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A proof of the stability of extremal graphs, Simonovits' stability from Szemerédi's regularity

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ABSTRACT

Let $T_{n,p}$ denote the complete p-partite graph of order n having the maximum number of edges. The following sharpening of Turán's theorem is proved. Every K_{p+1}-free graph with n vertices and $e(T_{n,p}) - t$ edges contains a p-partite subgraph with at least $e(T_{n,p}) - 2t$ edges.

As a corollary of this result we present a concise, contemporary proof (i.e., one applying the Removal Lemma, a corollary of Szemerédi's regularity lemma) for the classical stability result of Simonovits [25].

1. The Turán problem

Given a graph G with vertex set $V(G)$ and edge set $\mathcal{E}(G)$ its number of edges is denoted by $e(G)$. The neighborhood of a vertex $x \in V$ is denoted by $N(x)$, note that $x \notin N(x)$. For any $A \subset V$ the restricted neighborhood $N_G(x|A)$ stands for $N(x) \cap A$. Similarly, $\deg_G(x|A) := |N(x) \cap A|$. If the graph is well understood from the text we

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leave out subscripts. The *Turán graph* $T_{n,p}$ is the largest p-chromatic graph having n vertices, $n, p \geq 1$. Given a partition $(V_1, \ldots, V_p)$ of V the *complete multipartite* graph $K(V_1, \ldots, V_p)$ has vertex set V and all the edges joining distinct partite sets. $A \triangle B$ stands for the symmetric difference of the sets A and B. For further notations and notions undefined here see, e.g., the monograph of Bollobás [4].

Turán [27] proved that if an n-vertex graph G has at least $e(T_{n,p})$ edges then it contains a complete subgraph K_{p+1}, except if $G = T_{n,p}$. Given a class of graphs $\mathcal{L}$, a graph G is called $\mathcal{L}$*-free* if it does not contain any subgraph isomorphic to any member of $\mathcal{L}$. The *Turán number* $\mathrm{ex}(n, \mathcal{L})$ is defined as the largest size of an n-vertex, $\mathcal{L}$-free graph. Erdős and Simonovits [12] gave the following general asymptotic for the Turán number. Let $p + 1 := \min\{\chi(L) : L \in \mathcal{L}\}$. Then

$$\mathrm{ex}(n, \mathcal{L}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2) \quad \text{as} \quad n \to \infty. \tag{1}$$

They also showed that if G is an extremal graph, i.e., $e(G) = \mathrm{ex}(n, \mathcal{L})$, then it can be obtained from $T_{n,p}$ by adding and deleting at most $o(n^2)$ edges. This result is usually called Erdős–Stone–Simonovits theorem, although it was proved first in [12], but indeed (1) easily follows from a result of Erdős and Stone [13].

The aim of this paper is to present a new proof for the following stronger version of (1), a structural stability theorem, originally proved by Erdős and Simonovits [12], Erdős [6,7], and Simonovits [25].

Theorem 1. *For every $\varepsilon > 0$ and forbidden subgraph class $\mathcal{L}$ there is a $\delta > 0$, and n_0 such that if $n > n_0$ and G is an n-vertex, $\mathcal{L}$-free graph then*

$$e(G) \geq \left(1 - \frac{1}{p}\right)\binom{n}{2} - \delta n^2 \quad \textit{implies} \quad |\mathcal{E}(G_n) \triangle \mathcal{E}(T_{n,p})| \leq \varepsilon n^2. \tag{2}$$

I.e., one can change (add and delete) at most εn^2 edges of G to obtain a complete p-partite graph. In other words, if an n-vertex $\mathcal{L}$-free graph G is almost extremal, $\min\{\chi(L): L \in \mathcal{L}\} = p + 1$, then the structure of G is close to a p-partite Turán graph. This result is usually called *Simonovits' stability* of the extremum. Its simplest, elementary proof can be found in Lovász and Simonovits [20]. In Section 3 we use Szemerédi's regularity to give an even simpler, more transparent proof.

Our main tool is a very simple new proof for the case $\mathcal{L} = \{K_{p+1}\}$ in Section 2. It was known that this special case implies (2), and we present an elegant way to accomplish this in Section 3.

Stability results are usually more important than their extremal counterparts. That is why there are so many investigations concerning the *edit distance* of graphs. Let $G_1 = (V, \mathcal{E}_1)$ and $G_2 = (V, \mathcal{E}_2)$ be two (finite, undirected) graphs on the same vertex set. The *edit distance* from G_1 to G_2 is $\mathrm{ed}(G_1, G_2) := |\mathcal{E}_1 \triangle \mathcal{E}_2|$. Let $\mathcal{P}$ denote a class of graphs and G be a fixed graph. The edit distance from G to $\mathcal{P}$ is $\mathrm{ed}(G, \mathcal{P}) = \min\{\mathrm{ed}(G, F) :$

$F \in \mathcal{P}, V(G) = V(F)\}$. After several earlier results by different researchers, this notion was explicitly introduced in [3]. Alon and Stav [2] proved connections with Turán theory. For more recent results see Martin [21].

2. How to make a K_{p+1}-free graph p-chromatic

Ever since Erdős [8] observed that one can always delete at most $e/2$ edges from any graph G to make it bipartite there are many generalizations and applications of this (see, e.g., Alon [1] for a more precise form). Here we prove a version dealing with a narrower class of graphs. Recall that $e(T_{n,p}) := \max\{e(K(V_1, \ldots, V_p)) : \sum |V_i| = n\}$, the maximum size of a p-chromatic graph.

Theorem 2. *Suppose that $K_{p+1} \not\subset G$, $|V(G)| = n$, $t \geq 0$, and*

$$e(G) = e(T_{n,p}) - t.$$

Then there exists an (at most) p-chromatic subgraph H_0, $\mathcal{E}(H_0) \subset \mathcal{E}(G)$ such that

$$e(H_0) \geq e(G) - t.$$

Corollary 3 *(Stability of $\mathrm{ex}(n, K_{p+1})$). Suppose that G is K_{p+1}-free with $e(G) \geq e(T_{n,p}) - t$. Then there is a complete p-chromatic graph $K := K(V_1, \ldots, V_p)$ with $V(K) = V(G)$, such that*

$$|\mathcal{E}(G) \bigtriangleup \mathcal{E}(K)| \leq 3t.$$

Proof of Corollary 3. Delete t edges of G to obtain the p-chromatic H_0. Since $e(H_0) \geq e(T_{n,p}) - 2t$ one can add at most $2t$ edges to make it a complete p-partite graph. (Here $V_i = \emptyset$ is allowed.) □

There are other more exact stability results, e.g., Hanson and Toft [17] showed that for $t < \lfloor n/p \rfloor - 1$ the graph G itself is p-chromatic, there is no need to delete any edge. (Earlier Simonovits [25] proved this for $t < n/p - O_p(1)$.) Results of Győri [16] imply a stronger form, namely that $e(H_0) \geq e(G) - O(t^2/n^2)$. Erdős, Győri, and Simonovits [11] investigated dense triangle-free graphs. The advantage of our Theorem 2 is that it contains no ε, δ, n_0, it is true for every n, p and t.

The inequality in Corollary 3 is simple because we estimate the edit distance of G from a not necessarily balanced p-partite graph K. If we are interested in $\mathrm{ed}(G, T_{n,p})$ then we can use the following inequality obtained by a simple calculation. If $e(K((V_1, \ldots, V_p)) \geq e(T_{n,p}) - 2t$, then the sizes of V_i's should be 'close' to n/p. More exactly we get $4t \geq \sum_i (|V_i| - (n/p))^2$. Hence

$$\mathrm{ed}(K, T_{n,p}) \leq 2n\sqrt{t/p}\,. \tag{3}$$

Proof of Theorem 2. We find the large p-partite subgraph $H_0 \subset G$ by analyzing Erdős' degree majorization algorithm [9] to prove Turán's theorem. Our input is the K_{p+1}-free graph G and the output is a partition $V_1, V_2, \dots, V_p$ of $V(G)$ such that $\sum_i e(G|V_i) \le t$.

Let $x_1 \in V(G)$ be a vertex of maximum degree and let $V_1 := V \setminus N(x_1)$, $V_1^+ := N(x_1)$. Note that $x_1 \in V_1$ and $\deg(x) \le |V_1^+|$ for all $x \in V_1$. Hence

$$2e(G|V_1) + e(V_1, V_1^+) = \sum_{x \in V_1} \deg(x) \le |V_1||V_1^+|.$$

In general, define $V_0^+ := V(G)$ and let x_i be a vertex of maximum degree of the graph $G|V_{i-1}^+$, let $V_i := V_{i-1}^+ \setminus N(x_i)$, $V_i^+ := V_{i-1}^+ \cap N(x_i)$. We have $x_i \in V_i$, $\deg(x, V_{i-1}^+) \le |V_i^+|$ for all $x \in V_i$ and

$$2e(G|V_i) + e(V_i, V_i^+) = \sum_{x \in V_i} \deg(x|V_{i-1}^+) \le |V_i||V_i^+|. \tag{4}$$

The procedure stops in s steps when no more vertices left, i.e., if $V_1 \cup \dots \cup V_s = V(G)$. Note that $s \le p$ because $\{x_1, x_2, \dots, x_s\}$ span a complete graph.

Add up the left hand side of (4) for $1 \le i \le s$, we get $e(G) + (\sum_i e(G|V_i))$. The sum of the right hand sides is exactly $e(K(V_1, V_2, \dots, V_s))$. We obtain

$$e(T_{n,p}) - t + \left(\sum_i e(G|V_i)\right) = e(G) + \left(\sum_i e(G|V_i)\right) \le e(K(V_1, V_2, \dots, V_p)) \le e(T_{n,p})$$

implying $\sum_i e(G|V_i) \le t$. $\square$

Note that similar applications of Erdős' proof appear in the literature even in hypergraph settings (see, e.g., Mubayi [22]).

3. An application of the Removal Lemma

We only need a simple consequence of Szemerédi's regularity lemma [26]. Recall that the graph H contains a homomorphic image of F if there is a mapping $\varphi : V(F) \to V(H)$ such that the image of each F-edge is an H-edge. There is a homomorphism $\varphi : V(F) \to V(K_s)$ if and only if $s \ge \chi(F)$. If there is no any $\varphi : V(F) \to V(H)$ homomorphism then H is called hom(F)-free.

Lemma 4 *(A simple form of the Removal Lemma). For every $\alpha > 0$ and graph F there is an n_1 such that if $n > n_1$ and G is an n-vertex, F-free graph then it contains a* hom(F)*-free subgraph H with $e(H) > e(G) - \alpha n^2$.*

This means that H does not contain any homomorphic image of F as a subgraph, especially if $\chi(F) = p + 1$ then H is K_{p+1}-free. The Removal Lemma can be attributed to Ruzsa and Szemerédi [24]. It appears in a more explicit form in [10] and [15]. For a survey of applications of Szemerédi's regularity lemma see Komlós and Simonovits [19] or Komlós, Shokoufandeh, Simonovits, and Szemerédi [18]. In fact, the Removal Lemma can now be proved without the regularity lemma, with a much more reasonable bound on n_1 in Lemma 4, see Fox [14], Conlon and Fox [5].

Proof of (2). Using Lemma 4 and Corollary 3. Suppose that $F \in \mathcal{L}$, $\chi(F) = p+1$ and $\alpha > 0$ an arbitrary real. Suppose that G is F-free with $n > n_1(F, \alpha)$ and $e(G) > e(T_{n,p}) - \alpha n^2$. We have to show that the edit distance of G to $T_{n,p}$ is small. First we claim that the edit distance of G to a complete p-partite graph $K(V_1, \dots, V_p)$ is at most $7\alpha n^2$. Indeed, using the Removal Lemma we obtain a K_{p+1}-free subgraph H of G such that $e(H) > e(G) - \alpha n^2 > e(T_{n,p}) - 2\alpha n^2$. Apply Theorem 2 to H we get a p-partite subgraph H_0 with $e(H_0) > e(T_{n,p}) - 4\alpha n^2$. Then Corollary 3 yields a $K := K(V_1, \dots, V_p)$ with $\mathrm{ed}(K, H) < 6\alpha n^2$, giving $\mathrm{ed}(K, G) \le 7\alpha n^2$.

Since $e(K) \ge e(H_0) > e(T_{n,p}) - 4\alpha n^2$, we can use (3) with $t = 2\alpha n^2$ to get $\mathrm{ed}(K, T_{n,p}) \le n^2\sqrt{8\alpha/p}$. This completes the proof that $\mathrm{ed}(G, T_{n,p}) \le (7\alpha + \sqrt{8\alpha/p})n^2 < (8\sqrt{\alpha})n^2$. □

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