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Note

A variation of a theorem by Pósa

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ABSTRACT

A graph G is ℓ-hamiltonian if each linear forest F with ℓ edges contained in G can be extended to a hamiltonian cycle of G. We give a sharp upper bound for the maximum number of cliques of a fixed size in a non-ℓ-hamiltonian graph. Furthermore, we prove stability: if a non-ℓ-hamiltonian graph contains almost the maximum number of cliques, then it is a subgraph of one of two extremal graphs.

1. Background, ℓ-hamiltonian graphs

Let $V(G)$ denote the vertex set of a graph G, $E(G)$ the edge set of G, and $e(G) = |E(G)|$. If $v \in V(G)$, then $N(v)$ denotes the neighborhood of v and $\deg(v) = |N(v)|$. A graph G is *ℓ-hamiltonian* if every linear forest F, $E(F) \subset E(G)$ with $e(F) = \ell$ is contained in a hamiltonian cycle. In particular, being 0-hamiltonian is equivalent to being hamiltonian. Recall a sufficient condition for a graph G to be ℓ-hamiltonian:

Theorem 1 (*Pósa [9]*). *For $n \geq 3$ and $1 \leq \ell < n$, let G be an n-vertex graph such that $\deg(u) + \deg(v) \geq n + \ell$ for every non-edge uv in G. Then G is ℓ-hamiltonian.*

A family of extremal non-ℓ-hamiltonian graphs is as follows. For $n, d, \ell \in \mathbb{N}$ with $\ell < d \leq \left\lfloor \frac{n+\ell-1}{2} \right\rfloor$, let the graph $H_{n,d,\ell}$ be obtained from a copy of $K_{n-d+\ell}$, say with vertex set A, by adding $d - \ell$ vertices of degree d each of which is adjacent to the same set B of d vertices in A (see the left side of Fig. 1). Let

$$h(n, d, \ell) := |E(H_{n,d,\ell})| = \binom{n-d+\ell}{2} + (d-\ell)d. \tag{1}$$

For any $\ell < d \leq \lfloor (n + \ell - 1)/2 \rfloor$, the graph $H_{n,d,\ell}$ is not ℓ-hamiltonian: no linear forest F with ℓ edges in $G[B]$ can be completed to a hamiltonian cycle of $H_{n,d,\ell}$. Erdős [2] proved the following Turán-type result:

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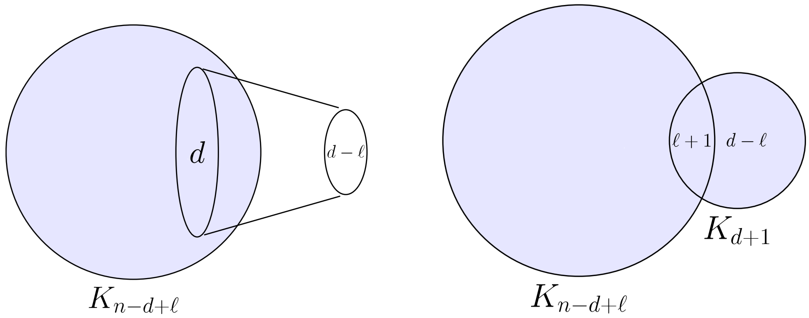

Fig. 1. Graphs $H_{n,d,\ell}$ and $H'_{n,d,\ell}$; shaded ovals denote complete graphs.

Theorem 2 (*Erdős [2]*). *Let n and d be integers with $0 < d \le \lfloor \frac{n-1}{2} \rfloor$. If G is a nonhamiltonian graph on n vertices with minimum degree $\delta(G) \ge d$, then*

$$e(G) \le \max\left\{h(n,d,0), h(n, \left\lfloor \frac{n-1}{2} \right\rfloor, 0)\right\}.$$

Graphs $H_{n,d,0}$ and $H_{n,\lfloor\frac{n-1}{2}\rfloor,0}$ show the sharpness of Theorem 2.

We also define another non-ℓ-hamiltonian graph with many edges. The graph $H'_{n,d,\ell}$ is obtained from a copy of $K_{n-d+\ell}$ and a copy of K_{d+1} by identifying $\ell+1$ vertices (see Fig. 1, on the right). No path of ℓ edges spanning the $\ell+1$ dominating vertices in $H'_{n,d,\ell}$ can be extended to a hamiltonian cycle.

Li and Ning [5] and independently the present authors [3] proved the following refinement of Theorem 2.

Theorem 3 (*[3,5]*). *Let $n \ge 3$ and $d \le \lfloor \frac{n-1}{2} \rfloor$. Suppose that G is an n-vertex nonhamiltonian graph with minimum degree $\delta(G) \ge d$ such that*

$$e(G) > \max\left\{h(n,d+1,0), h(n, \left\lfloor \frac{n-1}{2} \right\rfloor, 0)\right\}. \quad (2)$$

Then G is a subgraph of either $H_{n,d,0}$ or $H'_{n,d,0}$.

Recently, Ma and Ning [7] extended Theorem 3 to graphs with bounded circumference. Luo [6] and Ning and Peng [8] bounded the number cliques of given size in graphs with bounded circumference. The goal of this note is to refine and extend Theorem 3 in a different direction—for non-ℓ-hamiltonian graphs. One can view our results as an extension of Theorem 1. We state our results in the next section and prove them in the remaining two sections.

2. Cliques in non-ℓ-hamiltonian graphs

For a graph G, let $N(G, K_r)$ denote the number of copies of K_r in G. In particular, $N(G, K_2) = e(G)$. Let

$$h_r(n,d,\ell) := N(H_{n,d,\ell}, K_r) = \binom{n-d+\ell}{r} + (d-\ell)\binom{d}{r-1}.$$

We show that classical results easily imply the following extension of Theorem 2 for non-ℓ-hamiltonian graphs.

Theorem 4. *Let n, d, r, ℓ be integers with $r \ge 2$ and $0 \le \ell < d \le \lfloor \frac{n+\ell-1}{2} \rfloor$. If G is an n-vertex graph with minimum degree $\delta(G) \ge d$, and G is not ℓ-hamiltonian, then*

$$N(G, K_r) \le \max\left\{h_r(n,d,\ell), h_r(n, \left\lfloor \frac{n+\ell-1}{2} \right\rfloor, \ell)\right\}.$$

In particular,

$$e(G) \le \max\left\{h(n,d,\ell), h(n, \left\lfloor \frac{n+\ell-1}{2} \right\rfloor, \ell)\right\}.$$

The graphs $H_{n,d,\ell}$ and $H_{n,\lfloor(n+\ell-1)/2\rfloor,\ell}$ show that this bound is sharp for all $0 \le \ell < d \le \lfloor \frac{n+\ell-1}{2} \rfloor$.

Note that a partial case of Theorem 4 (when $\ell = 1$ and $r = 2$) was proved by Ma and Ning [7].

We also obtain a generalization of Theorem 3 which can be viewed as a stability version of Theorem 4.

Theorem 5. *Let $n \geq 3$, $r \geq 2$ and $\ell < d \leq \lfloor \frac{n+\ell-1}{2} \rfloor$. Suppose that G is an n-vertex not ℓ-hamiltonian graph with minimum degree $\delta(G) \geq d$ such that*

$$N(G, K_r) > \max\left\{h_r(n, d+1, \ell), h_r(n, \left\lfloor \frac{n+\ell-1}{2} \right\rfloor, \ell)\right\}. \tag{3}$$

Then G is a subgraph of either $H_{n,d,\ell}$ or $H'_{n,d,\ell}$.

3. Bound on the number of r-cliques: Proof of Theorem 4

Besides the Ore-type condition of Theorem 1, Pósa [9] and independently Kronk [4] proved the following degree sequence version.

Theorem 6 (*Pósa [9], Kronk [4]*)**.** *Let G be a graph on n vertices and let $0 \leq \ell \leq n-2$. The following two conditions (together) are sufficient for G to be ℓ-hamiltonian:*
(6.1) for all integers k with $\ell < k < \frac{n+\ell-1}{2}$, the number of points of degree not exceeding k is less than $k-\ell$,
(6.2) the number of points of degree not exceeding $\frac{n+\ell-1}{2}$ does not exceed $\frac{n-\ell-1}{2}$.

We will use the following claim.

Claim 7. *Let G be an n vertex graph with a set of s vertices with degree at most t. Then $N(G, K_r) \leq \binom{n-s}{r} + s\binom{t}{r-1}$.*

Proof. Let D be a set of s vertices with degree at most t. Then the number of K_r's disjoint from D is at most $\binom{n-s}{r}$. Since each vertex $v \in D$ has degree at most t, v is contained in at most $\binom{t}{r-1}$ copies of K_r. Summing up over all $v \in D$, we get the claim. □

The following lemma is a corollary of Theorem 6 using Claim 7.

Lemma 8. *Let G be an n-vertex, not ℓ-hamiltonian graph with $N(G, K_r) > h_r(n, \lfloor \frac{n+\ell-1}{2} \rfloor, \ell)$. Then*
(8.1) $V(G)$ contains a subset D of $k-\ell$ vertices of degree at most k for some k with $\ell < k \leq \lfloor \frac{n+\ell-1}{2} \rfloor$;
(8.2) $k < \lfloor \frac{n+\ell-1}{2} \rfloor$;
(8.3) $N(G, K_r) \leq h_r(n, k, \ell)$.

Proof. To estimate $N(G, K_r)$ in the cases (6.2) and (6.1) in Theorem 6, one can apply Claim 7 with the values $(s, t) = (\frac{n-\ell-1}{2}, \frac{n+\ell-1}{2})$ and $(s, t) = (\frac{n-\ell-2}{2}, \frac{n+\ell-2}{2})$, respectively. In both cases the upper bound for $N(G, K_r)$ is equal to $h(n, \lfloor \frac{n+\ell-1}{2} \rfloor, \ell)$. Eq. (8.3) is also implied by Claim 7. □

For each integer $p \geq 1$ and real x, define $\binom{x}{p}$ as $x(x-1)\dots(x-p+1)/p!$ for $x \geq p-1$ and 0 for $x < p-1$. This function is non-negative and convex (to see this, investigate the second derivative with respect to x when the function is positive).

For fixed integers n, ℓ, and r with $0 \leq \ell \leq n-1$ and $r \geq 2$, consider the function $h_r(n, x, \ell)$ in the closed interval $\left[\ell, \lfloor \frac{n+\ell-1}{2} \rfloor\right]$. One can show that this function is also convex in x (since both terms are convex), and it is strictly convex where it is positive. We obtained the following.

Claim 9. *Let $J \subseteq \left[\ell, \lfloor \frac{n+\ell-1}{2} \rfloor\right]$ be a closed interval. Then $h_r(n, x, \ell)$ is maximized on J at some of its endpoints.* □

Proof of Theorem 4. Let $1 \leq \ell \leq d \leq \lfloor \frac{n+\ell-1}{2} \rfloor$. Suppose that G is an n-vertex not ℓ-hamiltonian graph with minimum degree $\delta(G) \geq d$.

If $N(G, K_r) \leq h_r(n, \lfloor \frac{n+\ell-1}{2} \rfloor, \ell)$, then we are done. Otherwise, (8.3) in Lemma 8 implies that $N(G, K_r) \leq h_r(n, k, \ell)$ for some $\ell < k < \lfloor \frac{n+\ell-1}{2} \rfloor$. Since $k \geq \delta(G) \geq d$, Claim 9 gives

$$N(G, K_r) \leq \max_{k \in \left[d, \lfloor \frac{n+\ell-1}{2} \rfloor\right]} h_r(n, k, \ell) = \max\left\{h_r(n, d, \ell), h_r(n, \left\lfloor \frac{n+\ell-1}{2} \right\rfloor, \ell)\right\}. \quad \square$$

4. Stability: Proof of Theorem 5

Call an n-vertex graph G *ℓ-saturated* if G is not ℓ-hamiltonian, but each n-vertex graph obtained from G by adding an edge is ℓ-hamiltonian.

Theorem 10 (*Bondy and Chvátal (Theorem 9.11 in [1])*)**.** *Let G be an n-vertex ℓ-saturated graph G. Then for each non-edge $uv \notin E(G)$, $\deg(u) + \deg(v) \leq n - 1 + \ell$.*

Bondy and Chvátal observed that the proof by Pósa [9] yields the following fact: *If G is an n-vertex, not ℓ-hamiltonian graph, $uv \notin E(G)$, and $\deg(u)+\deg(v) \geq n+\ell$, then $G+uv$ is also not ℓ-hamiltonian.* Since this result implies both Theorems 1 and 6, to make this paper self-contained, we include here a sketch of their proof.

Suppose to the contrary, that G has no hamiltonian cycle containing some linear forest F with ℓ edges but $G+uv$ has a hamiltonian cycle through F. Then we can order the vertices so that G has a hamiltonian path $w_1w_2\dots w_n$ containing F, where $w_1 = u$ and $w_n = v$. Let $N_G(u) = \{w_{i_1},\dots,w_{i_k}\}$ where $k = \deg_G(u)$. If there exists a $1 \leq j \leq k$ such that $w_{i_j-1} \in N(v)$ and $w_{i_j}w_{i_j-1} \notin E(F)$, then

$$w_1w_2\dots w_{i_j-1}w_nw_{n-1}\dots w_{i_j}w_1$$

is a hamiltonian cycle in G containing F. So for each $1 \leq j \leq k$, either $w_{i_j-1}w_{i_j} \in F$, or $w_{i_j-1} \notin N(v)$. Since F has ℓ edges, there are $k-\ell$ choices of $w_{i_j-1} \in V(G)\setminus\{v\}$ satisfying $vw_{i_j-1} \notin E(G)$. This gives $\deg_G(v) \leq (n-1)-(k-\ell)$ yielding the contradiction $\deg(u)+\deg(v) \leq n+\ell-1$.

To complete the proof of Theorem 10 suppose that G is ℓ-saturated (note that G is not ℓ-hamiltonian) and suppose that $uv \notin E(G)$. If $\deg(u)+\deg(v) \geq n+\ell$, then $G+uv$ is not ℓ-hamiltonian, a contradiction. □

We show a useful feature of the structure of saturated graphs with many r-cliques.

Lemma 11. *Let G be an ℓ-saturated n-vertex graph with $N(G,K_r) > h_r(n, \lfloor \frac{n+\ell-1}{2} \rfloor, \ell)$. Then for some $\ell < k < \lfloor \frac{n+\ell-1}{2} \rfloor$, $V(G)$ contains a subset D of $k-\ell$ vertices of degree at most k such that $G-D$ is a complete graph.*

Proof. By Lemma 8 (8.1), there is a subset of $k-\ell$ vertices of degree at most k such that $\ell < k \leq \lfloor \frac{n+\ell-1}{2} \rfloor$. Choose the maximum such k, and let D be the set of the vertices with degree at most k. Then Lemma 8 (8.2) implies that $k < \lfloor \frac{n+\ell-1}{2} \rfloor$. The maximality of k gives $|D| = k-\ell$.

Suppose there exist $x, y \in V(G)-D$ such that $xy \notin E(G)$. Among all such pairs, choose x and y with the maximum $\deg(x)$. Let $D' := V(G)-N(x)-\{x\}$. We have $|D'| = n-\deg(x)-1 > 0$. By Theorem 10,

$$\deg(z) \leq n+\ell-1-\deg(x) = |D'|+\ell =: k' \quad \textit{for all} \quad z \in D'.$$

So D' is a set of $k'-\ell$ vertices of degree at most k'. Since $y \in D'\setminus D$, $k' \geq \deg(y) > k$. Thus by the maximality of k, we get $k' = n+\ell-1-\deg(x) > \lfloor \frac{n+\ell-1}{2} \rfloor$. Equivalently, $\deg(x) < \lceil \frac{n+\ell-1}{2} \rceil$, i.e., $\deg(x) \leq \lfloor \frac{n+\ell-1}{2} \rfloor$. We also obtain $|D'| = n-1-\deg(x) > n-1-\lceil \frac{n+\ell-1}{2} \rceil = \lfloor \frac{n+\ell-1}{2} \rfloor - \ell$.

For all $z \in D'$, either $z \in D$ where $\deg(z) \leq k \leq \lfloor \frac{n+\ell-1}{2} \rfloor$, or $z \in V(G)-D$, and so $\deg(z) \leq \deg(x) \leq \lfloor \frac{n+\ell-1}{2} \rfloor$. It follows that D' has a subset of $\lfloor \frac{n+\ell-1}{2} \rfloor - \ell$ vertices of degree at most $\lfloor \frac{n+\ell-1}{2} \rfloor$. This contradicts Lemma 8 (8.2).

Thus, $G-D$ is a complete graph. □

Lemma 12. *Under the conditions of Lemma 11, if $k = \delta(G)$, then $G = H_{n,\delta(G),\ell}$ or $G = H'_{n,\delta(G),\ell}$.*

Proof. Set $d := \delta(G)$, and let D be a set of $d-\ell$ vertices with degree at most d. Let $u \in D$. Since $\delta(G) \geq |D|+\ell = d$, u has a neighbor w outside of D. Consider any $v \in D-\{u\}$. By Lemma 11, w is adjacent to all of $V(G)-D-\{w\}$. It also is adjacent to u, therefore its degree is at least $(n-d+\ell-1)+1 = n-d+\ell$. We obtain

$$\deg(w)+\deg(v) \geq (n-d+\ell)+d = n+\ell.$$

Then by (10), w is adjacent to v, and hence w is adjacent to all vertices of D.

Let W be the set of vertices in $V(G)-D$ having a neighbor in D. We have obtained that $|W| \geq \ell+1$ and

$$N(u)\cap(V(G)-D) = W \textit{ for all } u \in D. \tag{4}$$

Let $G' = G[D\cup W]$. Since $|W| \leq \delta(G)$, $|V(G')| \leq 2d-\ell$. If $|V(G')| = 2d-\ell$, then by (4), each vertex $u \in D$ has the same d neighbors in $V(G)-D$. Because $\deg(u) = d$, D is an independent set. Thus $G = H_{n,d,\ell}$. Otherwise, $(\ell+1+d-\ell) \leq |V(G')| \leq 2d-\ell-1$.

If $|V(G')| = d+1$, then $|W| = \ell+1$. Because $\delta(G) \geq d$, each vertex in D has at least $d-1$ neighbors in D. But this implies that D is a clique, and $G = H'_{n,d,\ell}$.

So we may assume $d+2 \leq |V(G')| \leq 2d-\ell-1$. That is, $|W| \geq \ell+2$. We will show that in this case G is ℓ-hamiltonian, a contradiction.

Let F be a linear forest in G with ℓ edges. Set $F_1 := F\cap G'$ and $F_2 := F-F_1$. Let ab be an edge in $G[W]$ such that $ab \notin E(F)$ and $F\cup ab$ is a linear forest in G. Such an edge exists because $G[W]$ is a clique and either F_1 is a path that occupies at most $\ell+1$ vertices in W, or F_1 is a disjoint union of paths and we can choose ab to join endpoints of two different components of F_1.

For any $x, x' \in V(G')$,

$$\deg_{G'}(x)+\deg_{G'}(x') \geq d+d \geq |V(G')|+\ell+1.$$

Therefore by Theorem 1, G' has a hamiltonian cycle C that passes through $F_1\cup ab$. In particular, we obtain a hamiltonian (a,b)-path P_1 in G' that passes through F_1. Since $G'' := G-(V(G')-\{a,b\})$ is a complete graph, it contains a hamiltonian (a,b)-path P_2 that passes through F_2. Then $P_1\cup P_2$ is a hamiltonian cycle in G containing F, a contradiction. □

Proof of Theorem 5. Let G' be obtained by adding edges to G until it is ℓ-saturated. If $N(G', K_r) \le h_r(n, \lfloor \frac{n+\ell-1}{2} \rfloor, \ell)$, then we are done. Otherwise, by (8.2) in Lemma 8, G' contains a set of $k-\ell$ vertices with degree at most k where $\ell < k < \lfloor (n+\ell-1)/2 \rfloor$. If $k = d$, then $G' \in \{H_{n,d,\ell}, H'_{n,d,\ell}\}$ by Lemma 12, and thus G is a subgraph of one of these two graphs. If $k \ge d+1$, then $N(G, K_r) \le N(G', K_r) \le h_r(n, k, \ell)$ for some $d+1 \le k < \lfloor \frac{n+\ell-1}{2} \rfloor$ by (8.3) in Lemma 8.

So the convexity by Claim 9 gives that in both cases

$$N(G, K_r) \le \max_{k \in \left[d+1, \lfloor \frac{n+\ell-1}{2} \rfloor\right]} h_r(n, k, \ell) = \max\left\{ h_r(n, d+1, \ell), h_r(n, \left\lfloor \frac{n+\ell-1}{2} \right\rfloor, \ell) \right\},$$

a contradiction. □

Concluding remarks

In the proof of Theorem 4 for the upper bound for $N(G, K_r)$ we only use the degree sequence of G. It seems to be likely that one can obtain similar results for graph classes whose degree sequences are well understood.

Let P be a property defined on all graphs of order n and let k be a nonnegative integer. Bondy and Chvátal [1] call P to be *k-stable* if whenever $G+uv$ has property P and $\deg_G(u)+\deg_G(v) \ge k$, then G itself has property P. The k-closure $\mathrm{Cl}_k(G)$ of a graph G is the (unique) smallest graph H of order n such that $E(G) \subseteq E(H)$ and $\deg_H(u)+\deg_H(v) < k$ for all $uv \notin E(H)$. The k-closure can be obtained from G by a recursive procedure of joining nonadjacent vertices with degree-sum at least k. Thus, if P is k-stable and $\mathrm{Cl}_k(G)$ has property P, then G itself has property P. They prove k-stability (with appropriate values of k) for a series of graph properties, e.g., G contains C_s ($k = 2n-s$), G contains a path P_s ($k = n-1$), G contains a matching sK_2 ($k = 2s-1$), G contains a spanning s-regular subgraph ($k = n+2s-4$), G is s-connected ($k = n+s-2$), G is s-wise hamiltonian, i.e., every $n-s$ vertices span a C_{n-s} ($k = n+s-2$).

These graph classes are good candidates to find the maximum number of r-cliques in graphs not in them. But the proof of stability (like in Theorem 5) might require more insight.

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References

[1] J.A. Bondy, V. Chvátal, A method in graph theory, Discrete Math. 15 (1976) 111–135.
[2] P. Erdős, Remarks on a paper of Pósa, Magyar Tud. Akad. Mat. Kutató Int. Közl. 7 (1962) 227–229.
[3] Z. Füredi, A. Kostochka, R. Luo, A stability version for a theorem of Erdős on nonhamiltonian graphs, Discrete Math. 340 (2017) 2688–2690, arXiv:1608.05741.
[4] H.N. Kronk, Variations on a theorem of Pósa, in: The Many Facets of Graph Theory (Proc. Conf., Western Mich. Univ., Kalamazoo, Mich., 1968), Springer, Berlin, 1968, pp. 193–197.
[5] B. Li, B. Ning, Spectral analogues of Erdős' and Moon-Moser's theorems on Hamilton cycles, Linear Multilinear Algebra 64 (2016) 1152–1169.
[6] R. Luo, The maximum number of cliques in graphs without long cycles, J. Combin. Theory Ser. B 128 (2018) 219–226.
[7] J. Ma, B. Ning, Stability results on the circumference of a graph, arXiv:1708.00704, p. 31.
[8] B. Ning, X. Peng, Extensions of Erdős-Gallai theorem and Luo's theorem with applications, arXiv:1801.09981, p. 7.
[9] L. Pósa, On the circuits of finite graphs, Magyar. Tud. Akad. Mat. Kutató Int. Kőzl. 8 (1963/1964) 355–361.