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Avoiding long Berge cycles

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ABSTRACT

Let $n \geq k \geq r+3$ and $\mathcal{H}$ be an n-vertex r-uniform hypergraph. We show that if

$$|\mathcal{H}| > \frac{n-1}{k-2}\binom{k-1}{r}$$

then $\mathcal{H}$ contains a Berge cycle of length at least k. This bound is tight when $k-2$ divides $n-1$. We also show that the bound is attained only for connected r-uniform hypergraphs in which every block is the complete hypergraph $K_{k-1}^{(r)}$.

1. Definitions, Berge F subhypergraphs

An r-uniform hypergraph, or simply *r-graph*, is a family of r-element subsets of a finite set. We associate an r-graph $\mathcal{H}$ with its edge set and call its vertex set $V(\mathcal{H})$.

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Usually we take $V(\mathcal{H}) = [n]$, where $[n]$ is the set of first n integers, $[n] := \{1, 2, 3, \dots, n\}$. We also use the notation $\mathcal{H} \subseteq \binom{[n]}{r}$.

Definition 1.1 *(Anstee and Salazar [1], Gerbner and Palmer [7]).* For a graph F with vertex set $\{v_1, \dots, v_p\}$ and edge set $\{e_1, \dots, e_q\}$, a hypergraph $\mathcal{H}$ contains a **Berge** F if there exist distinct vertices $\{w_1, \dots, w_p\} \subseteq V(\mathcal{H})$ and edges $\{f_1, \dots, f_q\} \subseteq E(\mathcal{H})$, such that if $e_i = v_{i_1}v_{i_2}$, then $\{w_{i_1}, w_{i_2}\} \subseteq f_i$. The vertices $\{w_1, \dots, w_p\}$ are called the **base vertices** of the Berge F.

Of particular interest to us are Berge cycles.

Definition 1.2. A **Berge cycle** of length ℓ in a hypergraph is a set of ℓ distinct vertices $\{v_1, \dots, v_\ell\}$ and ℓ distinct edges $\{e_1, \dots, e_\ell\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ with indices taken modulo ℓ.

A **Berge path** of length ℓ in a hypergraph in a hypergraph is a set of $\ell + 1$ vertices $\{v_1, \dots, v_{\ell+1}\}$ and ℓ hyperedges $\{e_1, \dots, e_\ell\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for all $1 \le i \le \ell$.

Let $\mathcal{H}$ be a hypergraph and p be an integer. The *p-shadow*, $\partial_p\mathcal{H}$, is the collection of the p-sets that lie in some edge of $\mathcal{H}$. In particular, we will often consider the 2-shadow $\partial_2\mathcal{H}$ of a r-uniform hypergraph $\mathcal{H}$ in which each edge of $\mathcal{H}$ yields a clique on r vertices. We say a hypergraph is *connected* if its 2-shadow is a connected graph. A *component* of a hypergraph is a maximum connected subhypergraph.

2. Background

Erdős and Gallai [3] proved the following result on the Turán number of paths.

Theorem 2.1 *(Erdős and Gallai [3]). Let $k \ge 2$ and let G be an n-vertex graph with no path on k vertices. Then $e(G) \le (k-2)n/2$.*

This theorem is implied by a stronger result for graphs with no long cycles.

Theorem 2.2 *(Erdős and Gallai [3]). Let $k \ge 3$ and let G be an n-vertex graph with no cycle of length k or longer. Then $e(G) \le (k-1)(n-1)/2$.*

Győri, Katona, and Lemons [9] extended Theorem 2.1 to Berge paths in r-graphs. The bounds depend on the relationship of r and k.

Theorem 2.3 *(Győri, Katona, and Lemons [9]). Suppose that $\mathcal{H}$ is an n-vertex r-graph with no Berge path of length k. If $k \ge r + 2 \ge 5$, then $e(\mathcal{H}) \le \frac{n}{k}\binom{k}{r}$, and if $r \ge k \ge 3$, then $e(\mathcal{H}) \le \frac{n(k-1)}{r+1}$.*

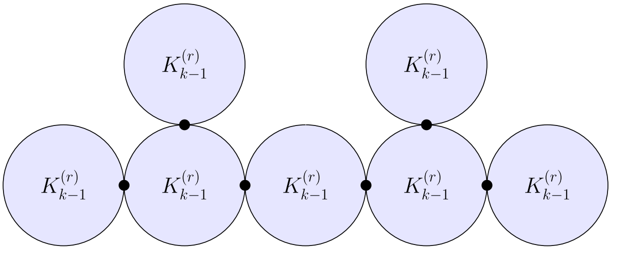
Both bounds in Theorem 2.3 are sharp for each k and r for infinitely many n. The remaining case of $k = r + 1$ was settled later by Davoodi, Győri, Methuku, and Tompkins [2]: *if $\mathcal{H}$ is an n-vertex r-graph with $|E(\mathcal{H})| > n$, then it contains a Berge path of length at least* $r+1$. Furthermore, Győri, Methuku, Salia, Tompkins and Vizer [10] have found a better upper bound on the number of edges in n-vertex *connected* r-graphs with no Berge path of length k. Their bound is asymptotically exact when r is fixed and k and n are sufficiently large.

The goal of this paper is to present similar results for cycles.

3. Main result: hypergraphs without long Berge cycles

Our main result is an analogue of the Erdős–Gallai theorem on cycles for r-graphs.

Theorem 3.1. *Let $r \geq 3$ and $k \geq r+3$, and suppose $\mathcal{H}$ is an n-vertex r-graph with no Berge cycle of length k or longer. Then $e(\mathcal{H}) \leq \frac{n-1}{k-2}\binom{k-1}{r}$. Moreover, equality is achieved if and only if $\partial_2\mathcal{H}$ is connected and for every block D of $\partial_2\mathcal{H}$, $D = K_{k-1}$ and $\mathcal{H}[D] = K_{k-1}^{(r)}$.*



Note that a Berge cycle can only be contained in the vertices of a single block of the 2-shadow. Hence the aforementioned sharpness examples cannot contain Berge cycles of length k or longer.

In the original version of this paper, we *conjectured* that the statement of Theorem 3.1 holds for $k = r + 2$ too. Very recently, Ergemlidze, Győri, Methuku, Salia, Tompkins, and Zamora [4] confirmed this and proved exact bounds for $k = r + 1$ as well.

Theorem 3.2 *(Ergemlidze et al. [4]). Let $k \geq 4$ and let $\mathcal{H}$ be an n-vertex r-graph with no Berge-cycles of length k or longer. If $k = r+1$, then $e(\mathcal{H}) \leq n-1$, and if $k = r+2$, then $e(\mathcal{H}) \leq \frac{n-1}{k-2}\binom{k-1}{r}$.*

Similarly to the situation with paths, the case of short cycles, $k \leq r + 1$, is different. Exact bounds for $k \leq r - 1$ and asymptotic bounds for $k = r$ were found in [12].

Theorem 3.3 *(Kostochka and Luo [12]). Let $r \geq 5$ and $r \geq k+1$, and suppose $\mathcal{H}$ is an n-vertex r-graph with no Berge cycle of length k or longer. Then $e(\mathcal{H}) \leq \frac{(k-1)(n-1)}{r}$.*

For convenience, below we will use notation

$$C_r(k) := \frac{1}{k-2}\binom{k-1}{r}. \tag{1}$$

(So $C_2(k)(n-1) = (k-1)(n-1)/2$.) Theorem 3.1 yields the following implication for paths.

Corollary 3.4. *Let $r \geq 3$ and $n \geq k+1 \geq r+4$. If $\mathcal{H}$ is a connected n-vertex r-graph with no Berge path of length k, then $e(\mathcal{H}) \leq C_r(k)(n-1)$.*

This gives a $\frac{k-2}{k-r}$ times stronger bound than Theorem 2.3 for connected r-graphs for all $r \geq 3$ and $n \geq k+1 \geq r+4$ and not only for sufficiently large k and n. In particular, Corollary 3.4 implies the following slight sharpening of Theorem 2.3 for $k \geq r+3$ in which we also describe the extremal hypergraphs.

Corollary 3.5. *Let $r \geq 3$ and $n \geq k \geq r+3$. If $\mathcal{H}$ is an n-vertex r-graph with no Berge path of length k, then $e(\mathcal{H}) \leq \frac{n}{k}\binom{k}{r}$ with equality only if every component of $\mathcal{H}$ is the complete r-graph $K_k^{(r)}$.*

In the next section, we introduce the notion of *representative pairs* and use it to derive useful properties of Berge F-free hypergraphs for rather general F. In Section 5, we cite Kopylov's Theorem and prove two useful inequalities. In Section 6 we prove our main result, Theorem 3.1, and in the final Section 7 we derive Corollaries 3.4 and 3.5.

4. Representative pairs, the structure of Berge F-free hypergraphs

Definition 4.1. For a hypergraph $\mathcal{H}$, a **system of distinct representative pairs (SDRP) of** $\mathcal{H}$ is a set of distinct pairs $A = \{\{x_1, y_1\}, \dots, \{x_s, y_s\}\}$ and a set of distinct hyperedges $\mathcal{A} = \{f_1, \dots f_s\}$ of $\mathcal{H}$ such that for all $1 \leq i \leq s$

— $\{x_i, y_i\} \subseteq f_i$, and
— $\{x_i, y_i\}$ is not contained in any $f \in \mathcal{H} - \{f_1, \dots, f_s\}$.

Lemma 4.2. *Let $\mathcal{H}$ be a hypergraph, let $(A, \mathcal{A})$ be an* SDRP *of $\mathcal{H}$ of maximum size. Let $\mathcal{B} := \mathcal{H} \setminus \mathcal{A}$ and let $B = \partial_2 \mathcal{B}$ be the 2-shadow of $\mathcal{B}$. For a subset $S \subseteq B$, let $\mathcal{B}_S$ denote the set of hyperedges that contain at least one edge of S. Then for all nonempty $S \subseteq B$, $|S| < |\mathcal{B}_S|$.*

Proof. Suppose there exists a nonempty set $S \subseteq B$ such that $|S| \geq |\mathcal{B}_S|$. Choose a smallest such S.

We claim that $|S| = |\mathcal{B}_S|$. Indeed, if $|S| > |\mathcal{B}_S|$ then $|S| \geq 2$ because $\mathcal{B}_S \neq \emptyset$ by definition. Take any edge $e \in S$. The set $S \setminus e$ is nonempty and $|S \setminus e| = |S| - 1 \geq |\mathcal{B}_S| \geq |\mathcal{B}_{S \setminus e}|$, a contradiction to the minimality of S.

Consider the case $|S| = |\mathcal{B}_S|$. By the minimality of S, each subset $S' \subset S$ satisfies $|S'| < |\mathcal{B}_{S'}|$. Therefore by Hall's theorem, one can find a bijective mapping of S to $\mathcal{B}_S$, where say the edge $e_i \in S$ gets mapped to hyperedge f_i in $\mathcal{B}_S$ for $1 \leq j \leq |S|$. Then $(A \cup \{e_i, \dots, e_{|S|}\}, \mathcal{A} \cup \{f_1, \dots, f_{|S|}\})$ is a larger SDRP of $\mathcal{H}$, a contradiction. □

Lemma 4.3. *Let $\mathcal{H}$ be a hypergraph and let $(A, \mathcal{A})$ be an* SDRP *of $\mathcal{H}$ of maximum size. Let $\mathcal{B} := \mathcal{H} \setminus \mathcal{A}$, $B = \partial_2\mathcal{B}$, and let G be the graph on $V(\mathcal{H})$ with edge set $A \cup B$. If G contains a copy of a graph F, then $\mathcal{H}$ contains a Berge F on the same base vertex set.*

Proof. Let $\{v_1, \dots, v_p\}$ and $\{e_1, \dots, e_q\}$ be a set of vertices and a set of edges forming a copy of F in G such that the edges $e_1, \dots, e_b$ belong to B. By Lemma 4.2, each subset S of $\{e_1, \dots, e_b\}$ satisfies $|S| < |\mathcal{B}_S|$. So we may apply Hall's Theorem to match each of these e_i's to a hyperedge $f_i \in \mathcal{B}$. The edges $e_i \in A$ can be matched to distinct edges of $\mathcal{A}$ given by the SDRP. Since $\mathcal{A} \cap \mathcal{B} = \emptyset$ this yields a Berge F in $\mathcal{H}$ on the same base vertex set. □

We note that this Lemma 4.3 was proved independently by Gerbner, Methuku, and Palmer [6].

We have $|\mathcal{H}| = |A| + |\mathcal{B}|$. Note that the number of r-edges in $\mathcal{B}$ is at most the number of copies of K_r in its 2-shadow. Therefore Lemma 4.3 gives a new proof for the following result of Gerbner and Palmer [8]: for any graph F,

$$\mathrm{ex}(n, K_r, F) \leq \mathrm{ex}_r(n, \text{Berge } F) \leq \mathrm{ex}(n, F) + \mathrm{ex}(n, K_r, F).$$

Here $\mathrm{ex}_r(n, \{\mathcal{F}_1, \mathcal{F}_2, \dots\})$ denotes the *Turán number of* $\{\mathcal{F}_1, \mathcal{F}_2, \dots\}$, the maximum number of edges in an r-uniform hypergraph on n vertices that does not contain a copy of any $\mathcal{F}_i$.

The *generalized Turán function* $\mathrm{ex}(n, K_r, F)$ is the maximum number of copies of K_r in an F-free graph on n vertices.

5. Kopylov's theorem and two inequalities

Definition. For a natural number α and a graph G, the *α-disintegration* of a graph G is the process of iteratively removing from G the vertices with degree at most α until the resulting graph has minimum degree at least $\alpha + 1$ or is empty. This resulting subgraph $H(G, \alpha)$ will be called the $(\alpha + 1)$-*core* of G. It is well known (and easy) that $H(G, \alpha)$ is unique and does not depend on the order of vertex deletion. If $H(G, \alpha)$ is the empty graph, then we say H is *α-disintegrable*.

The following theorem is a consequence of Kopylov [11] about the structure of graphs without long cycles. We state it in the form that we need.[3]

Theorem 5.1 *(Kopylov [11]). Let $n \geq k \geq 5$ and let $t = \lfloor \frac{k-1}{2} \rfloor$. Suppose that G is a 2-connected n-vertex graph with no cycle of length at least k. Suppose that it is saturated, i.e., for every nonedge xy the graph $G \cup \{xy\}$ has a cycle of length at least k. Then either*

(5.1.1) *the t-core $H(G,t)$ is empty, i.e., G is t-disintegrable; or*

(5.1.2) *$|H(G,t)| = s$ for some $t+2 \leq s \leq k-2$, it is a complete graph on s vertices, and $H(G,t) = H(G,k-s)$, i.e., the rest of the vertices can be removed by a $(k-s)$-disintegration.*

Note that in the second case $2 \leq k-s \leq t$.

Lemma 5.2. *Let k, r, t, s, a nonnegative integers, and suppose $k \geq r+3 \geq 6$, $t = \lfloor (k-1)/2 \rfloor$, and $0 \leq a \leq s \leq t$. Then*

$$a + \binom{s-a}{r-1} \leq \frac{1}{k-2}\binom{k-1}{r} := C_r(k).$$

This is the part of the proof where we use $k \geq r+3$ because this inequality does not hold for $k = r+2$ (then the right hand side is $(r+1)/r$ while the left hand side could be as large as $\lfloor (r+1)/2 \rfloor$).

Proof. Keeping k, r, t, s fixed the left hand side is a convex function of a (defined on the integers $0 \leq a \leq s$). It takes its maximum either at $a = s$ or $a = 0$. So the left hand side is at most $\max\{s, \binom{s}{r-1}\}$. This is at most $\max\{t, \binom{t}{r-1}\}$. We have eliminated the variables a and s.

We claim that $t \leq \frac{1}{k-2}\binom{k-1}{r}$. Indeed, keeping k, t fixed, the right hand side is minimized when $r = k-3$, and then it equals to $(k-1)/2$. This is at least $\lfloor (k-1)/2 \rfloor = t$.

Finally, we claim that $\binom{t}{r-1} \leq \frac{1}{k-2}\binom{k-1}{r}$. If $t < r-1$, then there is nothing to prove. For $t \geq r-1$ rearranging the inequality we get

$$r \leq \frac{k-1}{t} \times \frac{k-3}{t-1} \times \cdots \times \frac{k-r}{t-r+2}.$$

Each fraction on the right hand side is at least 2. Since $r < 2^{r-1}$, we are done. □

Lemma 5.3. *Let w, $r \geq 2$, $k \geq r+3$ and let $\mathcal{H}$ be a w-vertex r-graph. Let $\overline{\partial_2 \mathcal{H}}$ denote the family of pairs of $V(\mathcal{H})$ not contained in any member of $\mathcal{H}$ (i.e., the complement of the 2-shadow). Then*

[3] A proof and a recent application can be found in [13].

$$|\mathcal{H}| + |\overline{\partial_2\mathcal{H}}| \leq a_r(w) := \begin{cases} \binom{w}{2} & \textit{for } 2 \leq w \leq r+2, \\ \binom{w}{r} & \textit{for } r+2 \leq w. \end{cases}$$

Moreover, for $2 \leq w \leq k-1$, $|\mathcal{H}| + |\overline{\partial_2\mathcal{H}}| = a_r(w)$ *if and only if* $w = k-1$ *and either* $w > r+2$ *and* $\mathcal{H}$ *is complete, or* $w = r+2$ *and one of* $\mathcal{H}$ *or* $\overline{\partial_2\mathcal{H}}$ *is complete.*

Also, if $2 \leq w \leq k-1$, *we have* $a_r(w) \leq (w-1)\binom{k-1}{r}/(k-2) = C_r(k)(w-1)$.

Proof. The case of $w \geq r+2$ is a corollary of the classical Kruskal–Katona theorem, but one can give a direct proof by a double counting. If $\overline{\partial_2\mathcal{H}}$ is empty, then $|\mathcal{H}| = \binom{w}{r}$ if and only if $\mathcal{H} = \binom{V(\mathcal{H})}{r}$. Otherwise, let $\overline{\mathcal{H}}$ denote the r-subsets of $V(\mathcal{H})$ that are not members of $\mathcal{H}$, $\overline{\mathcal{H}} = \binom{V(\mathcal{H})}{r} \setminus \mathcal{H}$. Each pair of $\overline{\partial_2\mathcal{H}}$ is contained in $\binom{w-2}{r-2}$ members of $\overline{\mathcal{H}}$ and each $e \in \overline{\mathcal{H}}$ contains at most $\binom{r}{2}$ edges of $\overline{\partial_2\mathcal{H}}$. We obtain

$$|\overline{\partial_2\mathcal{H}}|\binom{w-2}{r-2} \leq |\overline{\mathcal{H}}|\binom{r}{2}.$$

Since $\binom{w-2}{r-2} \geq \binom{r}{r-2} = \binom{r}{2}$, $|\overline{\partial_2\mathcal{H}}| \leq |\overline{\mathcal{H}}|$ with equality only when $w = r+2$. Furthermore, if $\overline{\partial_2\mathcal{H}}$ and $\mathcal{H}$ are both nonempty, then for any $xy \in \overline{\partial_2\mathcal{H}}$ and $uv \in \partial_2\mathcal{H}$ (with possibly $x = u$), any r-tuple e containing $\{x,y\} \cup \{u,v\}$ is in $\overline{\mathcal{H}}$ but contributes strictly less than $\binom{r}{2}$ edges to $\overline{\partial_2\mathcal{H}}$, implying $|\overline{\partial_2\mathcal{H}}| < |\overline{\mathcal{H}}|$. This completes the proof of the case.

The case $w \leq r+1$ is easy, and the calculation showing $a_r(w) \leq C_r(k)(w-1)$ with equality only if $w = k-1$ is standard. □

6. Proof of Theorem 3.1, the main upper bound

Proof. Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices with no Berge cycle of length k or longer ($k \geq r+3 \geq 6$). Let $(A, \mathcal{A})$ be an SDRP of $\mathcal{H}$ of maximum size. Let $\mathcal{B} := \mathcal{H} \setminus \mathcal{A}$, $B = \partial_2\mathcal{B}$. By Lemma 4.3 the graph G with edge set $A \cup B$ does not contain a cycle of length k or longer.

Let $V_1, V_2, \ldots, V_p$ be the vertex sets of the standard (and unique) decomposition of G into 2-connected blocks of sizes $n_1, n_2, \ldots, n_p$. Then the graph $A \cup B$ restricted to V_i, denoted by G_i, is either a 2-connected graph or a single edge (in the latter case $n_i = 2$), each edge from $A \cup B$ is contained in a single G_i, and $\sum_{i=1}^{p}(n_i - 1) \leq (n-1)$.

This decomposition yields a decomposition of $A = A_1 \cup A_2 \cup \cdots \cup A_p$ and $B = B_1 \cup B_2 \cup \cdots \cup B_p$, $A_i \cup B_i = E(G_i)$. If an edge $e \in B_i$ is contained in $f \in \mathcal{B}$, then $f \subseteq V_i$ (because f induces a 2-connected graph K_r in B), so the block-decomposition of G naturally extends to $\mathcal{B}$, $\mathcal{B}_i := \{f \in \mathcal{B} : f \subseteq V_i\}$ and we have $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_p$, and $B_i = \partial_2\mathcal{B}_i$.

We claim that for each i,

$$|A_i| + |\mathcal{B}_i| \leq C_r(k)(n_i - 1), \tag{2}$$

and hence

$$|\mathcal{H}| = |A| + |\mathcal{B}| = \sum_{i=1}^{p} |A_i| + |\mathcal{B}_i| \leq \sum_{i=1}^{p} C_r(k)(n_i - 1) \leq C_r(k)(n-1),$$

completing the proof.

To prove (2) observe that the case $n_i \leq k-1$ immediately follows from Lemma 5.3. From now on, suppose that $n_i \geq k$.

Consider the graph G_i and, if necessary, add edges to it to make it a saturated graph with no cycle of length k or longer. Let the resulting graph be G'. Kopylov's Theorem (Theorem 5.1) can be applied to G'. If G' is t-disintegrable, then make $(n_i - k + 2)$ disintegration steps and let W be the remaining vertices of V_i ($|W| = k-2$). For the edges of A_i and $\mathcal{B}_i$ contained in W we use Lemma 5.3 to see that

$$|A_i[W]| + |\mathcal{B}_i[W]| < C_r(k)(|W| - 1).$$

In the t-disintegration steps, we iteratively remove vertices with degree at most t until we arrive to W. When we remove a vertex v with degree $s \leq t$ from G', a of its incident edges are from A, and the remaining $s-a$ incident edges eliminate at most $\binom{s-a}{r-1}$ hyperedges from $\mathcal{B}_i$ containing v. Therefore v contributes at most $a + \binom{s-a}{r-1} \leq C_r(k)$ (by Lemma 5.2) to $|\mathcal{B}_i| + |A_i|$.

It follows that

$$|A_i| + |\mathcal{B}_i| < \left(\sum_{v \in G' - W} C_r(k) \right) + C_r(k)(|W| - 1) = C_r(k)(n_i - 1).$$

This completes this case.

Next consider the case (5.1.2), $W := V(H(G,t))$, $|W| = s \leq k-2$. We proceed as in the previous case, making $(n_i - s)$ disintegration steps. Apply Lemma 5.3 for $|A_i[W]| + |\mathcal{B}_i[W]|$ and Lemma 5.2 for the $(k-s)$-disintegration steps (where $k - s \leq t$) to get the desired upper bound (with strict inequality). This completes the proof of (2).

The extremal systems. Suppose that $e(\mathcal{H}) = |A| + |\mathcal{B}| = C_r(k)(n-1)$. Then $\sum_{i=1}^{p}(n_i - 1) = n - 1$ (so $A \cup B$ is connected) and $|A_i| + |\mathcal{B}_i| = C_r(k)(n_i - 1)$ for each $1 \leq i \leq p$. From the previous proof and Lemma 5.3, we see that this holds if and only if for each i, $n_i = k-1$, and either $\mathcal{B}_i$ or A_i is complete. In particular, this implies that each block of $A \cup B$ is a K_{k-1}. We will show that each G_i corresponds to a block in $\mathcal{H}$ that is $K_{k-1}^{(r)}$ with vertex set V_i.

In the case that $\mathcal{B}_i$ is complete for all $1 \leq i \leq p$, we are done. Otherwise, if some A_i is complete ($n_i = k-1 = r+2$ by Lemma 5.3) then there are $\binom{k-1}{2} = \binom{k-1}{k-3} = \binom{k-1}{r}$ hyperedges in $\mathcal{A}$ intersecting V_i in at least two vertices. If all such hyperedges are contained in V_i, again we get $\mathcal{H}[V_i] = K_{k-1}^{(r)}$. So suppose there exists a $f \in \mathcal{A}$ which is paired with an edge $xy \in A_i$ in the SDRP, but for some $z \notin V_i$, $\{x, y, z\} \subseteq f$. Then z

belongs to another block G_j of $A \cup B$. In $A \cup B$, there exists a path from x to z covering $V_i \cup V_j$ which avoids the edge xy. Thus by Lemma 4.3, there is a Berge path from x to z with at least $2(k-1)-1$ base vertices which avoids the hyperedge f (since edge xy was avoided). Adding f to this path yields a Berge cycle of length $2(k-1)-1 > k$, a contradiction. □

7. Corollaries for paths

In order to be self-contained, we present a short proof of a lemma by Győri, Katona, and Lemons [9].

Lemma 7.1 *(Győri, Katona, and Lemons [9]). Let $\mathcal{H}$ be a connected hypergraph with no Berge path of length k. If there is a Berge cycle of length k on the vertices $v_1, \ldots, v_k$ then these vertices constitute a component of $\mathcal{H}$.*

Proof. Let $V = \{v_1, \ldots, v_k\}$, $E = \{e_1, \ldots, e_k\}$ form the Berge cycle in $\mathcal{H}$. If some edge, say e_1 contains a vertex v_0 outside of V, then we have a path with vertex set $\{v_0, v_1, \ldots, v_\ell\}$ and edge set E. Therefore each e_i is contained in V. Suppose $V \neq V(\mathcal{H})$. Since $\mathcal{H}$ is connected, there exists an edge $e_0 \in \mathcal{H}$ and a vertex $v_{k+1} \notin V$ such that for some $v_i \in V$, say $i = k$, $\{v_k, v_{k+1}\} \subseteq e_0$. Then $\{v_1, \ldots, v_k, v_{k+1}\}, \{e_1, \ldots, e_{k-1}, e_0\}$ is a Berge path of length k. □

Proof of Corollary 3.4. Suppose $n \geq k+1$ and $\mathcal{H}$ is a connected n-vertex r-graph with $e(\mathcal{H}) > C_r(k)(n-1)$. Then by Theorem 3.1, $\mathcal{H}$ has a Berge cycle of length $\ell \geq k$. If $\ell \geq k+1$, then removing any edge from the cycle yields a Berge path of length at least k. If $\ell = k$, then by Lemma 7.1, $\mathcal{H}$ again has a Berge path of length k. □

Now Theorem 3.1 together with Corollary 3.4 directly imply Corollary 3.5.

Proof of Corollary 3.5. Suppose $k \geq r+3 \geq 6$ and $\mathcal{H}$ is an r-graph. Let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_s$ be the connected components of $\mathcal{H}$ and $|V(\mathcal{H}_i)| = n_i$ for $i = 1, \ldots, s$.

If $n_i \leq k-1$, then $|\mathcal{H}_i| \leq \binom{n_i}{r} < \frac{n_i}{k}\binom{k}{r}$. If $n_i \geq k+1$, then by Corollary 3.4, $|\mathcal{H}_i| \leq C_r(k)(n_i - 1) < \frac{n_i}{k}\binom{k}{r}$. Finally, if $n_i = k$, then $|\mathcal{H}_i| \leq \binom{k}{r} = \frac{n_i}{k}\binom{k}{r}$, with equality only if $\mathcal{H}_i = K_k^{(r)}$. This proves the corollary. □

Concluding remarks and acknowledgments. In this paper, upper bounds were determined for the maximum number of edges in an r-uniform hypergraph with no Berge cycle of length k or greater when $k \geq r+3$. These bounds are sharp only when $n-1$ is divisible by $k-2$. Very recently in [5], the present authors developed and extended ideas from this paper to prove exact bounds and classify extremal examples for all n and $k \geq r+4$.

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