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Hypergraphs without exponents

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ABSTRACT

A short, concise proof is given for that for $k \geq 5$ there exists a k-uniform hypergraph H without exponent, i.e., when the Turán function is not polynomial in n. More precisely, we have $\mathrm{ex}(n, H) = o(n^{k-1})$ but it exceeds n^{k-1-c} for any positive c for $n > n_0(k, c)$. We conjecture that this is true for $k \in \{3, 4\}$ as well.

1. Introduction

We use standard notation. A *k-graph* (or *k-uniform hypergraph*) H is a pair (V, E) with $V = V(H)$ a set of vertices and $E = E(H)$ a collection of k-sets from V which are the hyperedges (or k-edges) of H. We may also use 'edge' for 'k-edge'. The *s-shadow*, $\partial_s H$, is the family of s-sets contained in the hyperedges of H. So $\partial_1 H$ is the set of non-isolated vertices, and $\partial_2 H$ is a graph. We write $[n]$ for $\{1, 2, \dots, n\}$. Given a set A and an integer k, we write $\binom{A}{k}$ for the set of k-sets of A.

The *complete* k-graph on n vertices is the k-graph $K_n^{(k)} = ([n], \binom{[n]}{k})$. Let $I_k(i)$ denote the k-uniform hypergraph consisting of two hyperedges sharing exactly i vertices. The k-graph H is *k-partite* if there exists a partition $\{P_1, \dots, P_k\}$ of $V(H)$ such that for

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every edge $e \in E(H)$ and part P_i we have $|e \cap P_i| = 1$. The *complete* k-partite k-graph $K_k(P_1, \dots, P_k)$ has all of such edges, $|E(K_k(P_1, \dots, P_k))| = |P_1| \times \dots \times |P_k|$.

Given a family of k-graphs $\mathcal{F}$, we say that a k-graph H is *$\mathcal{F}$-free* if it contains no member of $\mathcal{F}$ as a subgraph. We write $\mathrm{ex}(n, \mathcal{F})$ (or $\mathrm{ex}_k(n, \mathcal{F})$ if we want to emphasize k) for the maximum number of k-edges that can be present in an n-vertex $\mathcal{F}$-free k-graph. The function $\mathrm{ex}(n, \mathcal{F})$ is referred to as the *Turán number* of $\mathcal{F}$. We leave out parentheses whenever it is possible, e.g., in case of $|\mathcal{F}| = 1$ we write $\mathrm{ex}(n, F)$ instead of $\mathrm{ex}(n, \{F\})$.

Erdős and Simonovits (see [3,5]) *conjectured* that for any rational $1 \le \alpha \le 2$ there exists a graph F with $\mathrm{ex}_2(n, F) = \Theta(n^\alpha)$ and for every graph F we have $\mathrm{ex}_2(n, F) = \Theta(n^\alpha)$ for some rational α. Bukh and Conlon [2] showed that the first conjecture holds if we can forbid finite families of graphs. For a single graph, it is still unknown.

For hypergraphs Frankl [7] showed that all rationals occur as exponents of $\mathrm{ex}_k(n, \mathcal{F})$ for some k and for some finite family $\mathcal{F}$ of k-uniform hypergraphs. Fitch [6] showed that for a fixed k all rational numbers between 1 and k occur as exponents of $\mathrm{ex}_k(n, \mathcal{F})$ for some family $\mathcal{F}$ of k-uniform hypergraphs.

We say that a function $f(n) : \mathbb{N} \to \mathbb{R}$ has *no exponent* if there is no real α such that $f(n) = \Theta(n^\alpha)$. In other words, the order of magnitude of $f(n)$ is not a polynomial.

Brown, Erdős, and Sós [1] proposed the problem to determine (or estimate) $f_k(n, v, e)$, the maximum number of edges in a k-uniform, n-vertex hypergraph in which no v vertices span e or more edges. This is a Turán type problem: Let $\mathcal{G}_k(v, e)$ be the family of k-graphs, each member having e edges and at most v vertices, then $f_k(n, v, e) = \mathrm{ex}_k(n, \mathcal{G}_k(v, e))$.

Ruzsa and Szemerédi [16] showed that if a 3-uniform hypergraph does not contain three hyperedges on six vertices, then it has $o(n^2)$ edges, and they gave a construction with $n^{2-o(1)}$ hyperedges. This assumption is equivalent to forbidding the sub-hypergraphs $\{123, 124\}$ (a pair covered twice) and $\{123, 345, 561\}$ (a linear triangle). They proved

$$n^{2-o(1)} < \frac{1}{10} n r_3(n) < f_3(n, 6, 3) - (n/2) \le \mathrm{ex}_3(n, \{\{123, 124\}, \{123, 345, 561\}\}) \le f_3(n, 6, 3) = o(n^2). \quad (1.1)$$

(For the definition of $r_3(n)$, see (2.3) in Section 2). So they found a family of two 3-graphs such that not only its Turán number does not have a rational exponent, it does not have an exponent at all. This is the famous $(6, 3)$-theorem, $f_3(n, 6, 3)$ is non-polynomial.

Erdős, Frankl, and Rödl [4] extended this to every k proving $f_k(n, 3k - 3, 3) = o(n^2)$ but $\lim_{n\to\infty} f_k(n, 3k - 3, 3)/n^{2-\varepsilon} = \infty$ for all $\varepsilon > 0$ ($k \ge 3$ and ε are fixed, $n \to \infty$). The proofs of the upper bounds here and in (1.1) are based on Szemerédi's regularity lemma [18].

Answering a question of Erdős, a single 5-uniform hypergraph with no exponent was presented in [9]:

Theorem 1.1 *(Frankl and Füredi [9]). Let* $H = \{12346, 12457, 12358\}$*. Then* $\mathrm{ex}_5(n, H) = o(n^4)$ *but* $\mathrm{ex}_5(n, H) \neq O(n^{4-\varepsilon})$ *for any* $\varepsilon > 0$*.*

The aim of this paper is to give a short proof and a generalization for all $k \geq 5$. The original proof relied on the delta-system method, here we will use hypergraph regularity. We *conjecture* that examples with no exponents should exist for $k = 3$ and 4, too.

Definition 1.2. Let us consider three disjoint sets of vertices $A = \{a_1, \dots, a_{k-r}\}$, $B = \{b_1, \dots, b_r\}$ and $C = \{c_1, \dots, c_r\}$. Let $Q_k(r)$ denote the k-uniform hypergraph consisting of all the hyperedges of the form $A \cup (B \setminus \{b_i\}) \cup \{c_i\}$, for $1 \leq i \leq r$.

So $|E(Q_k(r)| = r$ and $|V(Q_k(r)| = k + r$. To avoid trivialities we suppose that $r \geq 2$. In this paper we study $\mathrm{ex}_k(n, Q_k(r))$ for every pair of values k and r, $k \geq r \geq 2$, and we either determine the order of magnitude or show that there is no exponent.

In the case of $r = 2$ we have $Q_k(2) = I_k(k-2)$, i.e., two k-edges meeting in $k-2$ elements. The study of the Turán number of $I_k(i)$ has been initiated by Erdős [3]. Frankl and Füredi [8] proved that $\mathrm{ex}_k(n, I_k(i)) = \Theta(n^{\max\{i, k-i-1\}})$. One obtains $\mathrm{ex}_k(n, Q_k(2)) = \Theta(n^{k-2})$ for $k \geq 3$ and $\mathrm{ex}_2(n, Q_2(2)) = \Theta(n)$.

Our main result is the following theorem.

Theorem 1.3. *If* $k \geq r \geq 3$ *and* $r \geq (k/2) + 1$*, then* $\mathrm{ex}_k(n, Q_k(r)) = \Theta(n^{k-1})$*.*

If $k \geq r \geq 3$ *and* $r \leq (k+1)/2$*, then* $\mathrm{ex}_k(n, Q_k(r)) = o(n^{k-1})$ *but* $\mathrm{ex}_k(n, Q_k(r)) \neq O(n^{k-1-\varepsilon})$ *for any* $\varepsilon > 0$*.*

Note that $Q_5(3) = \{12346, 12457, 12358\}$, so this Theorem is indeed an extension of Theorem 1.1. Since $Q_k(3) \subset \dots \subset Q_k(k)$, we have

$$\mathrm{ex}_k(n, Q_k(3)) \leq \mathrm{ex}_k(n, Q_k(4)) \leq \dots \leq \mathrm{ex}_k(n, Q_k(k)).$$

So to prove Theorem 1.3 we need to show that for $k \geq r \geq 3$ as $n \to \infty$ we have

(1.3.a) $\mathrm{ex}_k(n, Q_k(k)) = O(n^{k-1})$,
(1.3.b) $\mathrm{ex}_k(n, Q_k(r)) = \Omega(n^{k-1})$ if $k \leq 2r - 2$,
(1.3.c) $\mathrm{ex}_k(n, Q_k(r)) = o(n^{k-1})$ if $k \geq 2r - 1$,
(1.3.d) $\mathrm{ex}_k(n, Q_k(3)) = \Omega(n^{k-1-\varepsilon})$ if $k \geq 5$, $\forall \varepsilon > 0$ fixed.

We emphasize that to prove that $Q_k(3)$ has no exponent (for $k \geq 5$) we do not need the hypergraph removal lemma, we can only use the upper bound in the $(6, 3)$-theorem (1.1) and our new lower bound construction from Section 3.3.

Problem. Determine $\limsup_{n\to\infty} \mathrm{ex}_k(n, Q_k(r))/n^{k-1}$ for $4 \leq k \leq 2r - 2$.

The rest of the paper is organized as follows. In Section 2 the necessary tools are presented, and Section 3 contains the proof of Theorem 1.3.

2. Lemmas and tools

The following observation of Erdős and Kleitman is one of the basic tools to determine the order of magnitude of the size of a k-graph H: Every k-graph H has a k-partition of its vertices $V(H) = P_1 \cup \cdots \cup P_k$ into almost equal parts $\big(\big||P_i| - |P_j|\big| \le 1\big)$ such that for the k-partite subhypergraph H' with $E(H') := E(H) \cap E(K_k(P_1, \dots, P_k))$, one has

$$\frac{k!}{k^k}|E(H)| \le |E(H')| \le |E(H)|. \tag{2.1}$$

Suppose $n \ge r \ge t \ge 1$ are integers. An r-graph H on n vertices is called an (n, r, t)-*packing* if $|e \cap e'| < t$ holds for every $e, e' \in E(H)$, $e \ne e'$. The maximum of $|E(H)|$ is denoted by $P(n, r, t)$. Since $\binom{n}{t} \ge |\partial_t H| = \binom{r}{t}|E(H)|$, we have $P(n, k, t) \le \binom{n}{t}/\binom{r}{t}$. It is known that $P(n, r, t) = (1 + o(1))\binom{n}{t}/\binom{r}{t}$ when r and t are fixed and n tends to infinity. We only use the following easy statement: If r is fixed and $n \to \infty$ then

$$P(n, r, t) \ge \binom{n}{t} \Big/ \binom{r}{t}^2 = \Omega(n^t). \tag{2.2}$$

A set of numbers is called AP_k-*free* if it does not contain k distinct elements forming an arithmetic progression. Let $r_k(n)$ denote the maximum size of an AP_k-free subset of $[n]$. The celebrated Szemerédi's theorem [17] states that for a fixed k as $n \to \infty$ we have

$$r_k(n) = o(n). \tag{2.3}$$

(The case $r_3(n) = o(n)$ was proved much earlier by K. F. Roth).

Let k be an integer and p be a prime, $p > k$. We say that $S \subseteq \{0, \dots, p-1\}$ is k-*good* if for any $m_1, m_2, m_3 \in \{-k, -k+1, \dots, -1\} \cup \{1, \dots, k\}$ and $s_1, s_2, s_3 \in S$

$$\left.\begin{array}{l} m_1 + m_2 + m_3 = 0 \quad \text{and} \\ m_1 s_1 + m_2 s_2 + m_3 s_3 = 0 \end{array}\right\} \quad \text{imply} \quad s_1 = s_2 = s_3.$$

Here addition and multiplication are taken modulo p. Let $s_k(p)$ denote the size of the largest k-good set. The following result is an easy extension of Behrend's construction, see, e.g., Ruzsa [15]: There is a $c_k > 0$ such that

$$p \exp[-c_k \sqrt{\log p}] < s_k(p).$$

We only need that if k and $\varepsilon > 0$ are fixed and $p \to \infty$, then

$$s_k(p) > p^{1-\varepsilon}. \tag{2.4}$$

Note that a k-good set cannot contain a (strictly increasing) arithmetic progression of length 3, so $s_k(p) \le r_3(p)$ and $r_3(p) = o(p)$ by Roth's theorem, see (2.3).

We will use the hypergraph removal lemma. It was developed by several groups of researchers (together with different versions of hypergraph regularity), see [11–14].

Theorem 2.1 *(Hypergraph removal lemma). For any $\varepsilon > 0$ and integers $\ell \geq k$, there exist $\delta > 0$ and an integer n_0 such that the following statement holds. Suppose F is a k-uniform hypergraph on ℓ vertices and H is a k-uniform hypergraph on $n \geq n_0$ vertices, such that H contains at most $\delta\binom{n}{\ell}$ copies of F. Then one can delete at most $\varepsilon\binom{n}{k}$ hyperedges from H such that the resulting hypergraph is F-free.*

Recall that $I_k(i)$ denotes the k-uniform hypergraph consisting of two hyperedges sharing exactly i vertices. Frankl and Rödl [10] generalized the lower bound of the $(6,3)$-theorem (i.e., (1.1)) of Ruzsa and Szemerédi [16] as follows.

Theorem 2.2 *([10]). For any integer $k \geq 3$ there exists a $c'_k > 0$ such that for all $n \geq k$*

$$c'_k \times r_k(n) \times n^{k-2} \leq \mathrm{ex}_k(n, \{Q_k(k), I_k(k-1)\}).$$

They conjectured $\mathrm{ex}_k(n, \{Q_k(k), I_k(k-1)\}) = o(n^{k-1})$ and proved the case $k = 4$ (the case $k = 3$ is part of (1.1)). In order to prove $\mathrm{ex}_4(n, \{Q_4(4), I_4(3)\}) = o(n^3)$ they developed a hypergraph removal lemma for the 3-uniform case. They also described how the hypergraph removal lemma (Theorem 2.1) would imply the general upper bound $o(n^{k-1})$. Since then Theorem 2.1 has been proved, so we have the following statement.

Corollary 2.3. *For any $k \geq 2$ we have $\mathrm{ex}_k(n, \{Q_k(k), I_k(k-1)\}) = o(n^{k-1})$.*

Note that Theorem 2.2 and Corollary 2.3 imply Szemerédi's theorem: $r_k(n) = o(n)$.

Since the above corollary plays an important role in our main result, we include its few line proof from [10]. This is the only place where we need Theorem 2.1.

Proof Corollary 2.3. Let H be a $Q_k(k)$ and $I_k(k-1)$-free k-graph on n vertices. We will give an upper bound on its size. By (2.1) we may suppose that H is k-partite with parts $P_1, \ldots, P_k$. Consider its shadow ∂H, which is a $(k-1)$-uniform hypergraph. Since H is $I_k(k-1)$-free, each $f \in \partial H$ is contained in a unique $e(f) \in E(H)$. We get $\binom{k}{k-1}|E(H)| = |E(\partial H)|$. This already gives $|E(H)| = O(n^{k-1})$.

Every edge $e \in E(H)$ induces a complete subhypergraph $K_k^{(k-1)}$ in ∂H. We claim that these are the only cliques of size k in ∂H. Consider a copy K of $K_k^{(k-1)}$ in ∂H. Then $|P_i \cap V(K)| = 1$ for each P_i. If $e(f) = V(K)$ for some $f \in E(K)$ then K is the clique generated by $V(K) = e(f) \in E(H)$. Otherwise, when $e(f) \neq V(K)$ for each $f \in E(K)$, the k hyperedges $\{e(f) : f \in E(K)\}$ form a copy of $Q_k(k)$, a contradiction.

Therefore, the number of copies of $K_k^{(k-1)}$ in ∂H is $O(n^{k-1}) = o(n^{|V(K)|})$. Then by the hypergraph removal lemma (Theorem 2.1) there exists a subhypergraph H', $E(H') \subset E(\partial H)$, so that $E(H')$ meets every copy of $K_k^{(k-1)}$ in ∂H and $|E(H')| = o(n^{k-1})$. For such an H' we have $|E(H)| \leq |E(H')|$, finishing the proof. $\square$

3. Proof of Theorem 1.3

3.1. Upper bounds

Here we prove (1.3.a) and (1.3.c), the upper bounds for $\mathrm{ex}_k(n, Q_k(r))$.

Let H be a $Q_k(r)$-free k-graph on n vertices. We will give an upper bound on $|E(H)|$. By (2.1) we may suppose that H is k-partite with parts $P_1, \dots, P_k$. For a hyperedge $e \in E(H)$, let $D(e) \subseteq [k]$ denote the set of integers i such that there is another hyperedge $e' \in E(H)$ that differs from e only in P_i, $e \setminus P_i = e' \setminus P_i$. Note that $|D(e)| < r$ because H is $Q_k(r)$-free.

There is a set $D \subset \{1, \dots, k\}$ such that there are at least $|E(H)|/2^k$ hyperedges $e \in E(H)$ with $D(e) = D$. Let H' be the k-graph of these edges, $E(H') := \{e \in E(H) : D(e) = D\}$. Set $\ell := k - |D|$, we have $\ell \geq k - r + 1$, $\ell \geq 1$.

Let T be an edge of the complete $|D|$-partite hypergraph with parts $\{P_i : i \in D\}$, i.e., $|T| = |D|$ and $|T \cap P_i| = 1$ for each $i \in D$. (D might be the empty set). There are at most $O(n^{k-\ell})$ appropriate T. Define $H'[T]$ as the *link* of T in H', i.e., it is an ℓ-graph with edges $\{e \setminus T : T \subset e \in E(H')\}$.

Observe that $H'[T]$ is $I_\ell(\ell - 1)$-free. Indeed, two hyperedges of $H'[T]$ sharing $\ell - 1$ vertices would mean two hyperedges in H' sharing $k - 1$ vertices such that their only difference is in a part not belonging to D. So every $(\ell - 1)$-element set is contained in at most one hyperedge in $H'[T]$, thus $|H'[T]| \leq \binom{n}{\ell-1}$. Since $|E(H')| = \sum_T |E(H'[T])|$, we obtained

$$|E(H)| = O(|E(H')|) = O(n^{k-l})\binom{n}{\ell - 1} = O(n^{k-1}), \tag{3.1}$$

completing the proof of (1.3.a).

Finally, let us assume $k \geq 2r - 1$, i.e., $\ell \geq r$. We claim that in this case $H'[T]$ is also $Q_\ell(\ell)$-free. Indeed, if we add T to the hyperedges of a copy of $Q_\ell(\ell)$ from $H'[T]$, we obtain a $Q_k(\ell)$ in H'. Since $Q_k(\ell)$ contains a $Q_k(r)$, this is a contradiction. Thus we have $|E(H'[T])| = o(n^{\ell-1})$ by Corollary 2.3. We complete the proof as in (3.1)

$$|E(H)| = O(|E(H')|) = O(n^{k-l}) \times o(n^{\ell-1}) = o(n^{k-1}). \quad \square$$

3.2. Proof of Theorem 1.3, the polynomial range

In this subsection we prove the lower bound (1.3.b) by giving a construction.

Since $k \leq 2r - 2$, we have $r - 1 \geq k + 1 - r \geq 1$. Let X and Y be two disjoint sets, $|X| = \lfloor n/2 \rfloor$ and $|Y| = \lceil n/2 \rceil$. Let H^1 be an $(|X|, r - 1, r - 2)$-packing of maximum size, i.e., an $(r - 1)$-uniform hypergraph such that any two hyperedges share at most $r - 3$ vertices. By (2.2) we have $|E(H^1)| = \Theta(n^{r-2})$. Let H^2 be the complete $(k - r + 1)$-uniform hypergraph with vertex set Y. Finally, let H^3 be the k-graph with vertex set

$X \cup Y$ having as hyperedges all the k-sets that are unions of a hyperedge of H^1 and a hyperedge of H^2. Then H^3 has $\Theta(n^{k-1})$ hyperedges. We claim that H^3 is $Q_k(r)$-free.

Assume, on the contrary, that there is a copy of $Q_k(r)$ in H^3, $E(Q_k(r)) = \{f_1, f_2, \ldots, f_r\}$. Note that $|\cap f_i| = k - r < (k - r + 1) \leq r - 1$ and the symmetric differences $\{f_i \triangle f_j : 1 \leq i < j \leq r\}$ are all distinct 4-element sets. Consider, first, the case when for some $i \neq j$ we have $f_i \cap X = f_j \cap X$. Then all $f_t \cap X$ are identical. Indeed, if there exists an $f_t \cap X \neq f_i \cap X$, then these two $(r-1)$-sets have symmetric difference at least 4, so it should be exactly 4, and then $(f_i \cap X) \triangle (f_t \cap X)$ and $(f_j \cap X) \triangle (f_t \cap X)$ are identical 4-element sets, a contradiction. Then $|\cap f_i| \geq r - 1$, a contradiction.

From now on, we may suppose that the $(r-1)$-element sets $\{f_i \cap X\}$ are all distinct. Then, because $|(f_i \cap X) \triangle (f_j \cap X)| \geq 4$ we have that $f_i \cap Y = f_j \cap Y$ for all $1 \leq i < j \leq r$. Hence $|\cap f_i| \geq k - r + 1$, a final contradiction. □

3.3. *Proof of Theorem 1.3, a non-polynomial lower bound*

In this subsection we prove the lower bound (1.3.d) by giving a construction. We will show that if $n = kp$, where $k \geq 5$ and p is a prime, then $\mathrm{ex}(n, Q_k(3)) \geq p^{k-2} s_k(p)$. As $\mathrm{ex}(n, Q_k(3))$ is monotone in n and there is a prime between $n/2k$ and n/k, this and (2.4) give the desired bound $\Omega(n^{k-1-o(1)})$ for $\mathrm{ex}(n, Q_k(3))$.

Let the vertex set V consist of the pairs (i, j) with $1 \leq i \leq k$ and $0 \leq j \leq p-1$. Choose two integers $0 \leq \alpha, \beta \leq p - 1$ and a k-good set $S \subset \{0, \ldots, p-1\}$ of size $s_k(p)$. Suppose that $m_1, \ldots, m_k \in \{1, \ldots, k\}$ are distinct integers (i.e., a permutation of $[k]$). We define a k-partite k-graph $F = F(S, \alpha, \beta)$ on V with parts $P_i := \{(i, j) : 0 \leq j \leq p-1\}$. A k-set $\{(1, x_1), (2, x_2), \ldots, (k, x_k)\}$ is a hyperedge of F if the following two equations hold.

$$\left(\sum_{i=1}^{k} x_i\right) = \alpha \pmod{p},$$

$$\left(\sum_{i=1}^{k} m_i x_i\right) \in S + \beta \pmod{p}.$$

We have $|F(S, \alpha, \beta)| = p^{k-2} s_k(p)$. Indeed, we can pick an $s \in S$ and $k - 2$ values $x_3, \ldots, x_k$ arbitrarily, and since $m_1 \neq m_2$, the above two equations uniquely determine x_1 and x_2.

Claim 3.1. *F is $Q_k(3)$-free.*

Proof of Claim. Suppose, on the contrary, that there is a copy of $Q_k(3)$ in F, and let A, B, C be the sets of vertices as in Definition 1.2. Without loss of generality we may assume that $A = \{(i, x_i) : 4 \leq i \leq k\}$, $b_i = (i, x_i)$ $(i = 1, 2, 3)$, and $c_i = (i, y_i)$ $(i = 1, 2, 3)$. Then the constraints in the definition of F imply the following six equations.

$$\left(\sum_{i=1}^{k} x_i\right) + y_j - x_j = \alpha \pmod p \quad \text{for } j = 1,2,3$$

$$\left(\sum_{i=1}^{k} m_i x_i\right) + m_j(y_j - x_j) = s_j + \beta \pmod p \quad \text{for } j = 1,2,3$$

for some $s_1, s_2, s_3 \in S$. Define $u := \alpha - (\sum_{i=1}^{k} x_i)$ and $v := (\sum_{i=1}^{k} m_i x_i) - \beta$. We obtain

$$y_j - x_j = u, \quad \pmod p \quad \text{for } j = 1,2,3 \tag{3.2}$$

and

$$v + m_j u = s_j \pmod p \quad \text{for } j = 1,2,3. \tag{3.3}$$

These imply

$$(v + m_1 u - s_1)(m_2 - m_3) + (v + m_2 u - s_2)(m_3 - m_1) + (v + m_3 u - s_3)(m_1 - m_2) = 0.$$

Rearranging

$$(m_3 - m_2)s_1 + (m_1 - m_3)s_2 + (m_2 - m_1)s_3 = 0 \pmod p.$$

As S is a k-good set and $1 \le |m_i - m_j| \le k$, we have $s_1 = s_2 = s_3$. Then (3.3) gives $v + m_1 u = v + m_2 u = v + m_3 u$ implying $u = 0$. Then (3.2) gives $x_j = y_j$ (for $j = 1,2,3$), a contradiction. □

3.4. Another lower bound in the case of $k = 2r - 1$

We give another construction which gives the lower bound $\Omega(r_r(n)n^{k-2}) \le \mathrm{ex}_k(n, Q_k(r))$ in the case of $k = 2r - 1$. The construction in Section 3.3 yielded a slightly weaker lower bound $\Omega(s_k(n) \times n^{k-2})$.

We start with an r-graph H^1 with a set V_1 of $\lfloor n/2 \rfloor$ vertices and $\Omega(r_r(n)n^{r-2})$ hyperedges that is both $Q_r(r)$-free and $I_r(r-1)$-free. The existence of such hypergraphs was proved by Frankl and Rödl [10], see Theorem 2.2. Then we add a set V_2 of $\lceil n/2 \rceil$ new vertices and take all the k-edges which contain an r-edge of H^1 and $r - 1$ vertices from V_2. This hypergraph H obviously has $\Omega(r_r(n)n^{k-2})$ hyperedges. It is not difficult to see, like we did in Subsection 3.2, that H is $Q_k(r)$-free.

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