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Hypergraphs not containing a tight tree with a bounded trunk II: 3-trees with a trunk of size 2

Zoltán Füredi [a,*,1], Tao Jiang [b,2], Alexandr Kostochka [c,d,3], Dhruv Mubayi [e,4], Jacques Verstraëte [f,5]

[a] Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reáltanoda utca 13-15, H-1053, Budapest, Hungary
[b] Department of Mathematics, Miami University, Oxford, OH 45056, USA
[c] University of Illinois at Urbana–Champaign, Urbana, IL 61801, USA
[d] Sobolev Institute of Mathematics, Novosibirsk 630090, Russia
[e] Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, Chicago, IL 60607, USA
[f] Department of Mathematics, University of California at San Diego, 9500 Gilman Drive, La Jolla, CA 92093-0112, USA

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ABSTRACT

A *tight r-tree T* is an r-uniform hypergraph that has an edge-ordering $e_1, e_2, \dots, e_t$ such that for each $i \geq 2$, e_i has a vertex v_i that does not belong to any previous edge and $e_i - v_i$ is contained in e_j for some $j < i$. Kalai conjectured in 1984 that every n-vertex r-uniform hypergraph with more than $\frac{t-1}{r}\binom{n}{r-1}$ edges contains every tight r-tree T with t edges.

A *trunk* T' of a tight r-tree T is a tight subtree T' of T such that vertices in $V(T)\setminus V(T')$ are leaves in T. Kalai's Conjecture was proved (Frankl and Füredi, 1987) for tight r-trees that have a trunk of size one. In a previous paper (Füredi et al., 2019) we proved an asymptotic version for all tight r-trees that have a trunk of bounded size. In this paper we continue that work to establish the exact form of Kalai's Conjecture for all tight 3-trees of at least 8 edges that have a trunk of size two.

1. Introduction. Trees, trunks, and Kalai's conjecture

For an r-uniform hypergraph (r-*graph*, for short) H, the *Turán number* $\mathrm{ex}_r(n, H)$ is the largest m such that there exists an n-vertex r-graph G with m edges that does not contain H. Estimating $\mathrm{ex}_r(n, H)$ is a difficult problem even for r-graphs with a simple structure. Here we consider Turán-type problems for so called tight r-trees. A *tight r-tree* ($r \geq 2$) is an r-graph whose edges can be ordered so that each edge e apart from the first one contains a vertex v_e that does not

* Corresponding author.
E-mail addresses: z-furedi@math.uiuc.edu (Z. Füredi), jiangt@miamioh.edu (T. Jiang), kostochk@math.uiuc.edu (A. Kostochka), mubayi@uic.edu (D. Mubayi), jverstra@math.ucsd.edu (J. Verstraëte).

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belong to any preceding edge but the set $e - v_e$ is contained in some preceding edge. Such an ordering is called a *proper ordering* of the edges. A usual graph tree is a tight 2-tree.

A vertex v in a tight r-tree T is a *leaf* if it has degree one in T. A *trunk* T' of a tight r-tree T is a tight subtree of T such that in some proper ordering of the edges of T the edges of T' are listed first and vertices in $V(T) \setminus V(T')$ are leaves in T. Hence, each $e \in E(T) \setminus E(T')$ contains an $(r-1)$-subset of some $e' \in E(T')$ and a leaf in T (that lies outside $V(T')$). In the case of $r = 2$ each $e \in E(T) \setminus E(T')$ is a pendant edge. Every tight tree T with at least two edges has a trunk (for example, T minus the last edge in a proper ordering is a trunk). Let $c(T)$ denote the minimum size of a trunk of T. We write $e(H)$ for the number of edges in H.

In this paper we consider the following classical conjecture.

Conjecture 1.1 (*Kalai 1984, see in [1,3]*)**.** *Let T be a tight r-tree with t edges. Then*

$$\mathrm{ex}_r(n, T) \le \frac{t-1}{r}\binom{n}{r-1}.$$

The coefficient $(t-1)/r$ in this conjecture, if it is true, is optimal as one can see using constructions obtained from partial Steiner systems due to Rödl [4]. The conjecture turns out to be difficult even for very special cases of tight trees, in fact for $r = 2$ it is the famous Erdős–Sós conjecture. The following partial result on Kalai's conjecture was proved in 1987.

Theorem 1.2 (*[1]*)**.** *Let T be a tight r-tree with t edges and $c(T) = 1$. Suppose that G is an n-vertex r-graph with $e(G) > \frac{t-1}{r}\binom{n}{r-1}$. Then G contains a copy of T.*

A more detailed introduction and more references, the reader can find in our previous paper [2]. In that paper we also showed that Conjecture 1.1 holds *asymptotically* for tight r-trees with a trunk of a bounded size. Our result is as follows. Define $a(r, c) := (r^r + 1 - \frac{1}{r})(c - 1)$.

Theorem 1.3 (*[2]*)**.** *Let T be a tight r-tree with t edges and $c(T) \le c$. Then*

$$\mathrm{ex}_r(n, T) \le \left(\frac{t-1}{r} + a(r, c)\right)\binom{n}{r-1}.$$

The goal of this paper is to prove the conjecture in *exact* form for infinitely many 3-trees.

Theorem 1.4. *Let T be a tight 3-tree with t edges and $c(T) \le 2$. If $t \ge 8$ then*

$$\mathrm{ex}_3(n, T) \le \frac{t-1}{3}\binom{n}{2}.$$

Besides ideas and observations from [2], discharging is quite helpful here.

2. Notation and preliminaries. Shadows and default weights

In this section, we introduce some notation and list a couple of simple observations from [2]. For the sake of self-containment, we present their simple proofs as well.

The *shadow* of an r-graph G is $\partial(G) := \{S : |S| = r-1, \text{ and } S \subseteq e \text{ for some } e \in e(G)\}$.

The *link* of a set $D \subseteq V(G)$ in an r-graph G is defined as $L_G(D) := \{e \setminus D : e \in E(G), D \subseteq e\}$.

The *degree* of D, $d_G(D)$, is the number of the edges of G containing D. If G is an r-graph and $|D| = r - 1$, the elements of $L_G(D)$ are vertices. In this case, we also use $N_G(D)$ to denote $L_G(D)$. Many times we drop the subscript G. For $1 \le p \le r - 1$, the *minimum p-degree* of G is

$$\delta_p(G) := \min\{d_G(D) : |D| = p, \text{ and } D \subseteq e \text{ for some } e \in E(G)\}.$$

For an r-graph G and $D \in \partial(G)$, let $w(D) := \frac{1}{d_G(D)}$. For each $e \in E(G)$, let

$$w(e) := \sum_{D \in \binom{e}{r-1}} w(D) = \sum_{D \in \binom{e}{r-1}} \frac{1}{d_G(D)}.$$

We call w the *default weight function* on $E(G)$ and $\partial(G)$. Frankl and Füredi [1] (and later some others) used the following simple property of this function.

Proposition 2.1. *Let G be an r-graph. Let w be the default weight function on $E(G)$ and $\partial(G)$. Then*

$$\sum_{e \in E(G)} w(e) = |\partial(G)|.$$

Proof. By definition,

$$\sum_{e\in E(G)} w(e) = \sum_{e\in E(G)} \left(\sum_{D\in\binom{e}{r-1}} \frac{1}{d_G(D)} \right) = \sum_{D\in\partial(G)} \left(\sum_{e\in E(G), D\subseteq e} \frac{1}{d_G(D)} \right) = \sum_{D\in\partial(G)} 1 = |\partial(G)|. \quad \square$$

The following proposition is folklore.

Proposition 2.2. *Let q be a nonnegative real number and G be an r-graph with $e(G) > q|\partial(G)|$. Then G contains a subhypergraph G' with $\delta_{r-1}(G') \geq \lfloor q \rfloor + 1$.*

Proof. Starting from G, if there exists $D \in \partial(G)$ of degree at most $\lfloor q \rfloor$ in the current r-graph, we remove the edges of this r-graph containing D. Let G' be the final r-graph. Since we have deleted at most $q|\partial(G)| < e(G)$ edges, G' is nonempty. By the stopping rule, $\delta_{r-1}(G') \geq \lfloor q \rfloor + 1$. $\square$

In the calculations below we will frequently use the following observation.
If d is an integer, α is a real number with $d > \alpha$ then $d - 2 \geq \lfloor \alpha \rfloor - 1$.

3. The case of high degree central edge

We prove Theorem 1.4 in two major steps in this and the next section. In both cases the idea behind the proof is to find in the host 3-graph G a special pair of edges $\{e, f\}$ with good properties where we plan to map the trunk of size 2 of T.

We use two different weight arguments together with discharging to find such special pairs in the next lemma and in Lemma 4.1. It is natural to have a two part proof because in one case we need to start with a pair of triples e, f, $|e \cap f| = 2$ with high codegree ($\deg(e \cap f)$ is 'high') and in the other case, in Section 4, we need a pair e, f with a moderate codegree but high 'leaf' degrees.

Given edges $e = abc$ and $f = adc$ in a 3-graph G sharing pair ac, for a pair $\{x, y\} \subset \{a, b, c, d\}$, let $d'_{e,f}(x, y)$ denote the number of $z \in V(G) \setminus \{a, b, c, d\}$ such that $xyz \in G$. By definition

$$d'_{e,f}(x, y) \geq d(x, y) - 2 \qquad \text{for every} \{x, y\} \subset \{a, b, c, d\}. \tag{1}$$

Lemma 3.1. *Let $m \geq 4$ be a positive integer and let G be a 3-graph satisfying $e(G) > \frac{m}{3}|\partial(G)|$ and $\delta_2(G) > \frac{m}{3}$. Let w be the default weight function on $E(G)$ and $\partial(G)$. Then there exist edges $e = abc$ and $f = adc$ in G satisfying*

(a) *$w(e) < \frac{3}{m}$ and $w(ac) < \frac{1}{m}$,*
(b) *$\min\{d'_{e,f}(a, b), d'_{e,f}(c, b)\} \geq \left\lfloor \frac{m}{3} \right\rfloor$,*
(c) *$\max\{d'_{e,f}(a, b), d'_{e,f}(c, b)\} \geq \left\lfloor \frac{2m}{3} \right\rfloor$, and*
(d) *either $w(f) < \frac{3}{m} + \frac{1}{d(ac)}$ or $\max\{d'_{e,f}(a, d), d'_{e,f}(c, d)\} \geq m - 1$.*

Proof. For convenience, let $w_0 = \frac{3}{m}$. By Proposition 2.1, $\sum_{e\in G} w(e) = |\partial(G)|$. So,

$$\frac{1}{e(G)} \sum_{e\in G} w(e) = \frac{|\partial(G)|}{e(G)} < \frac{1}{m/3} = w_0. \tag{2}$$

Hence the average weight of an edge in G is less than w_0. We call an edge $e \in E(G)$ *light* if $w(e) < w_0$ and *heavy* otherwise. A pair $\{x, y\}$ of vertices in G is *good*, if $d(xy) \geq m + 1$.

To find the desired pair of edges e, f we first do some marking of edges. For every light edge e, fix an ordering, say a, b, c, of its vertices so that $d(ab) \leq d(bc) \leq d(ac)$. We call ab, bc, ac the *low, medium, high* sides of e, and might denote them by S_1, S_2, and S_3, respectively.

Since e is light, $w(e) = \sum_{1\leq i\leq 3}(1/d(S_i)) = \frac{1}{d(ab)} + \frac{1}{d(bc)} + \frac{1}{d(ac)} < w_0 = \frac{3}{m}$. It follows that

$$d(ac) > m, \quad d(bc) > \frac{2m}{3}, \quad d(ab) > \frac{m}{3}. \tag{3}$$

In particular, ac is good. We define markings involving e based on three cases.

Case M1: $d(ab) \geq \lfloor m/3 \rfloor + 2$ and $d(bc) \geq \lfloor 2m/3 \rfloor + 2$. In this case, we let e *mark* every edge containing ac apart from itself.

Case M2: $d(ab) \leq \lfloor m/3 \rfloor + 1$. By (3), $d(ab) = \lfloor m/3 \rfloor + 1$. We let e mark all the edges $acx \neq e$ containing ac such that abx is not an edge in G.

Case M3: $d(bc) \leq \lfloor 2m/3 \rfloor + 1$. By (3), $d(bc) = \lfloor 2m/3 \rfloor + 1$. Let e mark all the edges $acx \neq e$ containing ac such that bcx is not an edge in G.

If $m \geq 4$ and both M2 and M3 hold, then

$$\frac{3}{m} < \frac{1}{\lfloor m/3 \rfloor + 1} + \frac{1}{\lfloor 2m/3 \rfloor + 1} = \frac{1}{d(S_1)} + \frac{1}{d(S_2)} < w(e),$$

a contradiction. Thus for $m \geq 4$ each light edge abc satisfies exactly one of M1, M2 and M3.

We perform the above marking procedure for each light edge e. Suppose e marked $p = p(e)$ edges. In each of Cases M1, M2, M3, e marks at least one edge. Indeed, in case M1 we have $p(e) = d(ab) - 1 \geq m$; in case M2 we have $p(e) \geq d(ac) - d(ab) \geq (m+1) - (\lfloor m/3 \rfloor + 1) > 0$; and in case M3 we have $p(e) \geq d(ac) - d(bc) \geq (m+1) - (\lfloor 2m/3 \rfloor + 1) > 0$. So $p > 0$.

Claim 1. *If e is a light edge and f is an edge marked by e then (a)–(c) hold. Further, if f is light, then the lemma holds for (e, f).*

Proof of Claim 1. Suppose $e = abc$, where a, b, c are ordered as described earlier and suppose $f = acd$. Then (a) holds by e being light and by (3). Also (b) holds, since either $d(ab) \geq \lfloor m/3 \rfloor + 2$ or $d(ab) = \lfloor m/3 \rfloor + 1$ and $d'_{e,f}(a, b) = d(ab) - 1$ (because $abd \notin G$ by M2). Similarly, (c) holds, since either $d(bc) \geq \lfloor 2m/3 \rfloor + 2$ or $d(bc) = \lfloor 2m/3 \rfloor + 1$ and $d'_{e,f}(b, c) = d(bc) - 1$ (because $bcd \notin G$ by M3). Now, if f is also a light edge then (d) holds since $w(f) < \frac{3}{m}$. □

By Claim 1, we may henceforth assume that every marked edge is heavy. We will now use a discharging procedure to find our pair (e, f). Let the initial charge $ch(e)$ of every edge e in G equal to $w(e)$. Then $\sum_{e \in G} ch(e) = \sum_{e \in G} w(e) = |\partial(G)|$. We will redistribute charges among the edges of G so that the total sum of charges does not change and the resulting charge of each heavy edge remains at least w_0.

The discharging rule is as follows. Suppose a heavy edge f was marked by exactly $q = q(f)$ light edges. If $q = 0$, then let the new charge $ch^*(f)$ equal $ch(f)$. Otherwise, let f transfer to each light edge e that marks it a charge of $(ch(f) - w_0)/q$ so that $ch^*(f) = w_0$. The total charge does not change in this discharging process. Hence, by (2), there is an edge e with $ch^*(e) < w_0$. By our discharging rule, e must be a light edge.

Among all the p edges e marked, let f be one that gave the least charge to e. By definition, f gave e a charge of at most $(ch^*(e) - ch(e))/p < (w_0 - ch(e))/p$. We claim that the pair (e, f) satisfies the lemma. Suppose $e = abc$, where a, b, c are ordered as before, and suppose $f = acd$. By Claim 1, (a), (b), and (c) hold. It remains to prove (d). If all three pairs in f are good, then $w(f) < \frac{3}{m}$, contradicting f being heavy. So, at most two of the pairs in f are good. By our earlier discussion, ac is good. If one of ad and cd is also good, then the second part of (d) holds. So we may assume that ac is the only good pair in f. Let q be the number of the light edges that marked f. By the marking process, a light edge only marks edges containing its high side and the high side is a good pair. Since ac is the only good pair in f, each of the q light edges that marked f contains ac and has ac as its high side.

First, suppose that Case M1 was applied to e. Then all the edges containing ac other than e were marked, which by our assumption must be heavy. In particular, this implies that $q = 1$. By our rule, f gave e a charge of $ch(f) - w_0$. By our choice of f, each of the $p(e) = d(ac) - 1$ ($\geq m$) edges of G containing ac (other than e) gave e a charge of at least $ch(f) - w_0$. Hence, $w_0 > ch^*(e) \geq ch(e) + p(ch(f) - w_0)$, so

$$q(f) \times (w_0 - w(e)) > p(e) \times (w(f) - w_0). \tag{4}$$

Using $q(f) = 1$, $p(e) = d(ac) - 1 > 1$, $w_0 = 3/m$, we get

$$\frac{q(f)}{p(e)}(w_0 - w(e)) \leq \frac{1}{d(ac) - 1} \cdot \frac{3}{m} \leq \frac{1}{d(ac)},$$

from which the first part of (d) follows.

Note that (4) holds in cases M2 and M3, too.

Suppose that Case M2 was applied to e. Then $d(ab) = \lfloor m/3 \rfloor + 1$. If $q > \lfloor m/3 \rfloor + 1$, then one of light edges containing ac, say acx, satisfies that $abx \notin G$. By M2, e marked acx, contradicting our assumption that no light edge was marked. So $q \leq \lfloor m/3 \rfloor + 1$. We obtain

$$\begin{aligned} \frac{p(e)}{d(ac)} &\geq \frac{d(ac) - d(ab)}{d(ac)} = 1 - \frac{d(ab)}{d(ac)} \geq \frac{3(\lfloor m/3 \rfloor + 1) - m}{m} - \frac{d(ab)}{d(ac)} \\ &= d(ab)\left(\frac{3}{m} - \frac{1}{d(ab)} - \frac{1}{d(ac)}\right) > d(ab)\,(w_0 - w(e)) \geq q(f)\,(w_0 - w(e))\,. \end{aligned}$$

This and (4) imply the first part of (d).

Similarly if Case M3 was applied to e then $q \leq \lfloor 2m/3 \rfloor + 1 = d(bc)$. We obtain

$$\begin{aligned} \frac{p(e)}{d(ac)} &\geq \frac{d(ac) - d(bc)}{d(ac)} = 1 - \frac{d(bc)}{d(ac)} \geq \frac{3(\lfloor 2m/3 \rfloor + 1) - 2m}{m} - \frac{d(bc)}{d(ac)} \\ &= d(bc)\left(\frac{3}{m} - \frac{2}{d(bc)} - \frac{1}{d(ac)}\right) \geq d(bc)\,(w_0 - w(e)) \geq q(f)\,(w_0 - w(e))\,. \end{aligned}$$

This and (4) imply the first part of (d). □

Our next aim is to state a Proposition about how one can find a tight tree T in a 3-graph G considering only the degrees of the pairs in $\partial_2(G)$. We will do it by defining an embedding of T into G. An *embedding* of an r-graph H into an r-graph G is an injection $\phi : V(H) \to V(G)$ such that for each $e \in E(H)$, $\phi(e) \in E(G)$.

Consider a tight 3-tree T of size $t = m + 1$ with a trunk $\{e_1, e_2\}$. Suppose $e_1 = xyu$ and $e_2 = xyv$ so that $e_1 \cap e_2 = xy$. By our assumption, each edge in $E(T) \setminus \{e_1, e_2\}$ contains a pair in e_1 or e_2 and a vertex outside $e_1 \cup e_2$. For each pair A contained in e_1 or e_2, let $N'_T(A) = N_T(A) \setminus \{x, y, u, v\}$ and $\mu(A) = |N'_T(A)|$. Then $\mu(xy) = d_T(xy) - 2$, and $\mu(A) = d_T(A) - 1$ for each $A \in \{xu, xv, yu, yv\}$. By definition,

$$\mu(xy) + \mu(xu) + \mu(xv) + \mu(yu) + \mu(yv) = t - 2 = m - 1. \tag{5}$$

Suppose that G is a 3-graph, $e, f \in E(G)$, $e = acb$ and $f = acd$, so that $e \cap f = ac$. For each pair B contained in e or f, let $N'_G(B) = N_G(B) \setminus \{a, b, c, d\}$ and $d'_G(B) = |N'_G(B)|$. (In fact, $d'_G(B)$ is the same as $d'_{e,f}(B)$ but here we want to distinguish the two hypergraphs we consider).

Proposition 3.1. *Suppose there is a permutation $A_1, \dots, A_5$ of the five pairs contained in xyu or xyv, and another permutation $B_1, \dots, B_5$ of the five pairs contained in abc or abd, and a bijection $\phi : \{x, y, u, v\} \to \{a, b, c, d\}$ such that $\phi(A_\ell) = \phi(B_\ell)$ for each $1 \le \ell \le 5$, $\phi(\{x, y\}) = \{a, c\}$, and*

$$\sum_{1\le i\le \ell} \mu(A_i) \le d'_G(B_\ell) \ \text{ for } 1 \le \ell \le 5. \tag{6}$$

Then ϕ can be extended to an embedding of T into G.

Proof of Proposition 3.1. We can embed T into G as follows. Note that ϕ maps e_1 and e_2 to $\{e, f\}$. Then we map $N'_T(A_1)$ into $N'_G(B_1)$ followed by $N'_T(A_2)$ into $N'_G(B_2) \setminus \phi(N'_T(A_1))$. The five inequalities of (6) ensure that we can map $N'_T(A_\ell)$ into $N'_G(B_\ell) \setminus \phi(N'_T(A_1) \cup \cdots \cup N'_T(A_{\ell-1}))$. □

The next lemma proves a special case of Theorem 1.4.

Lemma 3.2. *Let T be a tight 3-tree with $t \ge 5$ edges. Suppose T has a trunk $\{e_1, e_2\}$ of size 2 such that $d_T(e_1 \cap e_2) \ge \lfloor \frac{t-1}{3} \rfloor + 2$. Let G be an n-vertex 3-graph that does not contain T. Then $e(G) \le \frac{t-1}{3}|\partial(G)|$.*

Proof of Lemma 3.2. For convenience, let $m = t - 1$. Let G be a 3-graph with $e(G) > \frac{m}{3}|\partial(G)|$. Then G contains a subgraph G' such that $e(G') > \frac{m}{3}|\partial(G')|$ and $\delta_2(G') > \frac{m}{3}$. For convenience, we assume G itself satisfies these two conditions. Let w be the default weight function on $E(G)$ and $\partial(G)$. Then G satisfies the conditions of Lemma 3.1. Let the edges $e = abc$ and $f = adc$ satisfy the claim of that lemma, where a, b, c are ordered as in Lemma 3.1. In particular, by (a), e is light and ac is good, i.e., $d(ac) \ge m + 1$. By our assumptions, $d(ab) \le d(bc)$. By parts (b) and (c),

$$d'_{e,f}(a, b) \ge \left\lfloor \frac{m}{3} \right\rfloor \quad \text{and} \quad d'_{e,f}(c, b) \ge \left\lfloor \frac{2m}{3} \right\rfloor. \tag{7}$$

We rename pairs $\{a, d\}$ and $\{c, d\}$ as D_1 and D_2 so that $d'_{e,f}(D_1) = \min\{d'_{e,f}(a, d), d'_{e,f}(c, d)\}$ and $d'_{e,f}(D_2) = \max\{d'_{e,f}(a, d), d'_{e,f}(c, d)\}$. We claim that in these terms,

$$d'_1 := d'_{e,f}(D_1) \ge \left\lfloor \frac{m}{3} \right\rfloor - 1 \quad \text{and} \quad d'_2 := d'_{e,f}(D_2) \ge \left\lfloor \frac{2m}{3} \right\rfloor - 1. \tag{8}$$

By (1) and the fact that $\delta_2(G) > \frac{m}{3}$, $d'_1, d'_2 \ge \lfloor \frac{m}{3} \rfloor - 1$. We will use part (d) of Lemma 3.1 to show the lower bound for d'_2. If the second part of (d) holds, then $d'_2 \ge m - 1$ and we are done. So suppose the first part of Lemma 3.1(d) holds instead, i.e., $(1/d(D_1)) + 1/(d(D_2)) < 3/m$. Then $\max\{d(D_1), d(D_2)\} > 2m/3$ implies $\max\{d'_1, d'_2\} \ge \lfloor \frac{2m}{3} \rfloor - 1$. Note that in (8) we cannot have equalities in both lower bounds simultaneously, because that one leads to the contradiction

$$\frac{1}{\lfloor m/3 \rfloor + 1} + \frac{1}{\lfloor 2m/3 \rfloor + 1} \le w(f) - \frac{1}{d(ac)} < \frac{3}{m}.$$

By our assumption, T has a trunk $\{e_1, e_2\}$ with $d_T(e_1 \cap e_2) \ge \lfloor \frac{m}{3} \rfloor + 2$. Suppose $e_1 = xyu$ and $e_2 = xyv$ so that $e_1 \cap e_2 = xy$. Recall that for each pair B contained in e_1 or e_2, $N'_T(B) = N_T(B) \setminus \{x, y, u, v\}$ and $\mu(B) = |N'_T(B)|$. Then $\mu(xy) = d_T(xy) - 2$, and $\mu(B) = d_T(B) - 1$ for each $B \in \{xu, xv, yu, yv\}$. Since $\mu(xy) = d_T(xy) - 2 \ge \lfloor \frac{m}{3} \rfloor > \frac{m}{3} - 1$, Eq. (5) implies

$$\mu(xu) + \mu(xv) + \mu(yu) + \mu(yv) < \frac{2m}{3}. \tag{9}$$

We consider four cases, and in each case we find an embedding of T into G using Proposition 3.1. We take $A_5 = xy$ (so $B_5 = ac$) so the case $\ell = 5$ always holds in condition (6) by $d_G(ac) \ge m + 1$.

Case 1.1. $d'_{e,f}(D_2) \ge \left\lfloor \frac{2m}{3} \right\rfloor$ and $D_2 = cd$. By symmetry we may assume that $\mu(yu) + \mu(yv) \ge \mu(xu) + \mu(xv)$ and that $\mu(xu) \ge \mu(xv)$. Then by (9), $\mu(xv) + \mu(xu) \le \left\lfloor \frac{m}{3} \right\rfloor$, and $\mu(xv) \le \left\lfloor \frac{m}{6} \right\rfloor \le \left\lfloor \frac{m}{3} \right\rfloor - 1$.

We embed T into G by mapping x, y, u, v to a, c, b, d, respectively, and define $A_1 = xv$ ($B_1 = ad = D_1$), $A_2 = xu$ ($B_2 = ab$), $\{A_3, A_4\} = \{yv, yu\}$ ($\{B_3, B_4\} = \{cd, bc\}$). Then the first parts of (8) and (7) imply that condition (6) holds for $\ell = 1, 2$ and the second part of (7), (i.e., $d'_{e,f}(c, b) \geq \lfloor \frac{2m}{3} \rfloor$) and our constraint in this case ($d'_{e,f}(c, d) \geq \lfloor \frac{2m}{3} \rfloor$) imply that (6) holds for $\ell = 3, 4$, too.

Case 1.2. $d'_{e,f}(D_2) \geq \lfloor \frac{2m}{3} \rfloor$ and $D_2 = ad$. By symmetry we may assume that $\mu(yu) + \mu(xv) \geq \mu(xu) + \mu(yv)$ and that $\mu(xu) \geq \mu(yv)$. Then by (9), $\mu(xu) + \mu(yv) \leq \lfloor \frac{m}{3} \rfloor$, and $\mu(yv) \leq \lfloor \frac{m}{6} \rfloor \leq \lfloor \frac{m}{3} \rfloor - 1$.

We embed T into G by mapping x, y, u, v to a, c, b, d, respectively, and define $A_1 = yv$ ($B_1 = cd = D_1$), $A_2 = xu$ ($B_2 = ab$), $\{A_3, A_4\} = \{xv, yu\}$ ($\{B_3, B_4\} = \{ad, bc\}$). Then we have the same upper bounds for $\sum_{1 \leq i \leq \ell} \mu(A_i)$ and the same lower bounds for $d'(B_\ell)$ as in Case 1.1 so (6) holds for each ℓ.

From now on, we may suppose that $d'_{e,f}(D_2) \leq \lfloor \frac{2m}{3} \rfloor - 1$. Then the inequality (8) and the note after it imply that $d'_{e,f}(D_2) = \lfloor \frac{2m}{3} \rfloor - 1$ and $d'_1 := d'_{e,f}(D_1) \geq \lfloor \frac{m}{3} \rfloor$.

Case 2.1. $d'_{e,f}(D_1) \geq \lfloor \frac{m}{3} \rfloor$ and $D_1 = ad$. By symmetry we may assume that $\mu(yu) + \mu(yv) \geq \mu(xu) + \mu(xv)$ and that $\mu(yu) \geq \mu(yv)$. Then by (9) $\mu(xv) + \mu(xu) \leq \lfloor \frac{m}{3} \rfloor$, and $\mu(yu) \geq 1$ so $\mu(xv) + \mu(xu) + \mu(yv) \leq \lfloor \frac{2m}{3} \rfloor - 1$.

We embed T into G by mapping x, y, u, v to a, c, b, d, respectively, and define $\{A_1, A_2\} = \{xv, xu\}$ ($\{B_1, B_2\} = \{ad, ab\}$) $A_3 = yv$ ($B_3 = cd = D_2$), and $A_4 = yu$ ($B_4 = bc$). Then the first part of (7) and our constraint in this case ($d'_{e,f}(ad) \geq \lfloor \frac{m}{3} \rfloor$) imply that condition (6) holds for $\ell = 1, 2$. The condition $d'_{e,f}(D_2) = \lfloor \frac{2m}{3} \rfloor - 1$ and the second part of (7), (i.e., $d'_{e,f}(c, b) \geq \lfloor \frac{2m}{3} \rfloor$) imply that (6) holds for $\ell = 3, 4$, too.

Case 2.2. $d'_{e,f}(D_1) \geq \lfloor \frac{m}{3} \rfloor$ and $D_1 = cd$. By symmetry we may assume that $\mu(yu) + \mu(xv) \geq \mu(xu) + \mu(yv)$ and that $\mu(yu) \geq \mu(xv)$. Then by (9), $\mu(xu) + \mu(yv) \leq \lfloor \frac{m}{3} \rfloor$, and $\mu(yu) \geq 1$ so $\mu(xu) + \mu(yv) + \mu(xv) \leq \lfloor \frac{2m}{3} \rfloor - 1$.

We embed T into G by mapping x, y, u, v to a, c, b, d, respectively, and define $\{A_1, A_2\} = \{xu, yv\}$ ($\{B_1, B_2\} = \{ab, cd\}$) $A_3 = xv$ ($B_3 = ad$), and $A_4 = yu$ ($B_4 = bc$). Then we have the same upper bounds for $\sum_{1 \leq i \leq \ell} \mu(A_i)$ and the same lower bounds for $d'(B_\ell)$ as in Case 2.1 so (6) holds for each ℓ. This completes the proof of Lemma 3.2. □

4. Proof of Theorem 1.4

We prove a shadow version of Theorem 1.4, which immediately implies Theorem 1.4.

Theorem 1.4′. *Let $t \geq 8$ be an integer. Let T be a tight 3-tree with t edges and $c(T) \leq 2$. If G is an r-graph that does not contain T, then $e(G) \leq \frac{t-1}{3}|\partial(G)|$.*

For an edge e, we denote by $d_{\min}(e)$ the *minimum codegree* over all three pairs of vertices in e.

Lemma 4.1. *Let G be a 3-graph satisfying $e(G) > \gamma|\partial(G)|$. Let w be the default weight function on $E(G)$ and $\partial(G)$. Then there exists a pair of edges e, f with $|e \cap f| = 2$ such that*

1. $w(e) < \frac{1}{\gamma}$,
2. $|e \cap f| = d_{\min}(e)$,
3. $w(f) < \frac{1}{\gamma} + \frac{3}{d_{\min}(e)-1}(\frac{1}{\gamma} - w(e))$.

Proof of Lemma 4.1. For convenience, let $w_0 = \frac{1}{\gamma}$. As in the proof of Lemma 3.1, call an edge e with $w(e) < w_0$ *light* and an edge e with $w(e) \geq w_0$ *heavy*. As before, the average of $w(e)$ over all e is $|\partial(G)|/e(G) < w_0$. For each light edge e, let us mark a pair of vertices in it of codegree $d_{\min}(e)$. If e is a light edge with a marked pair xy and f is another light edge containing xy, then our statements already hold. So we assume that no marked pair of any light edge lies in another light edge. Let us initially assign a charge of $w(e)$ to each edge e in G. Then the average charge of an edge in G is less than w_0. We now apply the following discharging rule. For each heavy edge f, transfer $\frac{1}{3}(w(f) - w_0)$ of the charge to each light edge e whose marked pair is contained in f. Note that for each f there are at most 3 such e. In particular, a heavy edge f still has charge at least w_0 after the discharging.

Since discharging does not change the total charge, there exists some edge e with charge less than w_0. By the previous sentence, e is a light edge in G. Let xy be its marked pair. There are $d_{\min}(e) - 1$ other edges containing it, each of which is heavy. Each such edge f has given a charge of $\frac{1}{3}(w(f) - w_0)$ to w_0. For e to still have a charge less than w_0, one of these edges f satisfies $\frac{1}{3}(w(f) - w_0) < \frac{w_0 - w(e)}{d_{\min}(e)-1}$. Hence $w(f) < \frac{1}{\gamma} + \frac{3}{d_{\min}(e)-1}(\frac{1}{\gamma} - w(e))$. □

To make the application of Lemma 4.1 smoother we collect all calculations in one statement.

Proposition 4.1. *Suppose that R_1, R_2, d, m are integers, $R_2 \geq R_1 \geq 2$, and $d \geq 2$ satisfying*

$$\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{d} < \frac{3}{m} + \frac{3}{d-1}\left(\frac{3}{m} - \frac{1}{d}\right). \tag{10}$$

If $m \geq 7$ then

$$R_2 \geq \left\lfloor \frac{2m}{3} \right\rfloor + 1, \qquad \max\{d, R_1\} \geq \left\lfloor \frac{m}{2} \right\rfloor + 1. \tag{11}$$

Moreover,

$$\frac{1}{R_1}+\frac{1}{m}+\frac{1}{d}<\frac{3}{m}+\frac{3}{d-1}\left(\frac{3}{m}-\frac{1}{m}-\frac{1}{d}\right), \tag{12}$$

$m \geq 7$, and $d \leq m$ imply

$$R_1 \geq \left\lfloor \frac{2m}{3} \right\rfloor + 1. \tag{13}$$

Finally,

$$\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{d}<\frac{3}{m}+\frac{3}{d-1}\left(\frac{3}{m}-\frac{1}{m}-\frac{1}{d}\right), \tag{14}$$

$m \geq 7$, and $d \leq m$ imply

$$R_2 \geq \left\lfloor \frac{5m}{6} \right\rfloor + 1. \tag{15}$$

Proof of Proposition 4.1. Standard algebra and calculus. To prove the first part of (11) define

$$h(d) := \frac{3}{d-1}\left(\frac{3}{m}-\frac{1}{d}\right)-\frac{1}{d}=\frac{9d-2m-md}{m(d^2-d)},$$

where we consider d to be a real variable, $d > 1$. Then (10) is equivalent to

$$\frac{1}{R_1}+\frac{1}{R_2}<\frac{3}{m}+h(d).$$

This and $R_2 \geq R_1 > 0$ imply $2/R_2 < (3/m) + h(d)$. For $m \geq 9$ and $d > 1$ we have $h(d) < 0$, so we get $2/R_2 < 3/m$, i.e., $R_2 > 2m/3$, as required. In the case of $m = 8$ we have $h(d) < 1/40$ for all $d > 1$ (it is equivalent to $0 < d^2 - 6d + 80$) so we get $2/R_2 < (3/8) + (1/40) = 2/5$, as required. In the case of $m = 7$ we can prove that $h(d) < 1/14$ for all $d > 1$ (it is equivalent to $0 < d^2 - 5d + 28$) so we get $2/R_2 < (3/4) + (1/14) = 2/4$ and we are done.

To prove the second part of (11) we calculate the derivative of $h(d)$

$$\frac{\partial}{\partial d}h(d)=\frac{(m-9)d^2+(4d-2)m}{m(d^2-d)^2}.$$

For $m \geq 9$ this is obviously positive for all $d > 1$. In the case $m = 8$ and $1 < d \leq 8$ both the numerator $(-d^2 + 32d - 16)$ and the denominator are positive. Similarly, for the case $m = 7$ and $1 < d \leq 7$ we get that the numerator $(-2d^2+28d-14)$ is positive. Hence $h(d)$ is strictly increasing for $1 < d \leq m$. So $h(d)$ takes its maximum in the interval $(1, \lfloor m/2 \rfloor]$ at the upper end. A simple calculation shows that $h(\lfloor m/2 \rfloor) \leq -1/m$. Suppose that $d \leq \lfloor m/2 \rfloor$. Then (12) and the monotonicity of h imply

$$\frac{1}{R_1}<\frac{1}{R_1}+\frac{1}{R_2}<\frac{3}{m}+h(d)\leq\frac{3}{m}+h\left(\left\lfloor\frac{m}{2}\right\rfloor\right)\leq\frac{2}{m},$$

and we are done.

To prove (13) define

$$h_2(d) := \frac{3}{d-1}\left(\frac{2}{m}-\frac{1}{d}\right)-\frac{1}{d}=\frac{6d-2m-dm}{m(d^2-d)},$$

where we consider d to be a real variable, $d > 1$. Then (12) is equivalent to

$$\frac{1}{R_1}<\frac{2}{m}+h_2(d).$$

Again calculus shows that $h_2(d)$ is strictly increasing for $1 < d \leq m$. So it takes its maximum in the interval $(1, m]$ at the upper end. We obtain $1/R_1 < (2/m) + h_2(m) = (m + 2)/(m^2 - m)$. This implies $R_1 > (m^2 - m)/(m + 2) > \lfloor 2m/3 \rfloor$ (for $m \geq 7$), as required.

Finally, to prove (15), the monotonicity of h_2 and (14) give

$$\frac{2}{R_2}\leq\frac{1}{R_1}+\frac{1}{R_2}<\frac{3}{m}+h_2(m)=\frac{2m+1}{m^2-m}.$$

This implies $R_2 > 2(m^2 - m)/(2m + 1) > \lfloor 5m/6 \rfloor$ (for $m \geq 7$) and we are done. □

Proof of Theorem 1.4′. Let T be a tight 3-tree with $t \geq 8$ edges that contains a trunk $\{e_1, e_2\}$ of size 2. For convenience, let $m = t - 1$. Suppose $e_1 = xyu$ and $e_2 = xyv$, so that $e_1 \cap e_2 = xy$. If $d_T(xy) \geq \lfloor \frac{m}{3} \rfloor + 2$, then we apply Lemma 3.2 and are done. Hence we may assume that

$$d_T(xy) \leq \left\lfloor \frac{m}{3} \right\rfloor + 1.$$

For each pair A contained in e_1 or e_2, let $N'_T(A) = N_T(A) \setminus \{x, y, u, v\}$ and let $\mu(A) = |N'_T(A)|$. Then $\mu(xy) = d_T(xy) - 2$ and $\mu(A) = d_T(A) - 1$ for the other pairs. Also, as we have seen in (5), we have

$$\mu(xu) + \mu(yu) + \mu(xv) + \mu(yv) + \mu(xy) = m - 1. \tag{16}$$

Since $\mu(xy) = d_T(xy) - 2 \leq \frac{m}{3} - 1$,

$$\mu(xy) + \frac{i}{4}(m - 1 - \mu(xy)) \leq \frac{m}{3} + \frac{im}{6} - 1 \qquad \forall i : 0 \leq i \leq 4. \tag{17}$$

Let G be a 3-graph with $e(G) > \frac{m}{3}|\partial(G)|$. We prove that G contains T. As before we may assume that $\delta_2(G) > \frac{m}{3}$. Let w be the default weight function on $E(G)$ and $\partial(G)$. There exist edges e and f in G such that they satisfy the properties proven in Lemma 4.1 with $\gamma = 3/m$, i.e, $d(e \cap f) = d_{\min}(e)$, $w(e) < \frac{3}{m}$, and so on. Suppose $e = acb$ and $f = acd$, so that $e \cap f = ac$. For each pair B contained in e or f, let $N'_G(B) = N_G(B) \setminus \{a, b, c, d\}$ and $d'_G(B) = |N'_G(B)|$. Then $d'_G(B) \geq d_G(B) - 2$ and for all B

$$d'(B) \geq \left\lfloor \frac{m}{3} \right\rfloor - 1. \tag{18}$$

Let us view e, f as glued together at ac with e on the left and f on the right. Let

$$L_{\min} = \min\{d_G(ab), d_G(bc)\}, \qquad R_{\min} = \min\{d_G(ad), d_G(cd)\},$$
$$L_{\max} = \max\{d_G(ab), d_G(bc)\}, \qquad R_{\max} = \max\{d_G(ad), d_G(cd)\}.$$

Since $d(ac) = d_{\min}(e)$, $L_{\max} \geq L_{\min} \geq d_G(ac)$. Since $w(e) < \frac{3}{m}$, we have

$$L_{\max} > m. \tag{19}$$

Lemma 4.1 gives

$$\frac{1}{R_{\min}} + \frac{1}{R_{\max}} + \frac{1}{d} = w(f) < \frac{3}{m} + \frac{3}{d-1}\left(\frac{3}{m} - w(e)\right) \leq \frac{3}{m} + \frac{3}{d-1}\left(\frac{3}{m} - \frac{1}{d}\right), \tag{20}$$

where $d = d(ac)$.

We consider three cases. In each case, we find an embedding of T into G.

Case 1. $L_{\min} > m$. This implies $d'_G(ab), d'_G(bc) \geq m - 1$. By symmetry, we may assume that $d_G(ad) \geq d_G(cd)$ so that $d_G(ad) = R_{\max}$ and $d_G(cd) = R_{\min}$. The inequality (20) shows that the condition (10) holds in Proposition 4.1 with $R_1 = R_{\min}$ and $R_2 = R_{\max}$. So (11) (and the lower bound (18)) give

$$d'_G(bc),\ d'_G(cd) \geq \left\lfloor \frac{m}{3} \right\rfloor - 1, \qquad \max\{d'_G(bc), d'_G(cd)\} \geq \left\lfloor \frac{m}{2} \right\rfloor - 1,$$
$$d'_G(ad) \geq \left\lfloor \frac{2m}{3} \right\rfloor - 1, \qquad d'_G(ab),\ d'_G(bc) \geq m - 1. \tag{21}$$

Now, consider T. By symmetry, we may assume that $\mu(xu) + \mu(yu) \geq \mu(xv) + \mu(yv)$ and that $\mu(xv) \geq \mu(yv)$. Then $\mu(yv) \leq \frac{1}{4}(m - 1 - \mu(xy))$. This, together with (17) implies

$$\mu(xy) \leq \left\lfloor \frac{m}{3} \right\rfloor - 1, \qquad \mu(yv) \leq \left\lfloor \frac{m-1}{4} \right\rfloor \leq \left\lfloor \frac{m}{3} \right\rfloor - 1,$$
$$\mu(xy) + \mu(yv) \leq \left\lfloor \frac{m}{2} \right\rfloor - 1, \qquad \mu(xy) + \mu(xv) + \mu(yv) \leq \left\lfloor \frac{2m}{3} \right\rfloor - 1. \tag{22}$$

Now we embed T into G by applying Proposition 3.1. First, we map x, y, u, v to a, c, b, d, respectively. This maps e_1 to e and e_2 to f. Then define $\{A_1, A_2\} = \{xy, yv\}$ (so $\{B_1, B_2\} = \{bc, cd\}$), $A_3 = xv$ (so $B_3 = ad$) and $\{A_4, A_5\} = \{ux, uy\}$ (so $\{B_4, B_5\} = \{bc, ab\}$). If we define A_2 so that $d'(B_1) \leq d'(B_2)$ then (6) holds, so this mapping can be extended to an embedding of T.

From now on, we may suppose that $L_{\min} \leq m$. By symmetry, we may assume that $d_G(ab) \geq d_G(bc)$ so that $d_G(ab) = L_{\max}$ and $d_G(bc) = L_{\min}$. Since $d(ac) = d_{\min}(e)$, $d_G(ac) \leq L_{\min} \leq m$. Since $\frac{1}{L_{\min}} + \frac{1}{d_G(ac)} < w(e) < \frac{3}{m}$ we get $\frac{1}{d_G(ac)} < \frac{2}{m}$. We also get $\frac{2}{L_{\min}} < \frac{3}{m}$. Summarizing (as in (3)) we have

$$d'_G(ab) \geq m - 1, \quad d'_G(bc) \geq \left\lfloor \frac{2m}{3} \right\rfloor - 1, \quad d'_G(ac) \geq \left\lfloor \frac{m}{2} \right\rfloor - 1. \tag{23}$$

Case 2. $L_{\min} \le m$ and $R_{\max} > m$. There will be two subcases. First suppose that $d_G(ad) \ge d_G(cd)$ (i.e., $d_G(ad) = R_{\max}$). Then, besides (23), from (18) we get

$$d'_G(ad) \ge m-1, \quad d'_G(cd) \ge \left\lfloor \frac{m}{3} \right\rfloor - 1. \tag{24}$$

Consider T. By symmetry, we may assume that $\mu(xu)+\mu(xv) \ge \mu(yu)+\mu(yv)$ and that $\mu(yu) \ge \mu(yv)$. Then by these assumptions and (17), we have

$$\mu(yv) \le \left\lfloor \frac{m-1}{4} \right\rfloor, \quad \mu(xy)+\mu(yv) \le \left\lfloor \frac{m}{2} \right\rfloor - 1, \quad \mu(xy)+\mu(yv)+\mu(yu) \le \left\lfloor \frac{2m}{3} \right\rfloor - 1. \tag{25}$$

Now we embed T into G by applying Proposition 3.1. First, we map x, y, u, v to a, c, b, d, respectively. Then define $A_1 = yv$ (so $B_1 = cd$), $A_2 = xy$ (so $B_2 = ac$), $A_3 = yu$ (so $B_3 = bc$) and $\{A_4, A_5\} = \{xu, xv\}$ (so $\{B_4, B_5\} = \{ab, ad\}$). So (6) holds and this mapping can be extended to an embedding of T.

Next, suppose that $d_G(cd) \ge d_G(ad)$. Then, instead of (24), we have

$$d'_G(cd) \ge m-1, \quad d'_G(ad) \ge \left\lfloor \frac{m}{3} \right\rfloor - 1. \tag{26}$$

Consider T. By symmetry, we may assume that $\mu(xu)+\mu(yv) \ge \mu(xv)+\mu(yu)$ and that $\mu(yu) \ge \mu(xv)$. By these assumptions and (17), we have

$$\mu(xv) \le \left\lfloor \frac{m-1}{4} \right\rfloor, \quad \mu(xy)+\mu(xv) \le \left\lfloor \frac{m}{2} \right\rfloor - 1, \quad \mu(xy)+\mu(xv)+\mu(yu) \le \left\lfloor \frac{2m}{3} \right\rfloor - 1. \tag{27}$$

Now we embed T into G as follows. First, we map x, y, u, v to a, c, b, d, respectively. Then define $A_1 = xv$ (so $B_1 = ad$), $A_2 = xy$ (so $B_2 = ac$), $A_3 = yu$ (so $B_3 = bc$) and $\{A_4, A_5\} = \{xu, yv\}$ (so $\{B_4, B_5\} = \{ab, cd\}$). So (6) holds by (23), (26) and (27) and this mapping can be extended to an embedding of T.

Case 3. $L_{\min} \le m$ and $R_{\max} \le m$.

Let $d := d_{\min}$, we have $d \le L_{\min} \le m$ so $w(e) \ge (1/m) + (1/d)$. Then Lemma 4.1 gives

$$\frac{1}{R_1} + \frac{1}{m} + \frac{1}{d} = w(F) < \frac{3}{m} + \frac{3}{d-1}\left(\frac{3}{m} - w(e)\right) < \frac{3}{m} + \frac{3}{d-1}\left(\frac{3}{m} - \frac{1}{m} - \frac{1}{d}\right).$$

Then Proposition 4.1 (13) yields a lower bound for $R_{\min}$. Similarly, from (15) we get a lower bound for $R_{\max}$, i.e.,

$$R_{\min} \ge \left\lfloor \frac{2m}{3} \right\rfloor + 1, \qquad R_{\max} \ge \left\lfloor \frac{5m}{6} \right\rfloor + 1. \tag{28}$$

There will be two subcases. First, suppose that $d_G(ad) \ge d_G(cd)$, (i.e., $d_G(ad) = R_{\max}$). Then, besides (23), from (28) we get

$$d'_G(ad) \ge \left\lfloor \frac{5m}{6} \right\rfloor - 1, \quad d'_G(cd) \ge \left\lfloor \frac{2m}{3} \right\rfloor - 1. \tag{29}$$

Consider T. By symmetry, we may assume that $\mu(xu)+\mu(xv) \ge \mu(yu)+\mu(yv)$ and that $\mu(xu) \ge \mu(xv)$. In particular,

$$\mu(xu) \ge \frac{1}{4}(m-1-\mu(xy)) \ge \frac{1}{4}\left(m-1-\frac{m}{3}+1\right) = \frac{m}{6}$$

and

$$\mu(xu)+\mu(xv) \ge \frac{1}{2}(m-1-\mu(xy)) \ge \frac{1}{4}\left(m-1-\frac{m}{3}+1\right) = \frac{m}{3}$$

These, together with $\mu(xy) \le (m/3) - 1$, give

$$\mu(xy) \le \left\lfloor \frac{m}{3} \right\rfloor - 1, \qquad \mu(xy)+\mu(yu)+\mu(yv) \le \left\lfloor \frac{2m}{3} \right\rfloor - 1,$$
$$\mu(xv)+\mu(xy)+\mu(yu)+\mu(yv) \le \left\lfloor \frac{5m}{6} \right\rfloor - 1. \tag{30}$$

By (23), (29), and (30), we can greedily embed T into G applying Proposition 3.1 by mapping x, y, u, v to a, c, b, d, respectively, and mapping in order $A_1 = xy$ to $B_1 = ac$, $\{A_2, A_3\} = \{yu, yv\}$ to $\{B_2, B_3\} = \{bc, cd\}$ (in arbitrary order) $A_4 = xv$ to $B_4 = ad$, and $A_5 = xu$ to $B_5 = ab$.

Next, suppose that $d_G(cd) \ge d_G(ad)$. Then, instead of (29), we have $d'(cd) \ge \left\lfloor \frac{5m}{6} \right\rfloor - 1$ and $d'(ad) \ge \left\lfloor \frac{2m}{3} \right\rfloor - 1$. By symmetry, we may assume that $\mu(xu)+\mu(yv) \ge \mu(xv)+\mu(yu)$ and that $\mu(xu) \ge \mu(yv)$. Again, $\mu(xu) \ge m/6$ holds. We can greedily embed T into G applying Proposition 3.1 by mapping x, y, u, v to a, c, b, d, respectively, and mapping $A_1 = xy$ to $B_1 = ac$, $\{A_2, A_3\} = \{yu, xv\}$ to $\{B_2, B_3\} = \{bc, ad\}$ (in arbitrary order), $A_4 = yv$ to $B_4 = cd$, and $A_5 = xu$ to $B_5 = ab$. $\square$

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