Discrete Mathematics 340 (2017) 2688–2690

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A stability version for a theorem of Erdős on nonhamiltonian graphs

Zoltán Füredi [a], Alexandr Kostochka [b,c,*], Ruth Luo [b]

[a] Alfréd Rényi Institute of Mathematics, Hungary
[b] University of Illinois at Urbana–Champaign, Urbana, IL 61801, United States
[c] Sobolev Institute of Mathematics, Novosibirsk 630090, Russia

ARTICLE INFO

Article history:
Received 27 April 2016
Accepted 28 August 2016
Available online 14 November 2016

Dedicated to the memory of Professor H. Sachs

Keywords:
Turán problem
Hamiltonian cycles
Extremal graph theory

ABSTRACT

Let n, d be integers with $1 \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, and set $h(n, d) := \binom{n-d}{2} + d^2$ and $e(n, d) := \max\{h(n, d), h(n, \left\lfloor \frac{n-1}{2} \right\rfloor)\}$. Because $h(n, d)$ is quadratic in d, there exists a $d_0(n) = (n/6) + O(1)$ such that

$$e(n, 1) > e(n, 2) > \cdots > e(n, d_0) = e(n, d_0 + 1) = \cdots = e\left(n, \left\lfloor \frac{n-1}{2} \right\rfloor\right).$$

A theorem by Erdős states that for $d \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, any n-vertex nonhamiltonian graph G with minimum degree $\delta(G) \geq d$ has at most $e(n, d)$ edges, and for $d > d_0(n)$ the unique sharpness example is simply the graph $K_n - E(K_{\lceil (n+1)/2 \rceil})$. Erdős also presented a sharpness example $H_{n,d}$ for each $1 \leq d \leq d_0(n)$.

We show that if $d < d_0(n)$ and a 2-connected, nonhamiltonian n-vertex graph G with $\delta(G) \geq d$ has more than $e(n, d + 1)$ edges, then G is a subgraph of $H_{n,d}$. Note that $e(n, d) - e(n, d + 1) = n - 3d - 2 \geq n/2$ whenever $d < d_0(n) - 1$.

1. Introduction

We use standard notation. In particular, $V(G)$ denotes the vertex set of a graph G, $E(G)$ denotes the edge set of G, and $e(G) = |E(G)|$. Also, if $v \in V(G)$, then $N(v)$ denotes the neighborhood of v and $d(v) = |N(v)|$. Ore [3] proved the following Turán-type result:

Theorem 1 (*Ore [3]*)**.** *If G is a nonhamiltonian graph on n vertices, then $e(G) \leq \binom{n-1}{2} + 1$.*

This bound is achieved only for the n-vertex graph obtained from the complete graph K_{n-1} by adding a vertex of degree 1. Erdős [2] refined the bound in terms of the minimum degree of the graph:

Theorem 2 (*Erdős [2]*)**.** *Let n, d be integers with $1 \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, and set $h(n, d) := \binom{n-d}{2} + d^2$. If G is a nonhamiltonian graph on n vertices with minimum degree $\delta(G) \geq d$, then*

$$e(G) \leq \max\left\{h(n, d), h\left(n, \left\lfloor \frac{n-1}{2} \right\rfloor\right)\right\} =: e(n, d).$$

This bound is sharp for all $1 \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor$.

* Corresponding author at: University of Illinois at Urbana–Champaign, Urbana, IL 61801, United States.
E-mail addresses: furedi.zoltan@renyi.mta.hu (Z. Füredi), kostochk@math.uiuc.edu (A. Kostochka), ruthluo2@illinois.edu (R. Luo).

http://dx.doi.org/10.1016/j.disc.2016.08.030

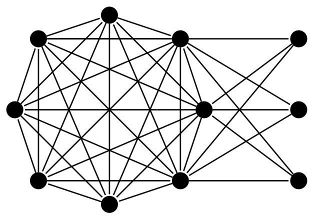

Fig. 1. $H_{11,3}$.

To show the sharpness of the bound, for $n, d \in \mathbb{N}$ with $d \le \lfloor \frac{n-1}{2} \rfloor$, consider the graph $H_{n,d}$ obtained from a copy of K_{n-d}, say with vertex set A, by adding d vertices of degree d each of which is adjacent to the same d vertices in A. An example of $H_{11,3}$ is given in Fig. 1.

By construction, $H_{n,d}$ has minimum degree d, is nonhamiltonian, and $e(H_{n,d}) = \binom{n-d}{2} + d^2 = h(n, d)$. Elementary calculation shows that $h(n, d) > h(n, \lfloor \frac{n-1}{2} \rfloor)$ in the range $1 \le d \le \lfloor \frac{n-1}{2} \rfloor$ if and only if $d < (n+1)/6$ and n is odd or $d < (n+4)/6$ and n is even. Hence there exists a $d_0 := d_0(n)$ such that

$$e(n, 1) > e(n, 2) > \cdots > e(n, d_0) = e(n, d_0 + 1) = \cdots = e\left(n, \left\lfloor \frac{n-1}{2} \right\rfloor\right),$$

where $d_0(n) := \lceil \frac{n+1}{6} \rceil$ if n is odd, and $d_0(n) := \lceil \frac{n+4}{6} \rceil$ if n is even. Let $H'_{n,d}$ denote the graph that is an edge-disjoint union of two complete graphs K_{n-d} and K_{d+1} sharing one vertex.

The result of this note is the following refinement of Theorem 2.

Theorem 3. *Let $n \ge 3$ and $d \le \lfloor \frac{n-1}{2} \rfloor$. Suppose that G is an n-vertex nonhamiltonian graph with minimum degree $\delta(G) \ge d$ such that*

$$e(G) > e(n, d+1) = \max\left\{h(n, d+1), h\left(n, \left\lfloor \frac{n-1}{2} \right\rfloor\right)\right\}. \tag{1}$$

(So we have $d < d_0(n)$.) Then G is a subgraph of either $H_{n,d}$ or $H'_{n,d}$.

This is a stability result in the sense that for $d < n/6$, each 2-connected, nonhamiltonian n-vertex graph with minimum degree at least d and "close" to $h(n, d)$ edges is a subgraph of the extremal graph $H_{n,d}$. Note that $h(n, d) - h(n, d+1) = n - 3d - 2$ is at least $n/2$ for $d < d_0 - 1$. Note also that $e(H'_{n,d}) > e(n, d+1)$ only when $d = O(\sqrt{n})$.

We will use the following well-known theorems of Pósa.

Theorem 4 (Pósa [4])**.** *Let $n \ge 3$. If G is a nonhamiltonian n-vertex graph, then there exists $1 \le k \le \lfloor \frac{n-1}{2} \rfloor$ such that G has a set of k vertices with degree at most k.*

Theorem 5 (Pósa [5])**.** *Let $n \ge 3$, $1 \le \ell < n$ and let G be an n-vertex graph such that*

$d(u) + d(v) \ge n + \ell$ for every non-edge uv in G. Then for every linear forest F with ℓ edges contained in G, the graph G has a hamiltonian cycle containing all edges of F.

2. Proof of Theorem 3

Call a graph G *saturated* if G is nonhamiltonian but for each $uv \notin E(G)$, $G + uv$ has a hamiltonian cycle. Ore's proof [3] of Dirac's Theorem [1] yields that

$$\text{for every } n\text{-vertex saturated graph } G \text{ and for each } uv \notin E(G),\ d(u) + d(v) \le n - 1. \tag{2}$$

First we show two facts on saturated graphs with many edges.

Lemma 6. *Let G be a saturated n-vertex graph with $e(G) > h(n, \lfloor \frac{n-1}{2} \rfloor)$. Then for some $1 \le k \le \lfloor \frac{n-1}{2} \rfloor$, $V(G)$ contains a subset D of k vertices of degree at most k such that $G - D$ is a complete graph.*

Proof. Since G is nonhamiltonian, by Theorem 4, there exists some $1 \le k \le \lfloor \frac{n-1}{2} \rfloor$ such that G has k vertices with degree at most k. Pick the maximum such k, and let D be the set of the vertices with degree at most k. Since $e(G) > h(n, \lfloor \frac{n-1}{2} \rfloor)$, $k < \lfloor \frac{n-1}{2} \rfloor$. So, by the maximality of k, $|D| = k$.

Suppose there exist $x, y \in V(G) - D$ such that $xy \notin E(G)$. Among all such pairs, choose x and y with the maximum $d(x)$. Since $y \notin D$, $d(y) > k$. Let $D' := V(G) - N(x) - \{x\}$ and $k' := |D'| = n - 1 - d(x)$. By (2),

$$d(z) \leq n - 1 - d(x) = k' \quad \text{for all } z \in D'. \tag{3}$$

So D' is a set of k' vertices of degree at most k'. Since $y \in D'$, $k' \geq d(y) > k$. Thus by the maximality of k, we get $k' = n - 1 - d(x) > \lfloor \frac{n-1}{2} \rfloor$. Equivalently, $d(x) < \lceil \frac{n-1}{2} \rceil$. For all $z \in D' + \{x\}$, either $z \in D$ where $d(z) \leq k \leq \lfloor \frac{n-1}{2} \rfloor$, or $z \in V(G) - D$, and so $d(z) \leq d(x) \leq \lfloor \frac{n-1}{2} \rfloor$. It follows that $e(G) \leq h(n, \lfloor \frac{n-1}{2} \rfloor)$, a contradiction. □

Lemma 7. *Under the conditions of Lemma 6, if $k = \delta(G)$, then $G = H_{n,\delta(G)}$ or $G = H'_{n,\delta(G)}$.*

Proof. Set $d := \delta(G)$, and let D be a set of d vertices with degree at most d. Let $u \in D$. Since $\delta(G) \geq |D| = d$, u has a neighbor $w \in V(G) - D$. Consider any $v \in D - \{u\}$. By Lemma 6, w is adjacent to all of $V(G) - D - \{w\}$. It also is adjacent to u, therefore its degree is at least $n - d$. We obtain

$$d(w) + d(v) \geq (n - d) + d = n.$$

Then by (2), w is adjacent to v, and hence w is adjacent to all vertices of D.

Let W be the set of vertices in $V(G) - D$ having a neighbor in D. We have obtained that $W \neq \emptyset$ and

$$N(u) \cap (V(G) - D) = W \text{ for all } u \in D. \tag{4}$$

Let $G' = G[D \cup W]$. If $|W| = 1$, then $G = H'_{n,d}$. If $|V(G')| = 2d$, then by (4), each vertex $u \in D$ has the same d neighbors in $V(G) - D$. Because $d(u) = d$, D is an independent set. Thus $G = H_{n,d}$. Otherwise, $d + 2 \leq |V(G')| \leq 2d - 1$, $|D| \geq 2$.

Fix a pair of vertices $w_1, w_2 \in W$. For any $x, y \in V(G')$,

$$d(x) + d(y) \geq d + d \geq |V(G')| + 1.$$

Therefore by Theorem 5, G' has a hamiltonian cycle C that uses the edge w_1w_2. Since $G'' := G - (V(G') - \{w_1, w_2\})$ is a complete graph, it contains a hamiltonian w_1, w_2-path P. Then $P \cup (C - w_1w_2)$ is a hamiltonian cycle of G, a contradiction. □

Proof of Theorem 3. Suppose that an n-vertex, nonhamiltonian graph G satisfies the constraints of Theorem 3 for some $1 \leq d \leq \lfloor \frac{n-1}{2} \rfloor$. We may assume G is saturated, since if a graph containing G is a subgraph of $H_{n,d}$ or $H'_{n,d}$, then G is as well.

By Lemma 6, G has a set D of $k \leq \lfloor \frac{n-1}{2} \rfloor$ vertices with degree at most k such that $G - D$ is a complete graph. Therefore $e(G) \leq \binom{n-k}{2} + k^2 = h(n, k)$. If $k \geq d + 1$, then $e(G) \leq \max\{h(n, d + 1), h(n, \lfloor \frac{n-1}{2} \rfloor)\} = e(n, d + 1)$, a contradiction. Thus $k \leq d$. Furthermore, $k \geq \delta(G) \geq d$, and hence $k = d$. Also, since $e(G) > h(n, \lfloor \frac{n-1}{2} \rfloor)$, we have $d + 1 \leq d_0(n) \leq (n + 8)/6$. Applying Lemma 7 completes the proof. □

Acknowledgment

We thank both referees for their helpful comments. The first author's research is supported in part by the Hungarian National Science Foundation OTKA 104343, by the Simons Foundation Collaboration Grant 317487, and by the European Research Council Advanced Investigators Grant 267195. Research of the second author is supported in part by NSF grants DMS-1266016 and DMS-1600592 and grants 15-01-05867 and 16-01-00499 of the Russian Foundation for Basic Research.

References

[1] G. Dirac, Some theorems on abstract graphs, Proc. Lond. Math. Soc. 2 (3) (1952) 69–81.
[2] P. Erdős, Remarks on a paper of Pósa, Magyar Tud. Akad. Mat. Kutató Int. Közl. 7 (1962) 227–229.
[3] O. Ore, Arc coverings of graphs, Acta Math. Acad. Sci. Hung. 10 (1959) 337–356.
[4] L. Pósa, A theorem concerning Hamilton lines, Magyar Tud. Akad. Mat. Kutató Int. Közl. 7 (1962) 225–226.
[5] L. Pósa, On the circuits of finite graphs, Magyar. Tud. Akad. Mat. Kutató Int. Kőzl. 8 (1963/1964) 355–361.