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Zoltán Füredi & András Gyárfás

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An Extension of Mantel's Theorem to k-Graphs

Zoltán Füredi and András Gyárfás

Abstract. According to Mantel's theorem, a triangle-free graph on n points has at most $n^2/4$ edges. A *linear k-graph* is a set of points together with some k-element subsets, called edges, such that any two edges intersect in at most one point. The k-graph F^k, called a *fan*, consists of k edges that pairwise intersect in exactly one point v, plus one more edge intersecting each of these edges in a point different from v. We extend Mantel's theorem as follows: fan-free linear k-graphs on n points have at most n^2/k^2 edges.

This extension nicely illustrates the difficulties of hypergraph Turán problems. The determination of the case of equality leads to transversal designs on n points with k groups—for $k = 3$ these are equivalent to Latin squares. However, in contrast to the graph case, new structures and open problems emerge when n is not divisible by k.

1. TRIANGLES AND FANS. Once upon a time, 111 years ago, Mantel proposed a problem in a Dutch journal. He (followed by four other solvers) provided a solution and the outcome is known in graph theory as Mantel's theorem. Let K_p denote the complete graph on $p \geq 2$ points; K_3 is often called a triangle. Graphs without a K_p subgraph are called K_p-free.

Theorem 1 (Mantel [4]). *Triangle-free graphs on n points have at most $n^2/4$ edges.*

It is not easy to select the winner of the beauty contest for the nicest proof of Mantel's theorem. A visit to the internet brings proofs, some presented as videos [9]. A natural proof by induction competes with the proof of Mantel, using Cauchy's inequality. The book of Aigner and Ziegler [1] exhibits seven different proofs, five of them for Turán's generalization, the flagship theorem of extremal graph theory.

Theorem 2 (Turán [6]). *K_p-free graphs on n points have at most $(1 - \frac{1}{p-1})\frac{n^2}{2}$ edges.*

Analogues of Mantel's theorem are also considered for *k-graphs*, where the edges are k-element subsets of the points. Turán [7] asked about the maximum number of edges among 3-graphs on n points that contain no tetrahedron, four triples on four points. This is known as a notoriously difficult question, where even the asymptotic answer is unknown (conjectured to be $\frac{5}{9}\binom{n}{3}$).

Here we have two aims. Namely, we present a more friendly extension of Mantel's theorem to k-graphs (Theorem 3). By doing so we illustrate many difficulties one can have encountering a Turán-type problem. Since many of these questions are unsolved, we investigate other important combinatorial structures between graphs and hypergraphs, like multigraphs (see, e.g., [3]), and linear hypergraphs. But even then, the solutions frequently lead to further unsolved problems.

A k-graph is called *linear* if any two edges intersect in at most one point. Note that graphs are linear 2-graphs. For any integer $k \geq 2$, the *fan*, denoted by F^k, is the k-graph

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having $k+1$ edges, $f_1, \ldots, f_k$, and g, such that $f_1, \ldots, f_k$ are disjoint apart from one common point v (*the center*) and an additional *crossing edge* g that intersects all f_i in points different from v. We prove the following extension of Mantel's theorem.

Theorem 3. *Fan-free linear k-graphs on n points have at most n^2/k^2 edges.*

Considering *all* k-graphs, a different extension of Mantel's theorem arises.

Theorem 4 (Mubayi and Pikhurko [5]). *Fan-free k-graphs on $n > n(k)$ points have at most $(n/k)^k$ edges.*

Proof of Theorem 3. We denote by $V(G)$, $E(G)$ the set of points and edges of a k-graph G. The number of edges containing a point $v \in V(G)$ is the *degree* of v, denoted by $d(v)$. If all points have the same degree m, the k-graph is called *m-regular*.

Assume that G is a fan-free linear k-graph with n points. For any point $v \in V(G)$, set $N_v := (\cup_{e \ni v} e) \setminus \{v\}$ (the open neighborhood of v), and $B_v := V(G) \setminus N_v$. Let $\Delta = \Delta(G)$ be the maximum degree of G and select $v \in V(G)$ such that $d(v) = \Delta$. Then $|N_v| = (k-1)\Delta$ and $|B_v| = n - (k-1)\Delta$. Since G is fan-free, every edge of G must intersect B_v. This and the inequality $d(x) \leq \Delta$ imply

$$|E(G)| \leq \sum_{x \in B_v} d(x) \leq \Delta |B_v| = \Delta(n - (k-1)\Delta). \tag{1}$$

On the other hand, obviously

$$|E(G)| \leq \frac{n\Delta}{k}. \tag{2}$$

If $\Delta \leq (n/k)$ we immediately get $|E(G)| \leq n^2/k^2$ from (2).

If $\Delta > (n/k)$ then (1) and the geometric–arithmetic mean inequality imply

$$|E(G)| \leq \Delta(n - (k-1)\Delta) \leq \frac{1}{4}(n - (k-2)\Delta)^2 < \frac{1}{4}\left(n - \frac{(k-2)n}{k}\right)^2 = \frac{n^2}{k^2}.$$

Thus in both cases $|E(G)| \leq n^2/k^2$ as claimed. ∎

2. EXTREMAL CONFIGURATIONS. It is interesting to see when we have equality in the bounds of the theorems of the previous section. Graphs (or k-graphs) attaining equality are called *extremal configurations.* In Theorem 2, the only extremal configuration is the balanced complete $(p-1)$-partite graph: the points of G are divided into $p-1$ groups, each with $\frac{n}{p-1}$ points and we have all edges joining two points from different groups. In particular, for $p = 3$, the only extremal configuration in Theorem 1 is the balanced complete bipartite graph with even number of vertices. The extremal k-graphs of Theorem 3 must satisfy $k|n$; this is assumed in the next subsection.

Transversal designs. The complexity of extremal configurations grows with k; they are the transversal designs, defined as follows. Assume that n is a multiple of k. A *transversal design* $T(n, k)$ is a linear k-graph on n points where the points are partitioned into k groups, each containing n/k points and where each pair of points from different groups belongs to exactly one edge. Note that for $k = 2$, $T(n, 2)$ is the complete bipartite graph with $n/2$ points in its partite classes. The next case, $k = 3$, is already more interesting. Assume that X, Y, Z are the groups of a transversal design $T(n, 3)$ with points $x_i \in X$, $y_i \in Y$, $z_i \in Z$ for $i = 1, 2, \ldots, n/3$. Consider the $\frac{n}{3} \times \frac{n}{3}$ matrix A defined with $a_{ij} = z_k$ where z_k is the (unique) point of Z for which $x_i y_j z_k$ is an edge of $T(n, 3)$. Each row and column of A contains different z_i's; such a matrix is called a *Latin square* (of order $n/3$). For the interested reader we note that in general, transversal designs $T(n, k)$ are equivalent to $k - 2$ *mutually orthogonal Latin squares*. The investigation of these combinatorial structures goes back to Euler. For a nice (and high-level) introduction, see van Lint and Wilson [**8**].

Proposition 1. *Any $T(n, k)$ is a fan-free linear k-graph with n^2/k^2 edges.*

Proof. Suppose that some $T(n, k)$ contains a k-fan F with center v in group i. Since the crossing edge g of F must have a point in group i different from v, g can intersect at most $k - 1$ edges from the k edges of F containing v. This is a contradiction. ∎

Theorem 5. *Equality holds in Theorem 3 only if G is a transversal design $T(n, k)$.*

Proof. We continue the proof of Theorem 3. If equality holds, then inequalities (1) and (2) are equalities. From (2) we have that $n = km$ and G is an m-regular k-graph. It is left to show that $|E(G)| = m^2$ implies that G is a transversal design with k groups. Since G is m-regular, we have $|B_v| = km - (k-1)m = m$ for every $v \in V(G)$.

Claim 1. *For every $v \in V(G)$, B_v is a strongly independent set, i.e., every edge intersects it in at most one point.*

To prove the claim, assume that $x, y \in B_v$ and that there is an edge $e \in E(G)$ containing x, y. Then (1) cannot be an equality, since e is counted from both x and y. This is a contradiction, proving the claim.

Applying the claim for the points of an arbitrary edge $e = \{v_1, v_2, \ldots, v_k\}$, we get the strongly independent sets $B_{v_1}, \ldots, B_{v_k}$. These sets must be pairwise disjoint, because if $x \in B_{v_i} \cap B_{v_j}$ then $B_{v_j} \cup \{v_i\} \subseteq B_x$, contradicting the fact that $|B_{v_j}| = |B_x|$. Thus $V(G)$ can be partitioned into k groups of size m, each forming a strongly independent set. The m^2 edges of G cover $m^2\binom{k}{2}$ pairs in $V(G)$ and this is equal to $\binom{mk}{2} - k\binom{m}{2}$, the number of pairs of $V(G)$ not covered by the groups $B_{v_1}, \ldots, B_{v_k}$. Thus each pair of points from different groups is covered exactly once, proving that G is a transversal design with k groups of size m. ∎

Truncated designs. What happens when n is not divisible by k? Turán [**6**] proved that the unique extremal configuration for K_p-free graphs on n points is the following graph (the Turán-graph): n points are divided into $p - 1$ groups as evenly as possible and the edges are all pairs of points from different groups. Considering the same question for fan-free k-graphs, we can answer only in the case $n \equiv -1 \pmod{k}$ for general k, as far as the maximum number of edges is concerned. A *truncated design* is obtained from a transversal design by removing one point (and all edges containing it).

Theorem 6. *Assume that $k \geq 2$, $n = k(m+1) - 1$, and G is a fan-free k-graph with n points. Then $|E(G)| \leq m^2 + m$. Truncated designs obtained from $T(n+1, k)$ are extremal configurations.*

Proof. It follows from Proposition 1 that any truncated design obtained from $T(n + 1, k)$ is a fan-free linear k-graph with $m^2 + m$ edges. To show that $|E(G)| \leq m^2 + m$

whenever G is a fan-free linear k-graph with $n = km + k - 1$ points, we follow the argument of the proof of Theorem 3 using the same notations.

If $\Delta \le m$, we immediately get from (2) that

$$|E(G)| \le \frac{((m+1)k-1)m}{k} = m^2 + m - \frac{m}{k} < m^2 + m.$$

If $\Delta \ge m+1$, then (1) and the geometric–arithmetic mean inequality imply

$$\begin{aligned}|E(G)| &\le \Delta(n-(k-1)\Delta) \le \frac{1}{4}(n-(k-2)\Delta)^2 \\ &\le \frac{1}{4}((m+1)k-1-(k-2)(m+1))^2 = m^2+m+\frac{1}{4}.\end{aligned}$$

Thus in both cases $|E(G)| \le m^2 + m$. ■

3. EXTREMAL TRIPLE SYSTEMS.

In this section, we refer to linear 3-graphs as triple systems.

Extensions of triangle-free graphs. To find all extremal configurations for $k = 3, n \equiv -1 \pmod 3$, we need a special case of a theorem of Andrásfai, Erdős, and Sós, stated here as a lemma.

Lemma 1 (Andrásfai, Erdős, and Sós [2]). *Assume that a nonbipartite graph G has n points and contains no triangles. Then the minimum degree of G is at most $2n/5$.*

The following graph, *the blown up five-cycle*, C_5^t, shows that Lemma 1 is sharp when $n \equiv 0 \pmod 5$. Take a five-cycle and replace its points with disjoint t-element sets of points, $A_1, \dots, A_5$ and replace its edges by complete bipartite graphs $[A_i, A_{i+1}]$ for $i = 1, \dots, 5 \pmod 5$.

There is an easy way to generate fan-free triple systems from triangle-free graphs. Consider a graph G with a proper edge-coloring, i.e., let the edge set of G be partitioned into d matchings (pairwise disjoint edges) $M_1, \dots, M_d$. The *extension* of G, $T(G)$, is the triple system obtained by extending $V(G)$ with d new points $v_1, \dots, v_d$ and extending every edge of M_i with the point v_i, for $i = 1, \dots, d$.

Proposition 2. *Assume that G is a triangle-free graph with a proper edge coloring. Then $T(G)$ is a fan-free triple system.*

Proof. No fan in $T(G)$ can be centered at v_i since three edges of M_i cannot be intersected by any edge of M_j for $j \ne i$. If a fan in $T(G)$ is centered at $w \in V(G)$ with triples $ww_1v_1, ww_2v_2, ww_3v_3$, then its crossing edge must be of the form $w_iw_jv_k$ with three different indices. Thus w, w_i, w_j is a triangle in G, contradicting the assumption. ■

An important special case of proper edge colorings is the 1-*factorization*, where all of the d matchings cover all points of the graph. Graphs having 1-factorizations are obviously d-regular but the converse statement is not true: odd cycles are easy examples. The most famous example is the *Petersen graph*.

We will apply Proposition 2 to two triangle-free nonbipartite graphs. One of them is the Wagner graph, C_8^*, the eight-cycle with its long diagonals; the other is C_5^2, defined earlier in this section.

Extremal triple systems for $n \equiv -1 \pmod 3$.

Theorem 7. *Assume that G is a linear fan-free triple system with $n = 3m + 2$ points. Then G has at most $m^2 + m$ edges. Equality holds only in the following cases.*

(7.1) *G is a truncated design obtained from a transversal design $T(3m + 3, 3)$,*

(7.2) *$m = 3$, G is the extension of a 1-factorization of C_8^*,*

(7.3) *$m = 4$, G is the extension of a 1-factorization of C_5^2.*

Proof. We use the notation of the proof of Theorem 6. Assume that G is a fan-free linear 3-graph, $|V(G)| = 3m + 2$, $|E(G)| = m^2 + m$.

From (2) it follows that $\Delta \geq m + 1$. The inequality (1) would give $|E(G)| < m^2 + m$ if any vertex of $v \in V(G)$ has degree larger than $m + 1$. Thus $\Delta(H) = m + 1$. Suppose $d(x) = m + 1 = \Delta$. Then for $B := B_x$ one has $|B| = m$, all points of B_x have degree $m + 1$, and B_x is a strongly independent set. It follows that $N_y = N_x$ for each $y \in B$, implying $d_H(w) = m$ for all $w \in W := V(G) \setminus B$.

Define the graph G^* on point set W, where $w_1, w_2 \in W$ is an edge in G^* if and only if $w_1 w_2 v$ is an edge of the 3-graph G for some $v \in B$. Then G^* is m-regular and can be written as the union of m matchings of size $m + 1$. If G^* is bipartite, then it must be isomorphic to $K_{m+1,m+1}$ with one matching removed; thus G is a truncated design obtained from three groups of size $m + 1$, the first possibility in Theorem 7.

We claim that G^* is a triangle-free graph. Indeed, if w_1, w_2, w_3 form a triangle in G^*, then G contains the edges

$$e = w_1 w_2 v_3, \quad f = w_1 w_3 v_2, \quad g = w_2 w_3 v_1$$

for $v_i \in B_v$. Because $d_G(v_1) = m + 1$, there is an edge $h = v_1 w_1 w_4$. Then e, f, g, h form a fan with center w_1 and with crossing edge g, a contradiction.

Thus we may suppose that G^* is a nonbipartite triangle-free graph. Applying Lemma 1 to our graph G^*, we get $m \leq 2(2m + 2)/5$; thus $m \leq 4$. We leave it to the reader to check that there are no m-regular nonbipartite triangle-free graphs on $2m + 2$ points for $m = 1, 2$, but for $m = 3, 4$ there are unique ones: C_8^*, C_5^2. Moreover, both graphs have 1-factorizations. Extending them with the points of B, we get the second and third possibilities in Theorem 7. ∎

Remark. Theorem 7 seemingly provides two exceptional extremal configurations. However, this is not right: there are *three*! The explanation is that the Wagner graph has two nonisomorphic 1-factorizations (and C_5^2 has only one).

4. CONCLUSION. It does not seem easy to find, for every $n \not\equiv 0 \pmod k$, the maximum number of edges in fan-free linear k-graphs on n points, let alone give a description of all extremal k-graphs attaining this maximum. We could answer the former question only in the case $n \equiv -1 \pmod k$ in Theorem 6 and the latter in the subcase $k = 3$ in Theorem 7. The study of the remaining cases might reveal some (possibly infinitely many) exceptional extremal configurations. We *conjecture* that the extremal number is m^2 for $m > m(k)$ and $n \not\equiv -1 \pmod k$.

We conclude with a *conjecture* that generalizes Theorems 3 and 4 as well: If $n = km$, $m > m(k)$, $1 \leq \ell \leq k$, then an F^k-free k-graph on n points with more than m^ℓ edges has two edges intersecting in at least ℓ points.

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Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences
furedi.zoltan@renyi.mta.hu

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences
gyarfas.andras@renyi.mta.hu