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# On 3-uniform hypergraphs without a cycle of a given length

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ABSTRACT

We study the maximum number of hyperedges in a 3-uniform hypergraph on $n$ vertices that does not contain a Berge cycle of a given length $\ell$. In particular we prove that the upper bound for $C_{2k+1}$-free hypergraphs is of the order $O(k^2 n^{1+1/k})$, improving the upper bound of Győri and Lemons (2012) by a factor of $\Theta(k^2)$. Similar bounds are shown for linear hypergraphs.



## 1. A generalization of the Turán problem

Counting substructures is a central topic of extremal combinatorics. Given two (hyper)graphs $G$ and $H$ let $N(G\,;H)$ denote the number of subgraphs of $G$ isomorphic to $H$. (Usually we consider a labeled host graph $G$.) Note that $N(G\,;K_2)=e(G)$, the number of edges of $G$. More generally, $N(\mathcal{G}\,;H)$ is the maximum of $N(G\,;H)$ where $G\in\mathcal{G}$, a class of graphs. In most cases, in Turán type problems, $\mathcal{G}$ is a set of $n$-vertex $\mathcal{F}$-free graphs, where $\mathcal{F}$ is a collection of forbidden subgraphs. This maximum is denoted by $N(n,\mathcal{F}\,;H)$. So $N(n,\mathcal{F}\,;H)$ is the maximum number of copies of $H$ in an $\mathcal{F}$-free graph on $n$ vertices. The Turán number $\mathrm{ex}(n,\mathcal{F})$ is defined as $N(n,\mathcal{F}\,;K_2)$. Let $\mathrm{ex}(m,n,\mathcal{F})$ be the maximum number edges in a bipartite graph with parts of order $m$ and $n$ vertices that do not contain any member of $\mathcal{F}$. $\mathcal{C}_\ell$ is the family of all cycles of length at most $\ell$. For any graph $G$ and any vertex $x$, we let $t(G)$ and $t(x)$ denote the number of triangles in $G$ and the number of triangles containing $x$, respectively. Let $t_\ell(n):=N(n,\mathcal{C}_\ell\,;K_3)$.

Our starting point is the Bondy–Simonovits [3] theorem, $\mathrm{ex}(n,C_{2k})\le 100kn^{1+1/k}$. Recall two contemporary versions due to Pikhurko [15], Bukh and Z. Jiang [4], respectively, and a classical result by Kővári, T. Sós, and Turán [14]. For all $k\ge 2$ and $n\ge 1$, we have

$$\mathrm{ex}(n,C_{2k})\le (k-1)n^{1+1/k}+16(k-1)n, \quad (1)$$

$$\mathrm{ex}(n,C_{2k})\le 80\sqrt{k\log k}\,n^{1+1/k}+10k^2n, \quad (2)$$

$$\mathrm{ex}(n,n,C_4)\le n^{3/2}+2n. \quad (3)$$

Erdős [6] conjectured that a triangle-free graph on $n$ vertices can have at most $(n/5)^5$ five cycles and that equality holds for the blown-up $C_5$ if $5|n$. Győri [9] showed that a triangle-free graph on $n$ vertices contains at most $c(n/5)^5$ copies of

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$C_5$, where $c < 1.03$. Grzesik [8], and independently, Hatami et al. [13] confirmed that Erdős' conjecture is true by using Razborov's method of flag algebras, i.e., $N(n, C_3; C_5) \le (n/5)^5$.

Bollobás and Győri [2] asked a related question: how many triangles can a graph have if it does not contain a $C_5$. They obtained the upper bound $t_5(n) \le (1+o(1))(5/4)n^{3/2}$ which yields the correct order of magnitude.

Later, Győri and Li [12] provided bounds on $t_{2k+1}(n)$.

$$\binom{k}{2} \operatorname{ex}\left(\frac{n}{k+1}, \frac{n}{k+1}, \mathcal{C}_{2k}\right) \le t_{2k+1}(n) \le \frac{(2k-1)(16k-2)}{3} \operatorname{ex}(n, C_{2k}). \tag{4}$$

The construction showing the lower bound in (4) is defined by considering a balanced bipartite $(X, Y)$-graph $G$ on $2n/(k+1)$ vertices which is extremal not containing any members of $\mathcal{C}_{2k}$. Each vertex $x$ in $X$ is replaced by $k$ vertices and connected to each other and to all neighbors of $x$, thus creating $\binom{k}{2}$ distinct triangles per each edge of $G$.

In Section 3 we improve the upper bound by a factor of $\Omega(k)$.

**Theorem 1.** *For $k \ge 2$,*

$$t_{2k+1}(n) := N(n, C_{2k+1}; K_3) \le 9(k-1) \operatorname{ex}\left(\left\lceil \frac{n}{3} \right\rceil, \left\lceil \frac{n}{3} \right\rceil, C_{2k}\right), \tag{5}$$

$$t_{2k}(n) \le \frac{2k-3}{3} \operatorname{ex}(n, C_{2k}). \tag{6}$$

The inequalities (1), (3) and (5) give $t_{2k+1}(n) \le 9(k-1)^2\left((2/3)n\right)^{1+1/k} + O(n)$ for $k \ge 3$ and $t_5(n) \le \sqrt{3}n^{3/2} + O(n)$. This latter one is not better than the Bollobás–Győri bound. However, our constant factor in Theorem 1 is the best possible in the following sense. It is widely believed that the Turán numbers in the above statements are 'smooth', i.e., there are constants $a_k, b_k$ depending only on $k$ such that $\operatorname{ex}(n, n, C_{2k}) = (a_k + o(1))n^{1+1/k}$ and $\operatorname{ex}(n, n, \mathcal{C}_{2k}) = (b_k + o(1))n^{1+1/k}$. If these are indeed true then the ratio of the upper bound in (5) and the lower bound in (4) is bounded by a constant factor of $O(a_k/b_k)$. It is also believed that the sequence $a_k/b_k$ is bounded (as $k \to \infty$), so further essential improvement is probably not possible.

Since the first version of this manuscript (2011) Alon and Shikhelman [1] improved the upper bound in Theorem 1 by a constant factor to $(16/3)(k-1)\operatorname{ex}(\lceil n/2 \rceil, C_{2k})$ and showed that $t_5(n) \le (1+o(1))(\sqrt{3}/2)n^{3/2}$. Nevertheless, we include our proof in Section 3 for completeness, and because we use Theorem 1 in our main result in the next section.

## 2. Berge cycles

A *Berge cycle* of length $k$ is a family of distinct hyperedges $H_0, \dots, H_{k-1}$ such that there are distinct vertices $v_0, \dots, v_{k-1}$ satisfying

$$v_i v_{i+1} \subset H_i \text{ for } 0 \le i \le k-1 \pmod{k}.$$

A hypergraph is *linear*, also called nearly disjoint, if every two edges meet in at most one vertex. Let $C_\ell^{(3)}$ be the collection of 3-uniform Berge cycles of length $\ell$.

We write $\operatorname{ex}_r(n, \mathcal{F})$ ($\operatorname{ex}_r^{\text{lin}}(n, \mathcal{F})$, resp.) to denote the maximum number of hyperedges in a $r$-uniform (and linear, resp.) hypergraph on $n$ vertices that does not contain any member of $\mathcal{F}$. Győri and Lemons [10] showed that

$$\operatorname{ex}\left(\left\lfloor \frac{n}{3} \right\rfloor, \left\lfloor \frac{n}{3} \right\rfloor, \mathcal{C}_{2k}\right) \le \operatorname{ex}_3(n, C_{2k+1}^{(3)}) < 4k^4 n^{1+\frac{1}{k}} + 15k^4 n + 10k^2 n. \tag{7}$$

The order of magnitude of the upper bound probably cannot be improved (as $k$ is fixed and $n \to \infty$).

Győri and Lemons [11] extended their result to $C_{2k}^{(3)}$-free 3-uniform hypergraphs (and also to $m$-uniform hypergraphs) by showing that the same lower bound as in (7) holds for $\operatorname{ex}_3(n, C_{2k}^{(3)})$ and that $\operatorname{ex}_3(n, C_{2k}^{(3)}) \le c(k)n^{1+\frac{1}{k}}$. The construction showing the lower bound in (7) is defined by considering a balanced bipartite graph $G$ on $n/3+n/3$ vertices which is extremal not containing any members of $\mathcal{C}_{2k}$. A 3-uniform $C_{2k}^{(3)}$-free hypergraph $\mathcal{H}$ is formed by doubling each vertex in one of the parts of $G$, thus turning each edge of $G$ to a hyperedge of $\mathcal{H}$. The number of hyperedges in $\mathcal{H}$ is $e(G) = \operatorname{ex}(n/3, n/3, \mathcal{C}_{2k})$.

In this paper, we make improvements on the bounds on $\operatorname{ex}_3(n, C_{2k+1}^{(3)})$ and $\operatorname{ex}_3(n, C_{2k}^{(3)})$. First, observe that trivially

$$t_{2k+1}(n) \le \operatorname{ex}_3(n, C_{2k+1}^{(3)}). \tag{8}$$

(Consider the triple system defined by the triangles of a $C_{2k+1}$-free graph.) So (4) gives a lower bound which (probably) improves the lower bound in (7) by a factor of $\Omega(k)$.

The aim of this paper is to improve the upper bound in (7) by a factor of (at least) $\Omega(k^2)$ and also to simplify the original proof. In Section 4 we reduce the upper bound into three subproblems as follows.

**Theorem 2.** *For $k \geq 2$ we have*

$$\mathrm{ex}_3(n, C^{(3)}_{2k+1}) \leq t_{2k+1}(n) + 4\,\mathrm{ex}(n, C_{2k}) + 12\,\mathrm{ex}^{\mathrm{lin}}_3(n, C^{(3)}_{2k+1}), \tag{9}$$

$$\mathrm{ex}_3(n, C^{(3)}_{2k}) \leq t_{2k}(n) + \mathrm{ex}(n, C_{2k}). \tag{10}$$

The first and the third terms in (9) are both lower bounds, and probably the middle term is the smallest one. In Section 5 we estimate the third term.

**Theorem 3.** *For $k \geq 2$ we have*

$$\mathrm{ex}^{\mathrm{lin}}_3(n, C^{(3)}_{2k+1}) \leq 2kn^{1+1/k} + 9kn. \tag{11}$$

We were not able to relate the left hand side directly to $\mathrm{ex}(n, C_{2k})$. In fact, just like in Győri and Lemons' proof [10], we reiterate a version of the original proof of Bondy and Simonovits [3] (as everybody else did in [16,15,5], and in [4]). Our rendering is much simpler than [10]. For the even case $\mathrm{ex}^{\mathrm{lin}}_3(n, C^{(3)}_{2k}) \leq \mathrm{ex}(n, C_{2k})$ is obvious by selecting a pair from each hyperedge in a linear $C_{2k}$-free triple system. We have no matching lower bound for $\mathrm{ex}^{\mathrm{lin}}_3(n, C^{(3)}_{\ell})$ other than what follows from the random method. Collier, Graber and Jiang [5] proved that $\mathrm{ex}^{\mathrm{lin}}_r(n, C^{(r)}_{2k+1}) \leq \alpha_{k,r} n^{1+1/k}$, but their $\alpha_{k,r}$ is greater than $r(2k)^r$. They find not only a Berge cycle but a *linear cycle*, i.e., a cyclic list of triples such that consecutive sets intersect in exactly one element and nonconsecutive sets are disjoint.

Theorems 1–3 together with (1) imply

$$\mathrm{ex}_3(n, C^{(3)}_{2k+1}) \leq (9k^2 + 10k + 5)n^{1+1/k} + O(k^2 n)$$

and $\mathrm{ex}_3(n, C^{(3)}_{2k}) \leq \frac{1}{3}(2k+9)(k-1)n^{1+1/k} + O(k^2 n)$. Using (2) one can lower the main coefficient to $O(k^{3/2}\sqrt{\log k})$. If the smoothness conjectures concerning $\mathrm{ex}(n, C_{2k})$ and $\mathrm{ex}(n, n, \mathcal{C}_{2k})$ hold, then the ratio of the upper bound (9) and lower bound (8) is of $O(a_k/b_k)$.

## 3. Counting triangles in $C_{2k}$-free and $C_{2k+1}$-free graphs

We need the following classical result of Erdős and Gallai [7] on paths.

$$\mathrm{ex}(n, P_k) \leq \frac{k-2}{2} n. \tag{12}$$

**Lemma 4.** *If $G$ is a $C_\ell$-free graph, then $t(G) \leq \frac{1}{3}(\ell - 3)e(G)$.*

**Proof.** For any vertex $x$, $t(x)$ is equal to the number of edges induced by $N(x)$. Therefore,

$$t(G) = \frac{1}{3}\sum_{x \in V(G)} t(x) = \frac{1}{3}\sum_{x \in V(G)} e(G[N(x)]).$$

The subgraph induced by $N(x)$ does not contain $P_{\ell-1}$, because $G$ is $C_\ell$-free. Therefore, by (12), we have

$$e(G[N(x)]) \leq \frac{1}{2}(\ell - 3)\deg(x).$$

We obtain

$$t(G) \leq \frac{1}{3}\sum_{x \in V(G)} \frac{1}{2}(\ell - 3)\deg(x) = \frac{1}{3}(\ell - 3)e(G). \quad \square$$

Note that Lemma 4 implies the upper bound (6) for $t_{2k}(n)$.

**Proof of Theorem 1.** Let $G$ be a $C_{2k+1}$-free graph, $k \geq 2$, with the $n$ element vertex set $V$. Let $\mathcal{H}$ be the family of triangles in $G$. Given any 3-partition (or 3-coloring) $\{V_1, V_2, V_3\}$ of $V$ let $\mathcal{H}(V_1, V_2, V_3)$ be the 3-partite induced subhypergraph of $\mathcal{H}$ with these parts, i.e., $\mathcal{H}(V_1, V_2, V_3) := \{T \in \mathcal{H} : |T \cap V_i| = 1 \text{ for all } 1 \leq i \leq 3\}$. Standard averaging argument shows that there is a partition such that each color class $V_i$ with color $i$ has size $\lfloor (n+i-1)/3 \rfloor$, $1 \leq i \leq 3$, and the number of triples in $\mathcal{H}' := \mathcal{H}(V_1, V_2, V_3)$ is at least 2/9'ths of the number of triples in $\mathcal{H}$. So we have $|\mathcal{H}| \leq (9/2)|\mathcal{H}'|$.

Let $G'$ be the edges of $G$ contained in any triple from $\mathcal{H}'$. Since $t(G) = |\mathcal{H}|$ and $t(G') = |\mathcal{H}'|$, we have $t(G) \leq (9/2)t(G')$. From now on, our aim is to give an upper estimate for $t(G')$. Since $t(G') \leq \frac{1}{3}(2k-2)e(G')$ by Lemma 4, we have that

$$t(G) \leq \frac{9}{2}t(G') \leq 3(k-1)e(G').$$

To complete the proof of Theorem 1 we only need an appropriate upper bound on $e(G')$.

Let $G_{ij}$ be the bipartite subgraph of $G'$ induced by the vertex set $V_i \cup V_j$, $1 \le i < j \le 3$. Assume that there exists a copy $L$ of $C_{2k}$ in $G_{ij}$ for some $i$ and $j$. Let $x$ and $y$ be two adjacent vertices in $L$. Since there exists a triangle in $G'$ with vertices $x, y, z$ for some $z \in V_k$ ($k \ne i, j$), there exists a copy of $C_{2k+1}$ in $G$ with the edge set $(E(L) - \{xy\}) \cup \{xz, yz\}$, a contradiction. Therefore, $G_{ij}$ is $C_{2k}$-free. We obtain

$$e(G') = \sum_{1 \le i < j \le 3} e(G_{i,j}) \le 3\,\mathrm{ex}(\lceil n/3 \rceil, \lceil n/3 \rceil, C_{2k}). \quad \square$$

## 4. $C_\ell^{(3)}$-free 3-uniform hypergraphs

**Proof of Theorem 2.** For a pair of vertices $u$ and $v$, $\deg_{\mathcal{H}}(u, v)$ (or just $\deg(u, v)$) denotes the number of hyperedges of $\mathcal{H}$ containing both $u$ and $v$.

**Proposition 5.** *Let $\mathcal{H}$ be a $C_\ell^{(3)}$-free hypergraph, $\ell \ge 3$. Let $G_2 := G_2(\mathcal{H})$ be the graph on the vertex set of $\mathcal{H}$ such that $E(G_2) := \{uv : \deg(u, v) \ge 2\}$. Then, $G_2$ is $C_\ell$-free.*

**Proof.** Suppose, on the contrary, that $L$ is a cycle of length $\ell$ in $G_2$. Let $\mathcal{H}(e)$ be the set of triples from $\mathcal{H}$ containing the pair $e$. Suppose that $\ell \ge 4$, the case $\ell = 3$ is trivial. Then every triple $E \in \mathcal{H}$ contains at most two edges from $E(L)$, but every $e \in E(L)$ is contained in at least two triples. Therefore, Hall condition holds, that is every $i$ edges of $E(L)$ (for $1 \le i \le \ell$) are contained in at least $i$ triples. So by Hall's theorem one can choose a distinct hyperedge from $\mathcal{H}(e)$ for each edge $e$ of $L$. These form a Berge cycle of length $\ell$, a contradiction. $\square$

*The upper bound on* $\mathrm{ex}_3(n, C_{2k+1}^{(3)})$.

Let $\mathcal{H}$ be a 3-uniform hypergraph that does not contain $C_{2k+1}^{(3)}$ as a subgraph. Let $G_2$ be defined as in Proposition 5. Then $G_2$ is $C_{2k+1}$-free. Let $\mathcal{H}_2$ be the collection of triples from $\mathcal{H}$ having all the three pairs covered at least twice. The edges of $\mathcal{H}_2$ induce triangles in $G_2$, hence we have

$$|\mathcal{H}_2| \le N(G_2; C_3) \le t_{2k+1}(n). \tag{13}$$

Let $\mathcal{H}_1$ be the set of triples $E$ from $\mathcal{H}$ having a pair $P(E)$ such that $P(E)$ is contained only in $E$. Note that $|\mathcal{H}| = |\mathcal{H}_1| + |\mathcal{H}_2|$. In the following, we find an upper bound for $|\mathcal{H}_1|$ by defining further subfamilies $\mathcal{H}_3, \ldots, \mathcal{H}_6$.

Color the vertices of $\mathcal{H}_1$ randomly with two colors. The probability that for an edge $E \in \mathcal{H}_1$ the pair $P(E)$ gets the same color and the vertex $E \setminus P(E)$ has the opposite color is $1/4$. This implies that there is a partition $V_1 \cup V_2$ of $V(\mathcal{H})$ and a subfamily $\mathcal{H}_3 \subset \mathcal{H}_1$ such that $|\mathcal{H}_3| \ge (1/4)|\mathcal{H}_1|$ and every edge $E$ of $\mathcal{H}_3$ has two vertices in $V_i$ and one vertex in $V_{3-i}$ for some $i \in \{1, 2\}$ such that $V_i \cap E = P(E)$. Split $\mathcal{H}_3$ into two subfamilies as follows.

$$\mathcal{H}_4 := \{\{u, v, w\} \in \mathcal{H}_3 : P(E) = \{u, v\} \subset V_i,\ w \in V_{3-i}, \max(\deg(w, u), \deg(w, v)) \ge 3,\ i \in \{1, 2\}\}$$

and let $\mathcal{H}_5 := \mathcal{H}_3 \setminus \mathcal{H}_4$.

We claim that the graph $G_4$ consisting of the pairs $P(E)$, $E \in \mathcal{H}_4$, is $C_{2k}$-free. Indeed, suppose, on the contrary, that $L = (v_1, \ldots, v_{2k})$ is a cycle of $G_4$. Since $G_4$ has no edge joining $V_1$ and $V_2$ we may suppose that $L \subset V_1$. Consider the triples of $\mathcal{H}_4$ containing the edges of $L$, $E_i := \{v_i, v_{i+1}, w_i\}$, $(1 \le i \le 2k - 1)$, and $E_{2k} := \{v_{2k}, v_1, w_{2k}\}$. The vertices $w_1, \ldots, w_{2k}$ are in $V_2$, so they are not on $L$. Assume that $\deg(v_1, w_1) \ge 3$. Then, there is a hyperedge $E_0 = \{v_1, w_1, u\} \in \mathcal{H}$ different from $E_1, \ldots, E_{2k}$. The hyperedges $\{E_0, E_1, E_2, \ldots, E_{2k}\}$ are containing the consecutive pairs $\{v_1, w_1, v_2, \ldots, v_{2k}\}$ in this cyclic order, so form a Berge cycle of length $2k + 1$. Thus,

$$|\mathcal{H}_4| = e(G_4) \le \mathrm{ex}(|V_1|, C_{2k}) + \mathrm{ex}(|V_2|, C_{2k}) \le \mathrm{ex}(n, C_{2k}). \tag{14}$$

Because the multiplicity of the pairs in any edge $E$ in $\mathcal{H}_5$ is at most 2, one can use a greedy algorithm to find a subfamily $\mathcal{H}_6 \subset \mathcal{H}_5$ such that $|\mathcal{H}_6| \ge (1/3)|\mathcal{H}_5|$, where $\mathcal{H}_6$ is linear, that is each vertex-pair is covered at most once by an edge of $\mathcal{H}_6$.

Finally,

$$\begin{aligned} |\mathcal{H}| &= |\mathcal{H}_1| + |\mathcal{H}_2| \le 4|\mathcal{H}_3| + |\mathcal{H}_2| \\ &= |\mathcal{H}_2| + 4|\mathcal{H}_4| + 4|\mathcal{H}_5| \le |\mathcal{H}_2| + 4|\mathcal{H}_4| + 12|\mathcal{H}_6|. \end{aligned}$$

This with (13), (14), and the linearity of $\mathcal{H}_6$ completes the proof of (9).

*The upper bound on* $\mathrm{ex}_3(n, C_{2k}^{(3)})$.

Let $\mathcal{H}$ be a 3-uniform hypergraph that does not contain $C_{2k}^{(3)}$ as a subgraph. Let $G_2, \mathcal{H}_1, \mathcal{H}_2$ be defined for $\mathcal{H}$ as before. By Proposition 5, $G_2$ is $C_{2k}$-free. Hence, $|\mathcal{H}_2| \le N(G_2; C_3) \le t_{2k}(n)$. Recall that for each hyperedge $E$ in $\mathcal{H}_1$, there exists a vertex-pair, $P(E)$, such that $P(E)$ is contained only in $E$ in $\mathcal{H}$. Let $G_1$ be the graph defined by its edge set as $E(G_1) := \{P(E) : E \in \mathcal{H}_1\}$. We have that $|\mathcal{H}_1| = e(G_1)$. Since $G_1$ is obviously $C_{2k}$-free we get

$$|\mathcal{H}| = |\mathcal{H}_1| + |\mathcal{H}_2| \le t_{2k}(n) + \mathrm{ex}(n, C_{2k}). \quad \square$$

## 5. $C_\ell^{(3)}$-free 3-uniform linear hypergraphs

A *theta graph* of order $\ell$, denoted by $\Theta_\ell$, is a cycle $C_\ell$ with a chord, where $\ell \geq 4$. The following result was used implicitly in [3] and is stated as a separate lemma in [16, Lemma 2] and also used in [4] and [15]. Let $F$ be a $\Theta$-graph of order $\ell$ and $\ell > t \geq 2$. Let $A \cup B$ be a partition of $V(F)$ with $A, B \neq \emptyset$ such that every path of length $t$ in $F$ that starts in $A$ necessarily ends in $A$. Then $F$ is bipartite with parts $A$ and $B$. We need the following corollary, whose proof is left to the reader.

**Corollary 6.** *Let $F$ be a $\Theta$-graph of order $\ell$, where $\ell > t \geq 1$ and $t$ is an odd integer. Let $A \cup B$ be a partition of $V(F)$, $A \neq \emptyset$ such that every path of length $t$ in $F$ that starts in $A$ necessarily ends in $A$. Then $A = V(F)$.* □

We also use the following easy fact, which is used in [3], [4] and [15], too. If the $n$-vertex graph $G$ contains no $\Theta$-graph of order at least $\ell \geq 4$, then $e(G) \leq (\ell - 2)n$. In other words

$$\mathrm{ex}(n, \Theta_{\geq \ell}) \leq (\ell - 2)n. \tag{15}$$

**Proof of the upper bound on $\mathrm{ex}_3^{\mathrm{lin}}(n, C_{2k+1}^{(3)})$ in Theorem 3.** Let $\mathcal{H}$ be a 3-uniform hypergraph on $n$ vertices such that no two hyperedges meet in two vertices. Suppose that $\mathcal{H}$ contains no $C_{2k+1}^{(3)}$ and let $\delta$ be the third of the average degree. We have $\sum_{v \in V(\mathcal{H})} \deg(v) = 3|\mathcal{H}| = 3\delta n$. Then, there exists a subhypergraph $\mathcal{H}'$ on $n'$ vertices such that the degree of each vertex of $\mathcal{H}'$ is at least $\delta$. Therefore, we may suppose that every degree of $\mathcal{H}$ is at least $\delta$, and also that $\delta \geq 11k$.

The mapping $\pi : \mathcal{H} \to \binom{[n]}{2} \cup \emptyset$ is called a *choice function* if $\pi(E) \subset E$ for each $E \in \mathcal{H}$. There are $4^{|\mathcal{H}|}$ such choice functions. Let $\partial\mathcal{H}$ be the set of vertex-pairs contained in the members of $\mathcal{H}$ and consider a coloring of $\partial\mathcal{H}$, where the color of each pair is given by the single hyperedge of $\mathcal{H}$ containing it. We call a subgraph $G$ of $\partial\mathcal{H}$ *multicolored*, if all edges of $G$ have different colors under this coloring. For a choice function $\pi$ on $\mathcal{H}$, define the graph $G_\pi$ as the graph induced by the edge set $\{\pi(E) : \pi(E) \neq \emptyset, E \in \mathcal{H}\}$. Because $\mathcal{H}$ is a linear hypergraph, for two different hyperedges $E$ and $E'$ in $\mathcal{H}$ we have $\pi(E) \neq \pi(E')$. First, we consider the properties of arbitrary multicolored $G_\pi$, later we will define a special $\pi$. Clearly, $G_\pi$ has no cycle $C_{2k+1}$.

**Lemma 7.** *Let $T$ be a subtree (not necessarily spanning) in $G_\pi$, let $x \in V(T)$ be an arbitrary vertex, and let $V_i := N_i(x)$ in $T$, the set of vertices of distance $i$ from $x$ in the tree $T$. Consider $G_i := G_\pi[V_i]$, the subgraph of $G_\pi$ restricted to $V_i$. Then $G_i$ has no $\Theta$-graph of order $2k$ or larger.*

**Corollary 8.** $e(G_i) \leq (2k-2)|V_i|$ *for* $1 \leq i \leq k$.

**Proof of Lemma 7.** We use induction on $i$. Since $V_0 = x$, and $V_1$ (more exactly $G_1$) contains no path of $2k$ vertices, it does not contain a $\Theta_{\geq 2k}$ either. From now on, we may suppose that $i \geq 2$.

Suppose, on the contrary, that $F$ is a $\Theta$ subgraph of $G_i$ of order $\ell \geq 2k$, $i \geq 2$. For arbitrary $y \in V_1$, let $V_i(y)$ be the subset of descendants of $y$ in $V_i$ in the tree $T$. Consider the partition of $V_i$ defined as $\{V_i(y) : y \in V_1\}$. There exists a $y_1 \in V_1$ such that $A := V(y_1) \cap V(F) \neq \emptyset$.

We claim that $F$ is contained in $V(y_1)$. Note that there is no path $P(a, b)$ of $F$ (neither of $G_i$) of length $2k+1-2i$ that starts in some vertex $a \in A \subset V_i(y_1)$ and ends in another vertex $b \in V_i \setminus V(y_1)$. Otherwise, the $xy_1a$ and $xb$ paths on $T$ have only a single common vertex (namely $x$), have lengths $i$ so together with $P(a, b)$ they form a $C_{2k+1}$ in $G_\pi$, a contradiction. Therefore, every path of length $2k+1-2i$ in $F$, that starts in $A$ ends in $A$. Corollary 6 implies that $A = V(F)$, i.e., $V(F) \subset V(y_1)$.

To finish the proof of Lemma 7 simply use induction to the subtree $T_1$ of $T$ consisting of all descendants of $y_1$. Then $N_{i-1}(y_1)$ in $T_1$ is exactly $V_i(y_1)$, so it does not contain any $\Theta_{\geq 2k}$. □

We say for two sets of sequences of integers $\alpha = (a_1, \ldots, a_k)$ and $\beta = (b_1, \ldots, b_k)$ that $\alpha > \beta$, if there is an $i$ such that $a_i > b_i$ and $a_j = b_j$ for all $j < i$. This is called the lexicographical ordering, and it is indeed a linear order.

We are ready to define a concrete $T$ and a choice function $\pi$. Fix a vertex $x \in V(\mathcal{H})$ arbitrarily, let $V_0 := \{x\}$. Consider all choice functions $\pi$ and all multicolored trees of $G_\pi$ with root and center $x$ and radius at most $k$. Let $T$ be such a tree for which the sequence of the neighborhood sizes $(|N_1(x)|, \ldots, |N_k(x)|)$ takes its maximum in the lexicographic order. Since $\mathcal{H}$ is linear we have $|N_1(x)| = \deg_{\mathcal{H}}(x)$. Recall that $N_i(x)$ is denoted by $V_i$, $0 \leq i \leq k$. Our aim is to prove that the sizes of the $|V_i|$'s increase rapidly as follows.

**Lemma 9.** *For $1 \leq i \leq k-1$ we have $|V_{i+1}| \geq \frac{\delta - 7k}{2k}|V_i|$.*

This lemma completes the proof, because we obtain $n \geq |V_k| \geq (\delta - 7k)^{k-1}(2k)^{-k+1}|V_1|$. This and $|V_1| = \deg_{\mathcal{H}}(x) \geq \delta$ give $2kn^{1/k} + 7k \geq \delta$.

**Proof of Lemma 9.** Let $\mathcal{H}_i$ be the hyperedges of $\mathcal{H}$ containing the edges of $T$ joining $V_i$ to $V_{i+1}$, $0 \leq i \leq k-1$, we have $|\mathcal{H}_i| = |V_{i+1}|$. If $uvw = E \in \mathcal{H}_i$ with $u \in V_i$, $v \in V_{i+1}$, then $w \notin V_j$ with $j < i$. Otherwise, leaving out the edge $uv$ from $T$ and joining $wv$ results in a multicolored tree preceding $T$ in the lexicographic order.

Let $\mathcal{B}_i$ be the set of hyperedges from $\mathcal{H} \setminus (\mathcal{H}_0 \cup \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_i)$ meeting $V_i$, but not meeting $\cup_{j<i} V_j$, $0 \le i \le k-1$. We have $\mathcal{B}_0 = \emptyset$. If $E \in \mathcal{B}_i$, then $E \subset V_i \cup V_{i+1}$. Otherwise, if $u \in E \cap V_i$ and $v \in E \setminus (V_i \cup V_{i+1})$ then truncating our tree at $V_0 \cup V_1 \cup \cdots \cup V_{i+1}$ and joining the edge $uv$ result in another tree lexicographically larger than $T$.

Let $\mathcal{B}_i^\alpha$, $0 \le i \le k-1$, be the set of those hyperedges from $\mathcal{B}_i$, that meet $V_i$ exactly in $\alpha$ vertices, $\alpha = 1, 2$ or 3. The graph $G_i$, for $1 \le i \le k-1$, is defined on the vertex set $V_i$ as follows. It contains exactly one vertex-pair from each member of $\mathcal{B}_i^3$ and the pairs $E \cap V_i$ for $E \in \mathcal{B}_i^2 \cup \mathcal{B}_{i-1}^1$. For $i = k$, the edge set of $G_k$ consists only of the sets $\{E \cap V_k : E \in \mathcal{B}_{k-1}^1\}$, since $\mathcal{B}_k$ is undefined. The graph $G_\pi$ consisting of the edges of $T$ and the $G_i$'s, $1 \le i \le k$, is a multicolored subgraph. So Corollary 8 implies that

$$e(G_i) \le (2k-2)|V_i|. \tag{16}$$

Consider the $\mathcal{H}$-degrees of the elements of $V_i$, $(1 \le i \le k-1)$. Their total sum is at least $\delta|V_i|$. Obviously,

$$\sum_{v \in V_i} \deg_{\mathcal{H}}(v) = \sum_{E \in \mathcal{H}} |E \cap V_i|.$$

The edges of $\mathcal{H}$ meeting $V_i$ belong to some $\mathcal{H}_j$, $j \le i$, or to $\mathcal{B}_{i-1} \cup \mathcal{B}_i$. An edge $E \in \mathcal{H}_j$ can meet $V_i$ in at least two elements, only if $j$ is equal to $i-1$ or $i$. We obtain for $1 \le i \le k-1$

$$\begin{aligned}\delta|V_i| &\le \sum_{v \in V_i} \deg(\mathcal{H})(v) = \sum_{E \in \mathcal{H}} |E \cap V_i| \\ &\le \left(\sum_{0 \le j \le i-2} |\mathcal{H}_j|\right) + 2|\mathcal{H}_{i-1}| + 2|\mathcal{H}_i| + |\mathcal{B}_{i-1}^2| + 2|\mathcal{B}_{i-1}^1| + 3|\mathcal{B}_i^3| + 2|\mathcal{B}_i^2| + |\mathcal{B}_i^1|.\end{aligned}$$

Inequality (16) implies that

$$\begin{aligned}|\mathcal{B}_{i-1}^2| &\le e(G_{i-1}) \le (2k-2)|V_{i-1}|, \\ 2|\mathcal{B}_{i-1}^1| + 3|\mathcal{B}_i^3| + 2|\mathcal{B}_i^2| &\le 3(|\mathcal{B}_{i-1}^1| + |\mathcal{B}_i^3| + |\mathcal{B}_i^2|) = 3e(G_i) \le (6k-6)|V_i|, \\ |\mathcal{B}_i^1| &\le e(G_{i+1}) \le (2k-2)|V_{i+1}|.\end{aligned}$$

Using these inequalities and the fact that $|\mathcal{H}_j| = |V_{j+1}|$ we obtain that

$$\delta|V_i| \le \left(\sum_{1 \le j \le i-1} |V_j|\right) + 2|V_i| + 2|V_{i+1}| + (2k-2)|V_{i-1}| + (6k-6)|V_i| + (2k-2)|V_{i+1}|.$$

By rearranging we have

$$(\delta - (6k-4))|V_i| \le \left(\sum_{1 \le j \le i-1} |V_j|\right) + (2k-2)|V_{i-1}| + 2k|V_{i+1}|. \tag{17}$$

For $i = 1$ the fact that $\mathcal{B}_0 = \emptyset$ implies the slightly stronger $(\delta - (6k-4))|V_1| \le 2k|V_2|$. So Lemma 9 holds for $i = 1$. For larger $i$ we use induction and (17) to prove first that $2|V_i| \le |V_{i+1}|$ for all $i < k$ and then the sharper inequality of Lemma 9. □

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