of $D(x, y)$ either intersect or $|N(x, y)| \leq 4$.*

Proof. A multiple loop $a_i a_i$ in $D(x, y)$ would give a C_4 in B with all edges containing a_i in their labels, this corresponds to an F_2 in H – a contradiction. A multiple edge $a_i a_j$ would give a $C_4 = (x, u_1, y, u_2)$ in B where $u_1, u_2 \in N(x, y), u_1 \neq u_2$ such that the consecutive edges of C_4 contain $a_i, a_j, a_j, a_i \in A_3$ in their labels. This would give an F_3 in H – a contradiction again, proving (1.1).

Two distinct loops $a_i a_i, a_j a_j$ in $D(x, y)$ can appear in two ways. Either we have a $C_4 = (x, u_1, y, u_2)$ in B where $u_1, u_2 \in N(x, y), u_1 \neq u_2$ such that the consecutive edges of C_4 contain $a_i, a_i, a_j, a_j \in A_3$ in their labels, this would give an F_3 in H, a contradiction. Otherwise $u = u_1 = u_2$ and we have two multiedges xu, yu both containing a_i, a_j in their labels, this gives an F_2 in H with u in its center, a contradiction again, proving (1.2).

Suppose that there exist two non-intersecting non-loop edges $a_i a_j, a_k a_l$ in $D(x, y)$. If these edges are defined by $u_1, u_2 \in N(x, y), u_1 \neq u_2$, we have a $C_4 = (x, u_1, y, u_2)$ in B with four distinct elements in their labels, giving an F_1 in H, a contradiction. Thus we may assume that $u_1 = u_2 = u$ and we have xu, yu in B with a_i, a_k and with a_j, a_l in their labels. Set

$$M = \{v \in N(x, y) : v \neq u, |L(v, x) \cup L(v, y)| \geq 2\}.$$

We claim that $|M| \leq 2$. Indeed, consider $v \in M$, there is $a_s, a_t \in A_3$ such that $a_s \neq a_t$ and xv, yv have labels containing a_s, a_t, respectively. Observe that either $\{s, t\} = \{i, k\}$ or $\{s, t\} = \{j, l\}$ otherwise there is a $C_4 = (x, u, y, v)$ with four distinct labels, giving an F_1 in H, a contradiction. This implies that $|M| \leq 4$. However, it cannot happen that for two distinct vertices $v, v' \in M$ the coincidence of the index pairs are $\{i, k\}$ and $\{j.l\}$, respectively, because it would result again in a $C_4 = (x, v, y, v')$ with four distinct labels, a contradiction as above. Thus $|M| \leq 2$ (equality is possible with edge pairs $a_i a_k, a_k a_i$ or $a_j a_l, a_l a_j$), proving the claim.

Observing that every vertex of $N(x, y) \setminus (\{u\} \cup M)$ defines a loop in $D(x, y)$, (1.1) and (1.2) implies that $|N(x, y)| \leq 4$, proving (1.3) and Claim 1. □

A *cherry* on $x \in A_2$ is defined as an incident edge pair, $ux, vx \in E(B)$ such that $u, v \in A_1, u \neq v$ and $L(u, x) \cap L(v, x) \neq \emptyset$. Let $C(x, y)$ be the number of cherries in the subgraph of B induced on $\{x, y\} \cup N(x, y)$. We have

$$C(x, y) \geq \sum_{a \in V(D(x,y))} d^+(a) + d^-(a) - 2, \tag{2}$$

because there are at least $d^+(a) - 1$ cherries on x with $L(u, x) \cap L(v, x) = \{a\}$ and at least $d^-(a) - 1$ cherries on y with $L(u, y) \cap L(v, y) = \{a\}$.

Claim 2. *For any two distinct vertices $x, y \in A_2$, $C(x, y) \geq |N(x, y)| - 4$.*

Proof. Claim 2 is certainly true for $|N(x, y)| \leq 4$. Otherwise, using (1.3) from Claim 1, we have pairwise intersecting edges in $D(x, y)$.

Case 1. The edges of $D(x, y)$ form a triangle (edges oriented two ways are allowed) plus at most one loop. Therefore $D(x, y)$ has at most seven edges thus $5 \leq |N(x, y)| \leq 7$. By (1.1) of Claim 1, $d^+(a) \leq 2$ for any vertex of the triangle. There are at least $|N(x, y)| - 1$ edges on the triangle, so there exists at least $|N(x, y)| - 4$ vertices a with $d^+(a) \geq 2$ resulting in at least $|N(x, y)| - 4$ cherries on x.

Case 2. All edges of $D(x, y)$ (apart from a possible loop) contain $a \in A_3$. For every $u \in N(x, y)$ (apart from one possible vertex which defines a loop) either ux or uy has label a. Thus $\sum_{a \in V(D(x,y))} d^+(a) + d^-(a) \geq |N(x, y)| - 1$ so (2) results in at least $|N(x, y)| - 3$ cherries on x or on y, completing the proof of Claim 2. □

Claim 3. $\sum_{x,y \in A_2} C(x, y) \leq \binom{|A_1|}{2}$.

Proof. Every cherry counted on the left hand side is on some pair of A_1. At most one cherry can be on any $(u, v) \in A_1$, otherwise (by (1.1) in Claim 1) we have one of F_2, F_3. □

Applying Claims 2, 3 we get

$$\sum_{x,y \in A_2} (|N(x, y)| - 4) \leq \sum_{x,y \in A_2} C(x, y) \leq \binom{|A_1|}{2},$$

thus

$$\sum_{x,y\in A_2} |N(x,y)| \le 4\binom{|A_2|}{2} + \binom{|A_1|}{2} \le O(n^2).$$

By convexity we get

$$|A_1|\binom{\frac{|E(B)|}{|A_1|}}{2} \le \sum_{u\in A_1}\binom{d(u)}{2} = \sum_{x,y\in A_2}|N(x,y)| \le O(n^2),$$

therefore $|E(B)| = O(n^{3/2})$, proving Lemma 1. □

To finish the proof of Theorem 1, we need to show that the presence of labels does not affect strongly the edge count of Lemma 1. Let B^* denote the subgraph of B with the edges of at least three-element labels.

Proposition 1. *If H is $\{F_1, F_2, F_3\}$-free then B^* is C_4-free.*

Proof. Assume $C = (x, u, y, v, x)$ is a four-cycle in B^*. From the definition of B^* there are three distinct elements, say a, b, c from the labels of three edges of C. The only way to avoid F_1 is that the fourth edge has label $\{a, b, c\}$. However, the same argument forces that all labels on C are equal to $\{a, b, c\}$ giving (many) F_3's. □

We can consider B^* as a bipartite multigraph obtained as the union of $|A_3|$ simple bipartite graphs as follows. Set

$$E(z) = \{(u, x) : u \in A_1, x \in A_2, (u, x, z) \in E(H) \text{ and } |L(u, x)| \ge 3\},$$

then $E(B^*) = \cup_{z\in A_3} E(z)$.

Proposition 2. *For every $z \in A_3$ there is no path in B^* with four edges such that its first and last edge is in $E(z)$.*

Proof. Suppose that edges e_1, e_2, e_3, e_4 form such a path for some $z \in A_3$. Since each edge of B^* has multiplicity at least three, we can replace e_2 by f_2 and e_3 by f_3 so that $f_2 \in E(z_1)$, $f_3 \in E(z_2)$ and z_1, z_2 are distinct and both different from z. Then the four triples of H,

$$e_1 \cup \{z\},\ f_2 \cup \{z_1\},\ f_3 \cup \{z_2\},\ e_4 \cup \{z\}$$

form an F_1, contradiction. □

For any vertex $x \in A_2$ let $L(x)$ denote the subset of A_3 that appears in some of the labels on edges of B^* incident to x.

Proposition 3. *For distinct vertices $x_1, x_2, x_3, x_4 \in A_2$,*

$$|L(x_1) \cap L(x_2) \cap L(x_3) \cap L(x_4)| \le 1.$$

Proof. Suppose on the contrary that we have $z_1, z_2 \in A_3$ such that for $i = 1, 2, 3, 4$, $e_i = \{z_1, x_i, u_{2i-1}\}$, $f_i = \{z_2, x_i, u_{2i}\}$ are all triples of H.

An F_1 is formed by the triples e_i, f_i, e_j, f_j if there is a pair i, j such that $u_{2i-1}, u_{2i}, u_{2j-1}, u_{2j}$ are all different. Thus, we may assume that for any pair $1 \le i < j \le 4$ there is an equality between elements $u_{2i-1}, u_{2i}, u_{2j-1}, u_{2j}$.

Let us call an equality $u_{2i-1} = u_{2i}$ *horizontal*, an equality $u_{2i} = u_{2j}$ or $u_{2i-1} = u_{2j-1}$ (for $i \ne j$) *vertical*, finally an equality $u_{2i-1} = u_{2j}$ (for $i \ne j$) *diagonal*. The terms to distinguish equalities refer to an arrangement of the vertices u_i into a 4×2 matrix with u_{2i-1}, u_{2i} in row i. Observe the following facts.

1. F_3 or F_2 is formed by the triples e_i, f_i, e_j, f_j if the pair $i \ne j$ have both horizontal equalities holding. Thus, at most one horizontal equality may hold.
2. If there is pair $i \ne j$ such that both vertical equalities hold, then a C_4 can be found in B^* contradicting to Proposition 1. Similarly,
3. if there is pair $i \ne j$ such that both diagonal equalities hold, we get a contradiction with Proposition 1.
4. We get a four edge path contradicting to Proposition 2 if there is a pair i, j such that exactly one vertical equality holds, that is $u_{2i} = u_{2j}$ and u_{2i-1}, u_{2j-1} are different and different from u_{2i} as well. (Symmetrically, if there are x_i, x_j such that $u_{2i-1} = u_{2j-1}$ and u_{2i}, u_{2j} are different and different from u_{2i-1} as well.)

Facts 1–4 imply that there exists a triple of indices, i, j, k such that we have exactly one diagonal equality on each pair of them. These are either in the form $u_{2i-1} = u_{2k}, u_{2i} = u_{2j-1}, u_{2j} = u_{2k-1}$ defining a six-cycle in B on the vertices $x_i, x_j, x_k, u_{2i}, u_{2j}, u_{2k}$, giving (three) F_1, for example e_i, f_k, f_j, e_j, or in the form $u_{2i-1} = u_{2k}, u_{2i} = u_{2j-1}, u_{2j-1} = u_{2k}$ that implies horizontal equality $u_{2i-1} = u_{2i}$, a contradiction. This proves Proposition 3. □

By Propositions 1, 2 the simple bipartite graph $B(z)$ with edge set $E(z)$ has no cycles or paths with four edges. Therefore each component of $B(z)$ is a double star. Thus each $B(z)$ can be written as the union of two graphs, $S(z), T(z)$ where each vertex of $S(z) \cap A_2$ and each vertex of $T(z) \cap A_1$ has degree one in $B(z)$. Set

$$S = \cup_{z \in A_3} S(z), T = \cup_{z \in A_3} T(z).$$

By the definition of S, for every vertex $x \in A_2$, we have $|L(x)| = d_S(x)$ where $d_S(x)$ is the degree of vertex x in the (multi) graph S. By Proposition 3

$$\sum_{x \in A_2} \binom{|L(x)|}{2} \leq 3\binom{|A_3|}{2}$$

therefore

$$\sum_{x \in A_2} \binom{d_S(x)}{2} \leq 3\binom{|A_3|}{2}.$$

Applying the same argument symmetrically for vertices of A_1 and for the graph T, we get

$$\sum_{u \in A_1} \binom{d_T(u)}{2} \leq 3\binom{|A_3|}{2}.$$

By the convexity argument, $|E(B^*)| = |E(S)| + |E(T)| = O(n^{3/2})$. By Lemma 1, we also have $|E(B)| = O(n^{3/2})$. Thus by (1) and the definition of B^*,

$$|E(H)| = \sum_{a_1a_2 \in E(B)} |L(a_1, a_2)| \leq 2|E(B)| + |E(B^*)| = O(n^{3/2}),$$

concluding the proof of Theorem 1.

3. Concluding remarks

Theorem 1 determines the order of magnitude ($\Theta(n^{3/2})$) for the 16 subsets of $\mathcal{F}$ containing F_1, F_2, F_3 and we pointed out that for all other choices $\mathcal{A} \subset \mathcal{F}$, $\mathrm{ex}(n, \mathcal{A})$ must be essentially larger. In this section we summarize what we know about these cases. There is a trivial case, when $\mathcal{A}$ is empty and $\mathrm{ex}(n, \mathcal{A}) = \binom{n}{3}$. Furthermore, as mentioned before, $\mathrm{ex}(n, F_2) = \Theta(n^{5/2})$ was proved by the first author (see in Mubayi [8]). Thus we have $2^7 - 2^4 - 2 = 110$ cases to consider. It turns out that in 92 cases the order of magnitude is $\Theta(n^2)$ (see Subsection 3.1) and only the remaining 18 cases are left unsolved (see Subsection 3.2).

A simple but useful lemma compares Turán numbers of closely related triple systems. Assume G is a triple system and $v, w \in V(G)$ is covered by $e \in E(G)$. The triple system obtained from G by removing e and adding the triple v, w, x where $x \notin V(G)$ is called a *fold out* of G. For example F_7 is a fold out of F_3, F_6 is a fold out of F_4.

Lemma 2. *(Fold out lemma.) If G is a triple system and G_1 is a fold out of G then* $\mathrm{ex}(n, G_1) \leq ex(n, G) + (|V(G)| - 2)\binom{n}{2}$.

Proof. Suppose that a triple system H has n vertices and has more than $\mathrm{ex}(n, G) + (|V(G)| - 2)\binom{n}{2}$ triples. A triple of H is called bad if it contains a pair of vertices that covered by at most $|V(G)| - 2$ triples of H, otherwise it is a good triple. Then H has more than $\mathrm{ex}(n, G)$ good triples thus contains a copy of G with all triples good. By definition, any pair of vertices in any triple of this copy of G is in more than $|V(G)| - 2$ triples of H so some of them is suitable to define the required fold out G_1 of G. □

3.1. When $\mathrm{ex}(n, \mathcal{A}) = \Theta(n^2)$

Here we collect all cases of $\mathcal{A} \subset \mathcal{F}$ when we can prove that $\mathrm{ex}(n, \mathcal{A}) = \Theta(n^2)$.

Proposition 4. *Assume that $\mathcal{A} \subset \mathcal{F} \setminus F_2$ and $\mathcal{A} \cap \{F_1, F_3, F_7\} \neq \emptyset$. Then* $\mathrm{ex}(n, \mathcal{A}) = \Theta(n^2)$.

Proof. The first condition ensures that the members of $\mathcal{A}$ cannot be pierced by one vertex, thus Construction (C3) shows that $\mathrm{ex}(n, \mathcal{A}) = \Omega(n^2)$. On the other hand, F_7 is a fold out of F_3 and F_1 is a fold out of F_7 thus by Lemma 2 (and by the second condition of the proposition)

$$\mathrm{ex}(n, F_1) \le \mathrm{ex}(n, F_7) + O(n^2) \le \mathrm{ex}(n, F_3) + O(n^2) \le \frac{7}{2}\binom{n}{2} + O(n^2)$$

where the upper bound of $\mathrm{ex}(n, F_3)$ is Füredi's result [4]. □

Proposition 5. *Assume that* $\mathcal{A} \subset \mathcal{F} \setminus F_3$ *and* $\mathcal{A} \cap \{F_1, F_7\} \neq \emptyset$. *Then* $\mathrm{ex}(n, \mathcal{A}) = \Theta(n^2)$.

Proof. To show that $\mathrm{ex}(n, F) = \Omega(n^2)$, consider the Construction (C3), it contains only F_3 from $\mathcal{F}$. The upper bound follows by the argument of Proposition 4. □

Proposition 6. *Assume that* $\{F_2, F_3\} \subset \mathcal{A} \subset \{F_2, F_3, F_4, F_5, F_7\}$. *Then* $\mathrm{ex}(n, \mathcal{A}) = \Theta(n^2)$.

Proof. To show that $\mathrm{ex}(n, \mathcal{A}) = \Omega(n^2)$, consider

- (C4) Steiner triple systems without F_4 (the Pasch configuration), they do not contain any member of $\mathcal{A}$.

The upper bound follows from [4] since $F_3 \in \mathcal{A}$. □

Proposition 7. *Assume that* $\{F_2, F_3, F_6\} \subset \mathcal{A} \subset \{F_2, F_3, F_5, F_6, F_7\}$. *Then* $\mathrm{ex}(n, \mathcal{A}) = \Theta(n^2)$.

Proof. To show that $\mathrm{ex}(n, \mathcal{A}) = \Omega(n^2)$, consider

- (C5) Steiner triple systems without F_6 (projective Steiner triple systems), they do not contain any member of $\mathcal{A}$.

The upper bound follows again from [4] since $F_3 \in \mathcal{A}$. □

Note that Proposition 4 covers 56 cases, Proposition 5 adds 24 further cases, Propositions 6, 7 add 8 plus 4 further cases. These 92 cases are the ones when $\mathrm{ex}(n, \mathcal{A}) = \Theta(n^2)$ follows from known results.

3.2. Unsolved cases

The 18 unsolved cases are grouped as follows.

- 1. $\mathrm{ex}(n, \{\mathcal{A} \cup F_6\})$ where $\mathcal{A} \subseteq \{F_2, F_4, F_5\}$ (8 cases)
- 2. $\mathrm{ex}(n, \{F_2, F_5\})$, $\mathrm{ex}(n, F_5)$
- 3. $\mathrm{ex}(n, \{F_2, F_4, F_5\})$, $\mathrm{ex}(n, \{F_4, F_5\})$
- 4. $\mathrm{ex}(n, \{F_2, F_4\})$, $\mathrm{ex}(n, F_4)$
- 5. $\mathrm{ex}(n, \mathcal{A})$ where $\{F_2, F_3, F_4, F_6\} \subseteq \mathcal{A} \subseteq \{F_2, F_3, F_4, F_5, F_6, F_7\}$ (4 cases)

The upper bounds for the unknown cases can be compared by using Lemma 2. For example, observing that F_6 is a fold out of F_4 and of F_5, moreover F_5 is a fold out of F_2, Lemma 2 implies

Proposition 8. *Let* $\mathcal{A}$ *be any subset of* $\{F_2, F_4, F_5\}$. *Then*

$$\mathrm{ex}(n, F_6) \le \mathrm{ex}(n, \{\mathcal{A} \cup F_6\}) \le \mathrm{ex}(n, F_6) + 7\binom{n}{2}.$$

A lower bound $\Omega(n^2)$ for the first four groups of unknown cases can be obtained from Construction (C3). Lower bounds for the fifth group of unknown cases can be given by well studied functions introduced in [1]. Let $\mathrm{ex}(n, (6, 3))$ be the maximum number of triples in a triple system that does not contain three triples inside any six vertices. Since all members of $\mathcal{F}$ except F_1 contain three triples inside six vertices an almost quadratic lower bound of Construction (C1) comes from [9] for the four unsolved cases in group 5. A quadratic upper bound is from [4] since $F_3 \in \mathcal{A}$. Thus we get

Proposition 9. *If* $\{F_2, F_3, F_4, F_6\} \subseteq \mathcal{A} \subseteq \{F_2, F_3, F_4, F_5, F_6, F_7\}$ *then*

$$\mathrm{ex}(n, (6, 3)) \le \mathrm{ex}(n, \mathcal{A}) = O(n^2).$$

In fact, the lower bound of Proposition 9 can be changed to $\mathrm{ex}(n,(7,4))$ (the maximum number of triples in a triple system that does not contain four triples inside any seven vertices).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The authors are indebted to an unknown referee for pointing out some inaccuracies in the manuscript.

The research of the first author was supported in part by the Hungarian National Research, Development and Innovation Office NKFIH grant KH-130371 and NKFI–133819. The research of the second and third authors was supported in part by the Hungarian National Research, Development and Innovation Office NKFIH grant K132696. The work of the third author is also connected to the scientific program of the "Development of quality-oriented and harmonized R+D+I strategy and functional model at BME" project, supported by the New Hungary Development Plan (Project ID: TAMOP-4.2.1/B-09/1/KMR-2010-0002).

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