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Stability in the Erdős–Gallai Theorem on cycles and paths, II[☆]

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ABSTRACT

The Erdős–Gallai Theorem states that for $k \geq 3$, any n-vertex graph with no cycle of length at least k has at most $\frac{1}{2}(k-1)(n-1)$ edges. A stronger version of the Erdős–Gallai Theorem was given by Kopylov: If G is a 2-connected n-vertex graph with no cycle of length at least k, then $e(G) \leq \max\{h(n, k, 2), h(n, k, \lfloor \frac{k-1}{2} \rfloor)\}$, where $h(n, k, a) := \binom{k-a}{2} + a(n-k+a)$. Furthermore, Kopylov presented the two possible extremal graphs, one with $h(n, k, 2)$ edges and one with $h(n, k, \lfloor \frac{k-1}{2} \rfloor)$ edges.

In this paper, we complete a stability theorem which strengthens Kopylov's result. In particular, we show that for $k \geq 3$ odd and all $n \geq k$, every n-vertex 2-connected graph G with no cycle of length at least k is a subgraph of one of the two extremal graphs or $e(G) \leq \max\{h(n, k, 3), h(n, k, \frac{k-3}{2})\}$. The upper bound for $e(G)$ here is tight.

1. Introduction

One of the fundamental questions in extremal graph theory is to determine the maximum number of edges in an n-vertex graph with no k-vertex path. According to [8], this problem was posed by Turán. A solution to the problem was obtained by Erdős and Gallai [4]:

Theorem 1.1 (*Erdős and Gallai [4]*)**.** *Let G be an n-vertex graph with more than $\frac{1}{2}(k-2)n$ edges, $k \geq 2$. Then G contains a k-vertex path P_k.*

Theorem 1.1 can be proved as a corollary of the following theorem about cycles in graphs:

Theorem 1.2 (*Erdős and Gallai [4]*)**.** *Fix $n, k \geq 3$. If G is an n-vertex graph that does not contain a cycle of length at least k, then $e(G) \leq \frac{1}{2}(k-1)(n-1)$.*

The bound of Theorem 1.2 is best possible for $n-1$ divisible by $k-2$. Indeed, any connected n-vertex graph in which every block is a K_{k-1} has $\frac{1}{2}(k-1)(n-1)$ edges and no cycles of length at least k. In the 1970s, some refinements and new proofs of Theorem 1.2 were obtained by Faudree and Schelp [6,5], Lewin [10], and Woodall [11]—see [8] for more details. The strongest version was proved by Kopylov [9]. His result uses the following n-vertex graphs $H_{n,k,a}$, where $n \geq k$ and $1 \leq a < \frac{1}{2}k$. The vertex set of $H_{n,k,a}$ is the union of three disjoint sets A, B, and C such that $|A| = a$, $|B| = n-k+a$ and

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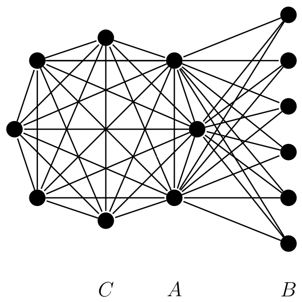

Fig. 1. $H_{14,11,3}$.

$|C| = k - 2a$, and the edge set of $H_{n,k,a}$ consists of all edges between A and B together with all edges in $A \cup C$ (Fig. 1 shows $H_{14,11,3}$). Let

$$h(n, k, a) := e(H_{n,k,a}) = \binom{k-a}{2} + a(n - k + a).$$

For a graph G containing a cycle, the *circumference*, $c(G)$, is the length of a longest cycle in G. Observe that $c(H_{n,k,a}) < k$: Since $|A \cup C| = k - a$, any cycle D of at length at least k has at least a vertices in B. But as B is independent and $2a < k$, D also has to contain at least $k + 1$ neighbors of the vertices in B, while only a vertices in A have neighbors in A. Kopylov [9] showed that the extremal 2-connected n-vertex graphs with no cycles of length at least k are $G = H_{n,k,2}$ and $G = H_{n,k,t}$ (see Fig. 2): the first has more edges for small n, and the second has more edges for large n.

Theorem 1.3 (*Kopylov [9]*). *Let $n \geq k \geq 5$ and $t = \lfloor \frac{1}{2}(k-1) \rfloor$. If G is an n-vertex 2-connected graph with $c(G) < k$, then*

$$e(G) \leq \max\{h(n, k, 2), h(n, k, t)\} \tag{1}$$

with equality only if $G = H_{n,k,2}$ or $G = H_{n,k,t}$.

Kopylov's theorem also implies Theorem 1.2 by applying induction to each block of a graph.

2. Results

2.1. A previous result

Recently, three of the present authors proved in [7] a stability version of Theorems 1.2 and 1.3 for n-vertex 2-connected graphs with $n \geq 3k/2$, but the problem remained open for $n < 3k/2$ when $k \geq 9$. The main result of [7] was the following:

Theorem 2.1 (*Füredi, Kostochka, Verstraëte [7]*). *Let $t \geq 2$ and $n \geq 3t$ and $k \in \{2t+1, 2t+2\}$. Let G be a 2-connected n-vertex graph $c(G) < k$. Then $e(G) \leq h(n, k, t-1)$ unless*

(*a*) *$k = 2t + 1$, $k \neq 7$, and $G \subseteq H_{n,k,t}$ or*
(*b*) *$k = 2t + 2$ or $k = 7$, and $G - A$ is a star forest for some $A \subseteq V(G)$ of size at most t.*

2.2. The essence of the main result

The paper [7] also describes the 2-connected n-vertex graphs G with $e(G) > h(n, k, t-1)$ and $c(G) < k \leq 8$ for all $n \geq k$. In particular, for $k < 8$, each such graph satisfies either (*a*) or (*b*) of Theorem 2.1.

Together with the cases for $k \leq 8$, this paper gives a full description of the 2-connected n-vertex graphs G with $c(G) < k$ and 'many' edges for all k and n. Our main result is:

Theorem 2.2. *Let $t \geq 4$ and $k \in \{2t+1, 2t+2\}$, so that $k \geq 9$. If G is a 2-connected graph on $n \geq k$ vertices and $c(G) < k$, then either $e(G) \leq \max\{h(n, k, t-1), h(n, k, 3)\}$ or*

(*a*) *$k = 2t + 1$ and $G \subseteq H_{n,k,t}$ or*
(*b*) *$k = 2t + 2$ and $G - A$ is a star forest for some $A \subseteq V(G)$ of size at most t.*
(*c*) *$G \subseteq H_{n,k,2}$.*

Note that

$$h(n, k, t) - h(n, k, t-1) = \begin{cases} n - t - 3 & \text{if } k = 2t + 1, \\ n - t - 5 & \text{if } k = 2t + 2, \end{cases}$$

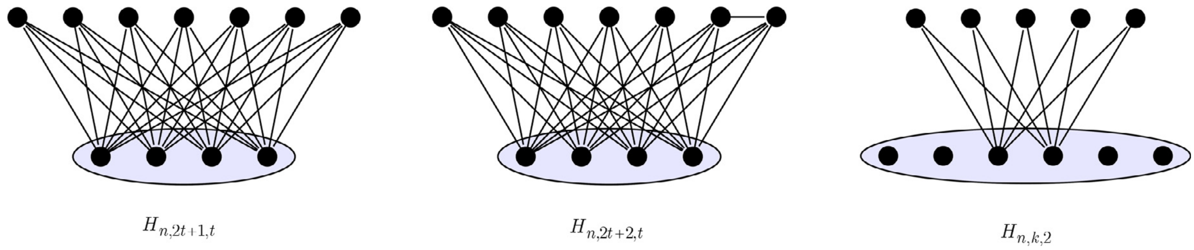

Fig. 2. Ovals denote complete subgraphs of order t, t, and $k-2$.

and

$$h(n,k,2) - h(n,k,3) = k - n - 3.$$

We consider the case $e(G) > h(n,k,t-1)$ whenever n is large compared to k (and t), and $e(G) > h(n,k,3)$ whenever n is small. We state these exact bounds in Section 3.

Also, note that the case $n < k$ is trivial and the case $k \le 8$ was fully resolved in [7].

We will reuse many slightly modified lemmas from [7] in the proof of the main result. As such, when introducing such lemmas, instead of repeating the proofs word-for-word, we provide brief proof sketches and a reference to the corresponding full proof in [7] for the interested reader.

2.3. A more detailed form of the main result

In order to prove Theorem 2.2, we need a more detailed description of the graphs satisfying (b) in the theorem that do not contain 'long' cycles. For this, we introduce four families of graphs $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, and $\mathcal{G}_4$ that (apart from $\mathcal{G}_1$) are identical to the families introduced in [7]. In the definitions below we use $t = \lfloor (k-1)/2 \rfloor$.

Let $\mathcal{G}_1(n,k) = \{H_{n,k,t}, H_{n,k,2}\}$. Each $G \in \mathcal{G}_2(n,k)$ is defined by a partition $V(G) = A \cup B \cup C$ and two vertices $a_1 \in A$, $b_1 \in B$ such that

- $|A| = t$,
- $G[A] = K_t$,
- $G[B]$ is the empty graph,
- $G(A,B)$ is a complete bipartite graph, and
- $N(c) = \{a_1, b_1\}$ for every $c \in C$.

Every graph $G \in \mathcal{G}_3(n,k)$ is defined by a partition $V(G) = A \cup B \cup J$ such that $|A| = t$, $G[A] = K_t$, $G(A,B)$ is a complete bipartite graph, and

- $G[J]$ has more than one component,
- all components of $G[J]$ are stars with at least two vertices each,
- there is a 2-element subset A' of A such that $N(J) \cap (A \cup B) = A'$,
- for every component S of $G[J]$ with at least 3 vertices, all leaves of S have degree 2 in G and are adjacent to the same vertex $a(S)$ in A'.

The class $\mathcal{G}_4(n,k)$ is empty unless $k = 10$. Each graph $H \in \mathcal{G}_4(n,10)$ has a 3-vertex set A such that $H[A] = K_3$ and $H - A$ is a star forest such that if a component S of $H - A$ has more than two vertices then all its leaves have degree 2 in H and are adjacent to the same vertex $a(S)$ in A.

These classes are illustrated in Fig. 3.

Now we define $\mathcal{G}(n,k)$ as follows:

(1) if k is odd, then $\mathcal{G}(n,k) = \mathcal{G}_1(n,k) = \{H_{n,k,t}, H_{n,k,2}\}$;
(2) if k is even and $k \ne 10$, then $\mathcal{G}(n,k) = \mathcal{G}_1(n,k) \cup \mathcal{G}_2(n,k) \cup \mathcal{G}_3(n,k)$;
(3) if $k = 10$, then $\mathcal{G}(n,k) = \mathcal{G}_1(n,10) \cup \mathcal{G}_2(n,10) \cup \mathcal{G}_3(n,10) \cup \mathcal{G}_4(n,10)$.

In these terms, we get the following refinement of Theorem 2.2:

Theorem 2.3 (*Main Theorem*)**.** *Let $k \ge 9$, $n \ge k$ and $t = \lfloor \frac{1}{2}(k-1) \rfloor$. Let G be an n-vertex 2-connected graph with no cycle of length at least k. Then either $e(G) \le \max\{h(n,k,t-1), h(n,k,3)\}$ or G is a subgraph of a graph in $\mathcal{G}(n,k)$.*

Since every graph in $\mathcal{G}_2(n,k) \cup \mathcal{G}_3(n,k)$ and many graphs in $\mathcal{G}_4(n,k)$ have a separating set of size 2 (see Fig. 4), the theorem implies the following simpler statement for 3-connected graphs:

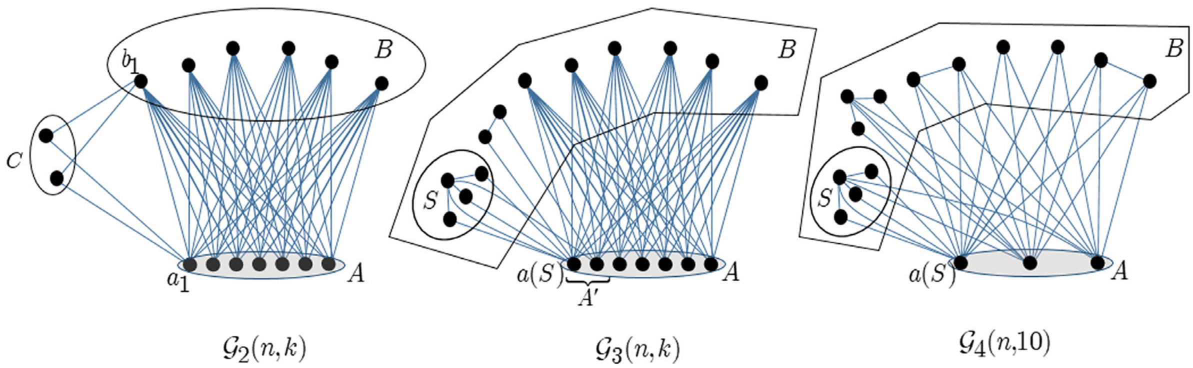

Fig. 3. Examples of graphs in classes $\mathcal{G}_2(n, k)$, $\mathcal{G}_3(n, k)$, and $\mathcal{G}_4(n, 10)$, respectively.

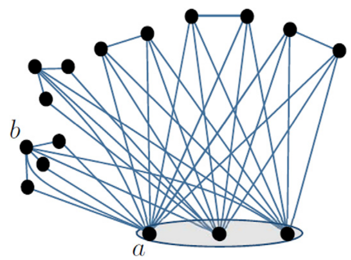

Fig. 4. The set $\{a, b\}$ forms a separating set of the graph.

Corollary 2.4. *Let $k \in \{2t+1, 2t+2\}$ where $k \geq 9$. If G is a 3-connected graph on $n \geq k$ vertices and $c(G) < k$, then either $e(G) \leq \max\{h(n, k, t-1), h(n, k, 3)\}$ or*

(1) *$G \subseteq H_{n,k,t}$, or*
(2) *$k = 10$ and G is a subgraph of some graph $H \in \mathcal{G}_4(n, 10)$ such that each component of $H - A$ has at most 2 vertices.*

3. The setup and ideas

3.1. Small dense subgraphs

First we define some more graph classes (also defined identically to [7]). For a graph F and a nonnegative integer s, we denote by $\mathcal{K}^{-s}(F)$ the family of graphs obtained from F by deleting at most s edges.

Let $F_0 = F_0(t)$ denote the complete bipartite graph $K_{t,t+1}$ with partite sets A and B where $|A| = t$ and $|B| = t+1$. Let $\mathcal{F}_0 = \mathcal{K}^{-t+3}(F_0)$, i.e., the family of subgraphs of $K_{t,t+1}$ with at least $t(t+1) - t + 3$ edges.

Let $F_1 = F_1(t)$ denote the complete bipartite graph $K_{t,t+2}$ with partite sets A and B where $|A| = t$ and $|B| = t+2$. Let $\mathcal{F}_1 = \mathcal{K}^{-t+4}(F_1)$, i.e., the family of subgraphs of $K_{t,t+2}$ with at least $t(t+2) - t + 4$ edges.

Let $\mathcal{F}_2$ denote the family of graphs obtained from a graph in $\mathcal{K}^{-t+4}(F_1)$ by subdividing an edge a_1b_1 with a new vertex c_1, where $a_1 \in A$ and $b_1 \in B$. Note that any member $H \in \mathcal{F}_2$ has at least $|A||B| - (t-3)$ edges between A and B and the pair a_1b_1 is not an edge.

Let $F_3 = F_3(t, t')$ denote the complete bipartite graph $K_{t,t'}$ with partite sets A and B where $|A| = t$ and $|B| = t'$. Take a graph from $\mathcal{K}^{-t+4}(F_3)$, select two non-empty subsets $A_1, A_2 \subseteq A$ with $|A_1 \cup A_2| \geq 3$ such that $A_1 \cap A_2 = \emptyset$ if $\min\{|A_1|, |A_2|\} = 1$, add two vertices c_1 and c_2, join them to each other and add the edges from c_i to the elements of A_i, $(i = 1, 2)$. The class of obtained graphs is denoted by $\mathcal{F}(A, B, A_1, A_2)$. The family $\mathcal{F}_3$ consists of these graphs when $|A| = |B| = t$, $|A_1| = |A_2| = 2$ and $A_1 \cap A_2 = \emptyset$. In particular, $\mathcal{F}_3(4)$ consists of exactly one graph, call it $F_3(4)$.

Graph F_4 has vertex set $A \cup B$, where $A = \{a_1, a_2, a_3\}$ and $B := \{b_1, b_2, \ldots, b_6\}$ are disjoint. Its edges are the edges of the complete bipartite graph $K(A, B)$ and three extra edges b_1b_2, b_3b_4, and b_5b_6 (see Fig. 4). Define F_4' as the (only) member of $\mathcal{F}(A, B, A_1, A_2)$ such that $|A| = |B| = t = 4$, $A_1 = A_2$, and $|A_i| = 3$. Let $\mathcal{F}_4 := \{F_4, F_4'\}$, which is defined only for $t = 4$ (see Fig. 5).

Define $\mathcal{F}(k) := \begin{cases} \mathcal{F}_0, & \text{if } k \text{ is odd,} \\ \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_4, & \text{if } k \text{ is even.} \end{cases}$

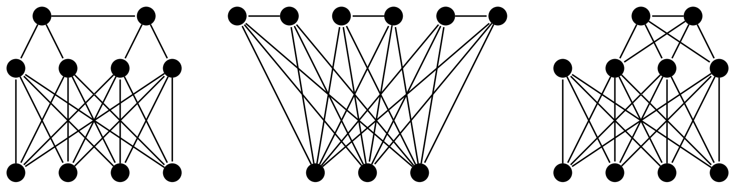

Fig. 5. Graphs $F_3(4)$, F_4, and F'_4.

3.2. Proof idea

In order to employ a stronger induction assumption, we will prove the following slightly stronger version of Theorem 2.3 claiming that the graphs in question contain dense graphs from $\mathcal{F}(k)$:

Theorem 2.3′. *Let $t \geq 4$, $k \in \{2t+1, 2t+2\}$, and $n \geq k$. Let G be an n-vertex 2-connected graph with no cycle of length at least k. Then either $e(G) \leq \max\{h(n,k,t-1), h(n,k,3)\}$ or*

(*a*) $G \subseteq H_{n,k,2}$, *or*
(*b*) *G is contained in a graph in $\mathcal{G}(n,k) - \{H_{n,k,2}\}$, and G contains a subgraph $H \in \mathcal{F}(k)$,*

where $\mathcal{G}(n,k)$ is as defined in Section 2.3.

The method of the proof is a variation of that of [7] for larger n as well as Kopylov's disintegration method for n close to k. We take an n-vertex graph G satisfying the hypothesis of Theorem 2.3′, and iteratively contract edges in a certain way so that each intermediate graph still satisfies the hypothesis. We consider the final graph of this process G_m on m vertices and show that G_m satisfies Theorem 2.3′. We will use two instrumental lemmas from [7].

Lemma 3.1 (*Main Lemma on Contraction, Lemma 4.9 in [7]*). *Let $k \geq 9$ and suppose F and F' are 2-connected graphs such that $F = F'/xy$ and $c(F') < k$. If F contains a subgraph $H \in \mathcal{F}(k)$, then F' also contains a subgraph $H' \in \mathcal{F}(k)$.*

This lemma shows that if G_m contains a subgraph $H \in \mathcal{F}(k)$, then the original graph G also contains a subgraph in $\mathcal{F}(k)$. The second result concludes that the original graph $G = G_n$ must satisfy (*b*) of Theorem 2.3′. For the full proof of the lemma, we refer the reader to [7]. Below we include a brief sketch of the proof.

Lemma 3.2 (*[7](Subsection 4.5)*). *Let $k \geq 9$, and let G be a 2-connected graph with $c(G) < k$ and $e(G) > h(n,k,t-1)$. If G contains a subgraph $H \in \mathcal{F}(k)$, then G is a subgraph of a graph in $\mathcal{G}(n,k) - \{H_{n,k,2}\}$.*

Sketch of proof. Consider a component of S of $G - H$. Because G is 2-connected, S has at least two neighbors, say x and y in H. Let ℓ be the length of a longest (x,y)-path P such that all internal vertices in P are in S. When k is odd, since H is "close" to $K_{t,t+1}$, it contains a long path P' from x to y. Thus if ℓ is too large, $P' \cup P$ yields a cycle of length k or longer, a contradiction. Then one can show that $\ell = 2$ (edges). That is, each path from H to H that goes through S has only one internal vertex. Thus $|V(S)| = 1$ and moreover, x and y both lie in the partite set of H of size t. This shows that $G \subseteq H_{n,k,t}$. The case for k even is handled similarly (but with more subcases; in particular we have $\ell \leq 3$). We obtain that either $G \subseteq H_{n,k,t}$ or the components of $G - H$ are star forests that connect to H in the ways described in the classes $\mathcal{G}_i(n,k)$, $i \in \{2,3,4\}$, otherwise G would contain a cycle of length k or longer. □

We will split the proof into the cases of small n and large n. The following observations can be obtained by simple calculations (for $t \geq 4$):

k	$h(n,k,3) \geq h(n,k,t-1)$	$h(n,k,2) \geq h(n,k,t-1)$
$2t+1$	If and only if $n \leq k + (t-5)/2$	If and only if $n \leq k + t/2 - 1$
$2t+2$	If and only if $n \leq k + (t-3)/2$	If and only if $n \leq k + t/2$

In the case of large n we will contract an edge such that the new graph still has more than $h(n-1,k,t-1)$ edges. In order to apply induction, we also need the number of edges to be greater than $h(n-1,k,3)$. To guarantee this, we pick the cutoffs for the two cases $n \leq k + (t-1)/2$ and $n > k + (t-1)/2$ (therefore $n - 1 > k + (t-3)/2$).

4. Tools

4.1. Classical theorems

Theorem 4.1 (*Erdős [3]*). *Let $d \geq 1$ and $n > 2d$ be integers, and*

$$\ell_{n,d} = \max\left\{\binom{n-d}{2} + d^2, \binom{\lceil \frac{n+1}{2} \rceil}{2} + \left\lfloor \frac{n-1}{2} \right\rfloor^2\right\}.$$

Then every n-vertex graph G with $\delta(G) \geq d$ and $e(G) > \ell_{n,d}$ is hamiltonian. □

Theorem 4.2 (*Chvátal [1]*). *Let $n \geq 3$ and G be an n-vertex graph with vertex degrees $d_1 \leq d_2 \leq \cdots \leq d_n$. If G is not hamiltonian, then there is some $i < n/2$ such that $d_i \leq i$ and $d_{n-i} < n - i$.* □

Theorem 4.3 (*Kopylov [9]*). *If G is 2-connected and P is an x, y-path of ℓ vertices, then $c(G) \geq \min\{\ell, d(x, P) + d(y, P)\}$.* □

4.2. Claims on contractions

A helpful tool will be the following lemma from [7] on contraction.

Lemma 4.4 (*Lemma 3.2 in [7]*). *Let $n \geq 4$ and let G be an n-vertex 2-connected graph. For every $v \in V(G)$, there exists $w \in N(v)$ such that G/vw is 2-connected.* □

For an edge xy in a graph H, let $T_H(xy)$ denote the number of triangles containing xy. Let $T(H) = \min\{T_H(xy) : xy \in E(H)\}$. When we contract an edge uv in a graph H, the degree of every $x \in V(H) - u - v$ either does not change or decreases by 1. Also if $u * v$ is the vertex created upon contraction, then the degree of $u * v$ in H/uv is at least $\max\{d_H(u), d_H(v)\} - 1$. Thus

$$d_{H/uv}(w) \geq d_H(w) - 1 \text{ for any } w \in V(H) \text{ and } uv \in E(H). \text{ Also } d_{H/uv}(u * v) \geq d_H(u) - 1. \tag{2}$$

Similarly,

$$T(H/uv) \geq T(H) - 1 \text{ for every graph } H \text{ and } uv \in E(H). \tag{3}$$

We will use the following analog of Lemma 3.3 in [7].[1]

Lemma 4.5. *Let h be a positive integer. Suppose a 2-connected graph G is obtained from a 2-connected graph G' by contracting edge xy into $x * y$ chosen using the following rules:*

(i) one of x, y, say x is a vertex of the minimum degree in G';

(ii) $T_{G'}(xy)$ is the minimum among the edges xu incident with x such that G'/xu is 2-connected. If G has at least h vertices of degree at most h, then either $G' = K_{h+2}$ or

(a) G' also has a vertex of degree at most h, and

(b) G' has at least $h + 1$ vertices of degree at most $h + 1$.

Proof. Note that in (*ii*), such edges exist by Lemma 4.4. Since G is 2-connected, $h \geq 2$.

Below for a positive integer s and a graph H, by $V_{\leq s}(H)$ we denote the set of vertices of degree at most s in H. Then by (2), each $v \in V_{\leq h}(G) - x * y$ is also in $V_{\leq h+1}(G')$. Moreover, then by (i),

$$x \in V_{\leq h+1}(G'). \tag{4}$$

Thus if $x * y \notin V_{\leq h}(G)$, then (b) follows. But if $x * y \in V_{\leq h}(G)$, then by (2), also $y \in V_{\leq h+1}(G')$. So, again (b) holds.

If $V_{\leq h-1}(G) \neq \emptyset$, then (a) holds by (2). So, if (a) does not hold, then

$$\text{each } v \in V_{\leq h}(G) - x * y \text{ has degree } h + 1 \text{ in } G' \text{ and is adjacent to both } x \text{ and } y \text{ in } G'. \tag{5}$$

Case 1: $|V_{\leq h}(G) - x * y| \geq h$. Then by (4), $d_{G'}(x) = h + 1$. This in turn yields $N_{G'}(x) = V_{\leq h}(G) + y$. Since G' is 2-connected, each $v \in V_{\leq h}(G) - x * y$ is not a cut vertex. Furthermore, $\{x, v\}$ is not a cut set. If it was, because y is a common neighbor of all neighbors of x, all neighbors of x must be in the same component as y in $G' - x - v$. It follows that

$$\text{for every } v \in V_{\leq h}(G) - x * y, G'/vx \text{ is 2-connected.} \tag{6}$$

If $uv \notin E(G)$ for some $u, v \in V_{\leq h}(G)$, then by (6) and (i), we would contract the edge xu rather than xy. Thus $G'[V_{\leq h}(G) \cup \{x, y\}] = K_{h+2}$ and so either $G' = K_{h+2}$ or y is a cut vertex in G', as claimed.

Case 2: $|V_{\leq h}(G) - x * y| = h - 1$. Then $x * y \in V_{\leq h}(G)$. This means $d_{G'}(x) = d_{G'}(y) = h + 1$ and $N_{G'}[x] = N_{G'}[y]$. So by (5), there is $z \in V(G)$ such that $N_{G'}[x] = N_{G'}[y] = V_{\leq h}(G) \cup \{x, y, z\}$. Again (6) holds (for the same reason that $N_{G'}[x] \subseteq N_{G'}[y]$). Thus similarly $vu \in E(G')$ for every $v \in V_{\leq h}(G) - x * y$ and every $u \in V_{\leq h}(G) + z$. Hence $G'[V_{\leq h}(G) \cup \{x, y, z\}] = K_{h+2}$ and either $G' = K_{h+2}$ or z is a cut vertex in G', as claimed. □

4.3. A property of graphs in $\mathcal{F}(k)$

A useful feature of graphs in $\mathcal{F}(k)$ is the following.

Lemma 4.6. *Let $k \geq 9$ and $n \geq k$. Let F be an n-vertex graph contained in $H_{n,k,t}$ with $e(F) > h(n, k, t - 1)$. Then F contains a graph in $\mathcal{F}(k)$.*

[1] The difference between our analog and the original Lemma 3.3 in [7] is small: the rules we are following are slightly different, and we prove the additional property (b).

Proof. Assume the sets A, B, C to be as in the definition of $H_{n,k,t}$. We will use induction on n.

Case 1: $k = 2t + 1$. If $n = k$, then $F \in \mathcal{K}^{-t+3}(H_{k,k,t})$ because $h(k, k, t) - h(k, k, t - 1) - 1 = t - 3$. Thus, since $H_{k,k,t} \supseteq F_0(t)$, F contains a subgraph in $\mathcal{F}_0$. Suppose now the lemma holds for all $k \le n' < n$. If $\delta(F) \ge t$, then each $v \in V(F) - A$ is adjacent to every $u \in A$. Hence F contains $K_{t,n-t}$. If $\delta(F) < t$, then since A is dominating and $n > 2t$, there is $v \in V(F) - A$ with $d_F(v) \le t - 1$. Then $F - v \subseteq H_{n-1,k,t}$, and we are done by induction.

Case 2: $k = 2t + 2$. Let $C = \{c_1, c_2\}$. If $n = k$ then as in Case 1,

$$e(H_{k,k,t}) - e(F) \le h(k, k, t) - h(k, k, t - 1) - 1 = t - 4,$$

i.e., $F \in \mathcal{K}^{-t+4}(H_{k,k,t})$. Since $F_1(t) \subseteq H_{k,k,t}$, F contains a subgraph in $\mathcal{F}_1$. Suppose now the lemma holds for all $k \le n' < n$. If $\delta(F) < t$, then there is $v \in V(F) - A$ with $d_F(v) \le t - 1$. Then $F - v \subseteq H_{n-1,k,t}$, and we are done by induction.

Finally, suppose $\delta(F) \ge t$. So, each $v \in B$ is adjacent to every $u \in A$ and each of c_1, c_2 has at least $t - 1$ neighbors in A. Since $|B \cup \{c_1\}| \ge n - t - 1 \ge t + 2$, F contains a member of $\mathcal{K}^{-1}(F_1(t))$. Thus F contains a member of $\mathcal{F}_1$ unless $t = 4$, $n = 2t + 3$ and c_1 has a nonneighbor $x \in A$. But then $c_1c_2 \in E(F)$, and so F contains either $F_3(4)$ or F_4'. □

5. Proof of Theorem 2.3′

Let $n \ge k \ge 9$ and suppose Theorem 2.3′ holds for all graphs with n' vertices where $k \le n' < n$. Suppose further that

$$G \text{ is an } n\text{-vertex 2-connected graph with } c(G) < k \text{ and } e(G) > \max\{h(n, k, t - 1), h(n, k, 3)\}. \quad (7)$$

5.1. Contraction procedures

If $n > k$, we iteratively construct a sequence of graphs $G_n, G_{n-1}, \dots, G_m$ where $G_n = G$ and $|V(G_j)| = j$ for all $m \le j \le n$. In [7], the following **Basic Procedure** (BP) was used:

At the beginning of each round, for some $j : k \le j \le n$, we have a j-vertex 2-connected graph G_j with $e(G_j) > h(j, k, t - 1)$.

(R1) If $j = k$, then we stop.
(R2) If there is an edge uv with $T_{G_j}(uv) \le t - 2$ such that G_j/uv is 2-connected, choose one such edge so that
 (*i*) $T_{G_j}(uv)$ is minimum, and subject to this
 (*ii*) uv is incident to a vertex of minimum possible degree.
 Then obtain G_{j-1} by contracting uv.
(R3) If (R2) does not hold, $j \ge k + t - 1$ and there is $xy \in E(G_j)$ such that $G_j - x - y$ has at least 3 components and one of the components, say H_1 is a K_{t-1}, then let $G_{j-t+1} = G_j - V(H_1)$.
(R4) If neither (R2) nor (R3) occurs, then we stop.

Remark 5.1. By definition, (R3) applies only when $j \ge k + t - 1$. As observed in [7], if $j \le 3t - 2$, then a j-vertex graph G_j with a 2-vertex set $\{x, y\}$ separating the graph into at least 3 components cannot have $T_{G_j}(uv) \ge t - 1$ for every edge uv. It also was calculated there that if $3t - 1 \le j \le 3t$, then any j-vertex graph G' with such 2-vertex set $\{x, y\}$ and $T_{G'}(uv) \ge t - 1$ for every edge uv has at most $h(j, k, t - 1)$ edges and so cannot be G_j.

In this paper, we use a quite similar **Modified Basic Procedure** (MBP): start with a 2-connected, n-vertex graph $G = G_n$ with $e(G) > h(n, k, t - 1)$ and $c(G) < k$. Then

(MR0) if $\delta(G_j) \ge t$, then apply the rules (R1)–(R4) of (BP) given above;
(MR1) if $\delta(G_j) \le t - 1$ and $j = k$, then stop;
(MR2) otherwise, pick a vertex v of smallest degree, contract an edge vu with the minimum $T_{G_j}(vu)$ among the edges vu such that G_j/vu is 2-connected, and set $G_{j-1} = G_j/uv$.

5.2. Proof of Theorem 2.3′ for the case $n \le k + (t - 1)/2$

Let G satisfy (7). Apply to G the Modified Basic Procedure (MBP) starting from $G_n = G$. Denote by G_m the terminating graph of MBP. By Remark 5.1, (R3) was never applied, since $k + (t - 1)/2 < k + t - 1$. Therefore

$$\text{for each } m \le j < n \text{, graph } G_j \text{ is obtained from } G_{j+1} \text{ by contracting an edge}. \quad (8)$$

Then G_j is 2-connected and $c(G_j) \le c(G) < k$ for each $m \le j \le n$. By construction, after each contraction, we lose at most $t - 1$ edges. It follows that $e(G_m) > h(m, k, t - 1)$.

Suppose first that $m > k$. Then the same argument as in [7] gives us the following structural result:

Lemma 5.1 (*Proposition 4.2 in [7]*). *Let $m > k \ge 9$ and $n \ge k$.*

- *If $k \ne 10$, then $G_m \subseteq H_{m,k,t}$.*
- *If $k = 10$, then $G_m \subseteq H_{m,k,t}$ or $G_m \supseteq F_4$.*

Again we sketch the proof briefly and refer the reader to [7] for the full proof.

Sketch of proof. If $\delta(G_m) \leq t-1$, then either Rule (R2) or Rule (MR2) applies to G_m, so Procedure MBP does not stop, contradicting the definition of m. Thus $\delta(G_m) \geq t$. Since G_m is 2-connected, $c(G_m) \geq 2\delta(G_m) \geq 2t$. So if k is even, $c(G_m) \in \{2t, 2t+1\}$, and if k is odd, $c(G_m) = 2t$. For simplicity in this sketch, we only consider the odd case.

Let $C = v_1, \ldots, v_{2t}$ be a longest cycle in G_m. Because we could not apply rule (R2), for each edge $v_i v_{i+1}$ in C, either $v_i v_{i+1}$ is contained in at least $t-1$ triangles, or the set $\{v_i, v_{i+1}\}$ is separating in G_m. In the latter case, we show that C can be extended to a longer cycle. Thus the former holds. If $v_i v_{i+1} z$ is a triangle, then $z \in V(C)$, otherwise we get a longer cycle by including z. Thus we have shown that the induced subgraph $G[V(C)]$ has many edges, and furthermore it can be shown that $G[V(C)]$ is 3-connected. We then apply a structural theorem for 3-connected graphs due to Enomoto [2] (see, e.g. Theorem 2.7 in [7]) that yields three possible cases for the structure of $G[V(C)]$. In the first case, $\overline{K_t} + \overline{K_t} \subseteq G_m[V(C)] \subseteq K_t + \overline{K_t}$. In this case, by considering the connected components of $G_m - V(C)$ and the ways they connect to C, similarly to the proof of Lemma 3.2, we obtain $G_m \subseteq H_{m,k,t}$. In the other two cases, we either obtain $c(G) \geq k$ or $q < 2t$, a contradiction. □

Since $F_4 \in \mathcal{F}(k)$, if $k = 10$ and $G_m \supseteq F_4$, then G_m contains a subgraph in $\mathcal{F}(k)$. Otherwise, by Lemmas 4.6 and 5.1, again G_m has a subgraph in $\mathcal{F}(k)$. Then by (8) and Lemma 3.1, for every $m \leq j \leq n$, graph G_j contains a subgraph $H_j \in \mathcal{F}(k)$. In particular, $G = G_n$ contains such a subgraph. Thus by Lemma 3.2, G satisfies Theorem 2.3′.

So, below we assume

$$m = k. \tag{9}$$

Since $c(G_k) < k$, G_k does not have a hamiltonian cycle. Let $d_1 \leq d_2 \leq \cdots \leq d_k$ be the vertex degrees of G_k. By Theorem 4.2, there exists some $2 \leq i \leq t$ such that $d_i \leq i$ and $d_{k-i} < k-i$. Let $r = r(G_k)$ be the smallest such i.

Let R be a set of r vertices of degree at most r in G_k. Then

$$e(G_k) \leq r^2 + e(G_k - R) \leq r^2 + \binom{k-r}{2}.$$

For $k = 2t+1$, $r^2 + \binom{k-r}{2} > h(n, k, t-1)$ only when $r = t$ or $r < (t+4)/3$, and for $k = 2t+2$, when $r = t$ or $r < (t+6)/3$. If $r = r(G_k) = t$, then repeating the argument in [7] yields:

Lemma 5.2 (*Lemma 4.4 in [7]*). *If $r(G_k) = t$ then $G_k \subseteq H_{k,k,t}$.*

Sketch of proof. Since $c(G_k) < k$, G_k is nonhamiltonian. Let G' be the hamiltonian closure of G_k. Then $r(G')$ exists, and furthermore, $r(G') \geq r(G_k)$. Thus $r(G') = t$. Our goal is to show that $G' \subseteq H_{k,k,t}$. Let $V(G') = \{v_1, \ldots, v_k\}$ and $d'_i = d_{G'}(v_i)$ for $i = 1, \ldots, k$. Rename the vertices of G' so that $d'_1 \leq \cdots \leq d'_k$. By the definition of $r(G') = t$, $d'_1 \leq \cdots d'_t \leq t$. Let $A = \{v_k, v_{k-1}, \ldots, v_{k-t+1}\}$. If any vertex in A has too small degree, then we show $e(G_k) \leq h(k, k, t-1)$, a contradiction. Since G' is hamiltonian-closed, for each nonedge $xy \notin E(G')$,

$$d(x) + d(y) \leq |V(G')| - 1 = k - 1. \tag{10}$$

Using this, we show that $G'[A] = K_t$. Next, we consider the edges between $G' - A$ and A. If there are many non-edges, then applying (10) for each non-edge yields that $e(G') \leq h(k, k, t-1)$, so we finally show that every vertex in A but at most one is adjacent to every other vertex in G'. We focus here on the case that every vertex in A is adjacent to every other vertex. Then the neighborhood of every vertex of degree at most t is exactly A. If k is odd, we show that also $d'_{t+1} = t$ and so $G' = H_{k,k,t}$, since the vertices of $G' - A$ must form an independent set. The even case is proved similarly, but with more subcases. □

By Lemmas 4.6, 3.1, and 3.2, $G \subseteq H_{n,k,t}$ and contains some subgraph in $\mathcal{F}(k)$. This finishes the case $r = t$.

So we may assume that

$$\text{if } k = 2t+1 \text{ then } r < (t+4)/3, \text{ and if } k = 2t+2 \text{ then } r < (t+6)/3. \tag{11}$$

Our next goal is to show that G contains a large "core", i.e., a subgraph with large minimum degree. For this, we recall the notion of *disintegration* used by Kopylov [9].

Definition. For a natural number α and a graph G, the *α-disintegration* of a graph G is the process of iteratively removing from G the vertices with degree at most α until the resulting graph has minimum degree at least $\alpha + 1$. This resulting subgraph $H = H(G, \alpha)$ will be called the *α-core* of G.

It is well known that $H(G, \alpha)$ is unique and does not depend on the order of vertex deletion.

Claim 5.3. *The t-core $H(G, t)$ of G is nonempty.*

Proof of Claim 5.3. We may assume that for all $m \le j < n$, graph G_j was obtained from G_{j+1} by contracting edge x_jy_j, where $d_{G_{j+1}}(x_j) \le d_{G_{j+1}}(y_j)$. By Rule (MR2), $d_{G_{j+1}}(x_j) = \delta(G_{j+1})$, provided that $\delta(G_{j+1}) \le t-1$.

By definition, $|V_{\le r}(G_k)| \ge r$. So by Lemma 4.5 (applied several times), for each $k+1 \le j \le k+t-r$, because each G_j is not a complete graph (otherwise it would have a hamiltonian cycle),

$$\delta(G_j) \le j-k+r-1 \text{ and } |V_{\le j-k+r}(G_j)| \ge j-k+r. \tag{12}$$

To show that

$$\delta(G_j) \le t-1 \text{ for all } k \le j \le n, \tag{13}$$

by (12) and (11), it is enough to observe that

$$\delta(G_j) \le j-k+r-1 \le (n-k)+r-1 \le \frac{t-1}{2} + \frac{t+6}{3} - 1 = \frac{5t+3}{6} < t.$$

We will apply a version of t-disintegration in which we first manually remove a sequence of vertices and count the number of edges they cover. By (13) and (MR2), $d_{G_n}(x_{n-1}) = \delta(G_n) \le n-k+r-1$. Let $v_n := x_{n-1}$. Then $G - v_n$ is a subgraph of G_{n-1}. If $x_{n-2} \ne x_{n-1} * y_{n-1}$ in G_{n-1}, then let $v_{n-1} := x_{n-2}$, otherwise let $v_{n-1} := y_{n-1}$. In both cases, $d_{G-v_n}(v_{n-1}) \le n-k+r-2$. We continue in this way until $j = k$: each time we delete from the graph $G - v_n - \cdots - v_{j+1}$ the unique survived vertex v_j that was in the preimage of x_{j-1} when we obtained G_{j-1} from G_j. Graph $G - v_n - \cdots - v_{k+1}$ has $r \ge 2$ vertices of degree at most r. We additionally delete 2 such vertices v_k and v_{k-1}. Altogether, we have lost at most $(r+n-k-1)+(r+n-k-2)+\cdots+r+2r$ edges in the deletions.

Finally, apply t-disintegration to the remaining graph on $k-2 \in \{2t-1, 2t\}$ vertices. Suppose that the resulting graph is empty.

Case 1: $n = k$. Then

$$e(G) \le r + r + t(2t-1-t) + \binom{t}{2},$$

where $r + r$ edges are from v_k and v_{k-1}, and after deleting v_k and v_{k-1}, every vertex deleted removes at most t edges, until we reach the final t vertices which altogether span at most $\binom{t}{2}$ edges.

For $k = 2t+1$,

$$h(k,k,t-1) - e(G) \ge \binom{2t+1-(t-1)}{2} + (t-1)^2 - \left[r + r + t(2t-1-t) + \binom{t}{2}\right] = t+2-2r,$$

which is nonnegative for $r < (t+3)/3$. Therefore $e(G) \le h(k,k,t-1)$, a contradiction.

Similarly, if $k = 2t+2$,

$$e(G) \le r + r + t(2t-t) + \binom{t}{2},$$

and

$$h(k,k,t-1) - e(G) \ge \binom{2t+2-(t-1)}{2} + (t-1)^2 - [r + r + t(2t-t) + \binom{t}{2}] = t+4-2r,$$

which is nonnegative when $r < (t+6)/3$.

Case 2: $k < n \le k + (t-1)/2$. Then for $k = 2t+1$,

$$\begin{aligned} e(G) &\le \left[(r+n-k-1) + (r+n-k-2) + \cdots + r\right] + 2r + t(2t-1-t) + \binom{t}{2} \\ &\le \left[(t-1) + (t-1) + \cdots + (t-1)\right] + h(k,k,t-1) \\ &= (t-1)(n-k) + h(k,k,t-1) \\ &= h(n,k,t-1), \end{aligned}$$

where the last inequality holds because $r+n-k-1 \le t-1$.

Similarly, for $k = 2t+2$,

$$\begin{aligned} e(G) &\le \left[(r+n-k-1) + (r+n-k-2) + \cdots + r\right] + 2r + t(2t-t) + \binom{t}{2} \\ &\le (n-k)(t-1) + h(k,k,t-1) \\ &= h(n,k,t-1). \end{aligned}$$

This contradiction completes the proof of Claim 5.3. □

For the rest of the proof of Theorem 2.3′ for $n \le k + (t-1)/2$, we will follow the method of Kopylov in [9] to show that $G \subseteq H_{n,k,2}$. Let G^* be the k-closure of G. That is, add edges to G until adding any additional edge creates a cycle of length at least k. In particular, for any non-edge xy of G^*, there is an (x, y)-path in G^* with at least $k-1$ edges.

Because G has a nonempty t-core, and G^* contains G as a subgraph, G^* also has a nonempty t-core (which contains the t-core of G). Let $H = H(G^*, t)$ denote the t-core of G^*. We will show that

$$H \text{ is a complete graph}. \tag{14}$$

Indeed, suppose (14) does not hold. Choose a longest path P of G^* whose terminal vertices $x \in V(H)$ and $y \in V(H)$ are nonadjacent. By the maximality of P, every neighbor of x in H is in P. The same holds for y. Hence $d_P(x) + d_P(y) = d_H(x) + d_H(y) \ge 2(t+1) > k$, and also P has $k-1$ edges. By Theorem 4.3, $c(G^*) \ge k$, a contradiction. This proves (14).

Let $\ell = |V(H)|$. Because every vertex in H has degree at least $t+1$, $\ell \ge t+2$. Furthermore, if $\ell \ge k-1$, then G^* has a clique K of size at least $k-1$. Because G^* is 2-connected, we can extend a $(k-1)$-cycle of K to include at least one vertex in $G^* - H'$, giving us a cycle of length at least k. It follows that

$$t + 2 \le \ell \le k - 2, \tag{15}$$

and therefore $k - \ell < t$. Apply $(k-\ell)$-disintegration to G^*, and denote by H' the resulting graph. By construction, $H \subseteq H'$.

Case 1: There exists $v \in V(H') - V(H)$. Since $v \notin V(H)$, there exists a nonedge between a vertex in H and a vertex in $H' - H$. Pick a longest path P with terminal vertices $x \in V(H')$ and $y \in V(H)$. Then $d_P(x) + d_P(y) \ge (k - \ell + 1) + (\ell - 1) = k$, and therefore $c(G^*) \ge k$.

Case 2: $H = H'$. Then

$$e(G^*) \le \binom{\ell}{2} + (n-\ell)(k-\ell) = h(n, k, k-\ell).$$

If $3 \le (k-\ell) \le t-1$, then $e(G) \le \max\{h(n, k, 3), h(n, k, t-1)\}$, so by (15), $k - \ell = 2$, and H is the complete graph with $k-2$ vertices. Let $D = V(G^*) - V(H)$. If there is an edge xy in $G^*[D]$, then because G^* is 2-connected, there exist two vertex-disjoint paths, P_1 and P_2, from $\{x, y\}$ to H such that P_1 and P_2 only intersect $\{x, y\} \cup H$ at the beginning and end of the paths. Let a and b be the terminal vertices of P_1 and P_2 respectively that lie in H. Let P be any (a, b)-hamiltonian path of H. Then $P_1 \cup P \cup P_2 + xy$ is a cycle of length at least k in G^*, a contradiction.

Therefore D is an independent set, and since G^* is 2-connected, each vertex of D has degree 2. Suppose there exists $u, v \in D$ where $N(u) \ne N(v)$. Let $N(u) = \{a, b\}$, $N(v) = \{c, d\}$ where it is possible that $b = c$. Then we can find a cycle C of H that covers $V(H)$ which contains edges ab and cd. Then $C - ab - cd + ua + ub + vc + vd$ is a cycle of length k in G^*. Thus for every $v \in D$, $N(v) = \{a, b\}$ for some $a, b \in H$. I.e., $G^* = H_{n,k,2}$, and thus $G \subseteq H_{n,k,2}$. This completes the proof of Theorem 2.3′ for the case $n \le k + (t-1)/2$. □

5.3. Proof of *Theorem 2.3′* for all *n*

We use induction on n with the base case $n \le k + (t-1)/2$. Suppose $n \ge k + t/2$ and for all $k \le n' < n$, Theorem 2.3′ holds. Let G be a 2-connected graph G with n vertices such that

$$e(G) > \max\{h(n, k, t-1), h(n, k, 3)\} \text{ and } c(G) < k. \tag{16}$$

Apply one step of Procedure BP. If (R4) was applied (so neither (R2) nor (R3) applies to G), then $G_m = G$ (with G_m defined as in the previous case). By Lemmas 5.1, 4.6, and 3.2, the theorem holds.

Therefore we may assume that either (R2) or (R3) was applied. Let G^- be the resulting graph. Then $c(G^-) < k$, and G^- is 2-connected.

Claim 5.4.

$$e(G^-) > \max\{h(|V(G^-)|, k, t-1), h(|V(G^-)|, k, 3)\}. \tag{17}$$

Proof. If (R2) was applied, i.e., $G^- = G/uv$ for some edge uv, then

$$e(G^-) \ge e(G) - (t-1) > h(n-1, k, t-1) \ge h(n-1, k, 3),$$

so (17) holds. Therefore we may assume that (R3) was applied to obtain G^-. Then $n \ge k+t-1$ and $e(G) - e(G^-) = \binom{t+1}{2} - 1$. So by (16),

$$e(G^-) > h(n, k, t-1) - \binom{t+1}{2} + 1. \tag{18}$$

The right hand side of (18) equals $h(n-(t-1), k, t-1) + t^2/2 - 5t/2 + 2$ which is at least $h(n-(t-1), k, t-1)$ for $t \ge 4$, proving the first part of (17).

We now show that also $e(G^-) > h(n-(t-1), k, 3)$. Indeed, for $k = 2t+1$,

$$e(G^-) - h(n-(t-1), k, 3) > \binom{t+2}{2} + (t-1)(n-t-2) - \binom{t+1}{2} + 1$$

$$-\left[\binom{2t-2}{2} + 3(n-(t-1)-(2t-2))\right] \geq 0 \text{ when } n \geq 3t.$$

Similarly, for $k = 2t+2$,

$$e(G^-) - h(n-(t-1), k, 3) > \binom{t+3}{2} + (t-1)(n-t-3) - \binom{t+1}{2} + 1$$

$$-\left[\binom{2t-1}{2} + 3(n-(t-1)-(2t-1))\right] > 0 \text{ when } n \geq 3t+1.$$

Thus if $n \geq 3t+1$, then (17) is proved. But if $n \in \{3t-1, 3t\}$ then by Remark 5.1, no graph to which (R3) was applied may have more than $h(n, k, t-1)$ edges. □

By (17), we may apply induction to G^-. So G^- satisfies either (a) $G^- \subseteq H_{|V(G^-)|,k,2}$, or (b) G^- is contained in a graph in $\mathcal{G}(n, k) - H_{|V(G^-)|,k,2}$ and contains a subgraph $H \in \mathcal{F}(k)$.

Suppose first that G^- satisfies (b). If (R3) was applied to obtain G^- from G, then because G^- contains a subgraph $H \in \mathcal{F}(k)$ and $G^- \subseteq G$, G also contains H. If (R2) was applied, then by Lemma 3.1, G contains a subgraph $H' \in \mathcal{F}(k)$. In either case, Lemma 3.2 implies that G is a subgraph of a graph in $\mathcal{G}(n, k) - H_{n,k,2}$.

So we may assume that (a) holds, that is, G^- is a subgraph of $H_{|V(G^-)|,k,2}$. Because $\delta(G^-) \leq 2$, $\delta(G) \leq 3$, and so G has edges in at most $2 \leq t-2$ triangles. Therefore (R2) was applied to obtain G^-, where $G/uv = G^-$. Let D be an independent set of vertices of G^- of size $(n-1)-(k-2)$ with $N(D) = \{a, b\}$ for some $a, b \in V(G^-)$. Since $T_{G^-}(xa), T_{G^-}(xb) \leq 1$ for every $x \in D$, we have that $T_G(uv) \leq 2$ with equality only if $T(G) = 2$ where $T(G) = \min_{xy \in E(G)} T_G(xy)$.

We want to show that $T_G(uv) \leq 1$. If not, suppose first that $u * v \in D \subseteq V(G^-)$. Then there exists $x \in D - u * v$, and x and $u * v$ are not adjacent in G^-. Therefore x was not in a triangle with u and v in G, and hence $T_G(xa) = T_{G^-}(xa) \leq 1$, so the edge xa should have been contracted instead. Otherwise if $u * v \notin D$, at least one of $\{a, b\}$, say a, is not $u * v$. If $T(G) = 2$, then for every $x \in D \subseteq V(G)$, $T_G(xa) = 2$, therefore each such edge xa was in a triangle with uv in G. Then $T_G(uv) \geq |D| = (n-1)-(k-2) \geq k + t/2 - 1 - k + 2 \geq 3$, a contradiction.

Thus $T_G(uv) \leq 1$ and $e(G) \leq 2 + e(G^-) \leq 2 + h(n-1, k, 2) = h(n, k, 2)$. But for $n \geq k + t/2$, we have $h(n, k, t-1) \geq h(n, k, 2)$, a contradiction. This completes the proof of Theorem 2.3′ and therefore the proof of the main result. □

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