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A discrete isodiametric result: The Erdős–Ko–Rado theorem for multisets

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ABSTRACT

There are many generalizations of the Erdős–Ko–Rado theorem. Here the new results (and problems) concern families of t-intersecting k-element multisets of an n-set. We point out connections to coding theory and geometry. We verify the conjecture that for $n \geq t(k-t)+2$ such a family can have at most $\binom{n+k-t-1}{k-t}$ members.

1. Introduction

1.1. The isodiametric problem

In 1963 Mel'nikov [12] proved that the ball has the maximal volume among all sets with a given diameter in every Banach space of finite dimension. We call the problem of finding the maximal volume among the sets with given diameter in a metric space the *isodiametric problem*. Various results have been achieved concerning the discrete versions of this problem.

Kleitman [10] as a slight generalization of a theorem of Katona [9] determined the maximal volume among subsets with diameter of r in $\{0, 1\}^n$ with the Hamming distance (that is, the distance of $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \{0, 1\}^n$ is $|\{i \leq n : x_i \neq y_i\}|$) and proved that it is achieved if the subset is a ball of radius $r/2$ if r is even. Ahlswede and Khachatrian [1] generalized this result to $\{0, 1, \ldots, q-1\}^n$ and solved the isodiametric problem for all q, n and diameter r.

Du and Kleitman [5] considered and Bollobás and Leader [3] completely solved the isodiametric problem in $[k]^n$ with the ℓ_1 distance. Here $[k] := \{1, 2, \ldots, k\}$ and the distance of $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in [k]^n$ is $\sum_{i=1}^n |x_i - y_i|$.

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1.2. Erdős–Ko–Rado type theorems

Let us call a set system $\mathcal{F}$ *intersecting* if $|F_1 \cap F_2| \geq 1$ for all $F_1, F_2 \in \mathcal{F}$. It is easy to see that the cardinality of an intersecting set system of subsets of $[n]$ is at most 2^{n-1}. By restrictions on the sizes of the subsets, the problem becomes more difficult.

Let us use the following notation $\binom{[n]}{k} := \{A \subseteq [n] : |A| = k\}$. In the 1930's Erdős, Ko and Rado proved (and published in 1961) the following theorem:

Theorem 1.1 ([6]). *If $n \geq 2k$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ intersecting then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

Observe that if one considers the indicator functions of subsets of $[n]$ as elements of $\{0, 1\}^n$ (Hamming distance and ℓ_1 distance are the same in this case), then the intersecting property of $\mathcal{F} \subseteq \binom{[n]}{k}$ is equivalent with the fact that the diameter of the set of the indicator functions of the elements of $\mathcal{F}$ is at most $2k - 1$. So as the inequality is sharp in the Erdős–Ko–Rado theorem, it solves an isodiametric problem.

A set system $\mathcal{F}$ is called *t-intersecting* if $|F_1 \cap F_2| \geq t$ for all $F_1, F_2 \in \mathcal{F}$. Erdős, Ko and Rado also proved in the same article that if n is large enough, every member of the largest t-intersecting family of k-subsets of $[n]$ contains a fixed t-element set. They did not give the optimal threshold. Frankl [7] showed for $t \geq 15$ and Wilson [13] for every t that the optimal threshold is $n = (k - t + 1)(t + 1)$. Finally, Ahlswede and Khachatrian [2] determined the maximal families for all values of n. For $0 \leq t \leq k \leq n$ and $0 \leq i \leq k - t$ let

$$\mathcal{A}_{n,k,t,i} := \{A : A \subseteq [n], |A| = k, |A \cap [t + 2i]| \geq t + i\}.$$

Theorem 1.2 ([2]). *Let $0 \leq t \leq k \leq n$. If $\mathcal{F} \subseteq \binom{[n]}{k}$ is t-intersecting then*

$$|\mathcal{F}| \leq \max_{0 \leq i \leq k-t} |\mathcal{A}_{n,k,t,i}| =: AK(n, k, t).$$

This result is also a solution to an isodiametric problem.

1.3. Multiset context, definitions, notation

We think of k-multisets as choosing k elements of $[n]$ with repetition and without ordering, so there are $\binom{n+k-1}{k}$ k-multisets. Let $m(i, F)$ show how many times we chose the element i. We define two further equivalent representations.

Definition 1.3. A *multiset* F of $[n]$ is a sequence $(m(1, F), m(2, F), \ldots, m(n, F)) \in \mathbb{R}^n$ of n natural (i.e., non-negative integer) numbers. We call $m(i, F)$ the *multiplicity* of i in F, $\sum_{i=1}^n m(i, F)$ the *cardinality* of F.

We denote the cardinality of F by $|F|$ and we say that F is a *k-multiset* if $|F| = k$.

The *intersection* of two multisets G and F is a multiset defined as

$$(\min\{m(1, F), m(1, G)\}, \min\{m(2, F), m(2, G)\}, \ldots, \min\{m(n, F), m(n, G)\}).$$

We will use the notation $\mathbb{M}(n, k, t) := \{\mathcal{F} : \mathcal{F}$ is t-intersecting family of k-multisets of $[n]\}$.

If it does not cause any misunderstanding to simplify notations we use the same letter F for a multiset from $\binom{[n]}{k}$ and the corresponding vector from $\mathbb{R}^n$. For example $F = (3, 1, 2, 0, 0)$, $G = (2, 2, 0, 1, 1)$ and $F \cap G = (2, 1, 0, 0, 0)$ with vector notation if $F = \{a, a, a, b, c, c\}$ and $G = \{a, a, b, b, d, e\}$ are 6-multisets of a five element set $\{a, b, c, d, e\}$ (and $F \cap G = \{a, a, b\}$).

By definition the k-multisets of $[n]$ lie on the intersection of $\{0, 1, \ldots, k\}^n$ and the hyperplane $\sum_{i=1}^n x_i = k$. Concerning cardinality of $F \cap G$ we have

$$
\begin{aligned}
|F \cap G| &= \sum_{i=1}^{n} \min\{m(i,F), m(i,G)\} \\
&= \sum_{i=1}^{n} \frac{1}{2}(m(i,F) + m(i,G) - |m(i,F) - m(i,G)|) = k - \frac{1}{2} d_{\ell_1}(F,G),
\end{aligned}
$$

where d_{ℓ_1} denotes the ℓ_1 distance. This implies that a lower bound on the cardinality of the intersection of two elements gives an upper bound on their ℓ_1 distance. So again, an upper bound on the cardinality of a t-intersecting family of k-multisets gives a result for an isodiametric problem (in the metric space of the intersection of a hyperplane and $\{0, 1, \ldots, k\}^n$).

We give a third representation of multisets, which we will use most in the proofs of our results. Let n and ℓ be positive integers and let $M(n,\ell) := \{(i,j) : 1 \le i \le n, 1 \le j \le \ell\}$ be a $\ell \times n$ rectangle with ℓ rows and n columns. Due to practical reasons we changed the 'usual' indexing of rows and columns. We call $A \subseteq M(n,\ell)$ a *k-multiset* if the cardinality of A is k and $(i,j) \in A$ implies $(i,j') \in A$ for all $j' \le j$. Certainly $m(i,F) = \max\{s : (i,s) \in F\}$ gives the equivalence with our original vector definition.

1.4. The history of t-intersecting k-multisets

Brockman and Kay [4] stated the following conjecture:

Conjecture 1.4 (*[4], Conjecture 5.2*)**.** *There is an $n_0(k,t)$ such that if $n \ge n_0(k,t)$ and $\mathcal{F} \in \mathbb{M}(n,k,t)$, then*

$$|\mathcal{F}| \le \binom{n+k-t-1}{k-t}.$$

Furthermore, equality is achieved if and only if each member of $\mathcal{F}$ contains a fixed t-multiset of $M(n,k)$.

Meagher and Purdy [11] answered the case $t = 1$.

Theorem 1.5 (*[11]*)**.** *If $n \ge k+1$ and $\mathcal{F} \in \mathbb{M}(n,k,1)$, then*

$$|\mathcal{F}| \le \binom{n+k-2}{k-1}.$$

If $n > k+1$, then equality holds if and only if all members of $\mathcal{F}$ contain a fixed element of $M(n,k)$.

They also gave a possible candidate for the threshold $n_0(k,t)$.

Conjecture 1.6 (*[11], Conjecture 4.1*)**.** *Let k, n and t be positive integers with $t \le k$, $t(k-t)+2 \le n$ and $\mathcal{F} \in \mathbb{M}(n,k,t)$, then*

$$|\mathcal{F}| \le \binom{n+k-t-1}{k-t}.$$

Moreover, if $n > t(k-t)+2$, then equality holds if and only if all members of $\mathcal{F}$ contain a fixed t-multiset of $M(n,k)$.

If $n < t(k-t)+2$, then the family consisting of all multisets of $M(n,k)$ containing a fixed t-multiset of $M(n,k)$ still has cardinality $\binom{n+k-t-1}{k-t}$, but it cannot be the largest one for $n \ge t+2$. Indeed, if we fix a $(t+2)$-element set T and consider the family of the multisets F with $|F \cap T| \ge t+1$, we get a larger family.

1.5. The main result: extremal families have kernels

A multiset T is called a *t-kernel* of the multiset family $\mathcal{F}$ if $|F_1 \cap F_2 \cap T| \ge t$ holds for all $F_1, F_2 \in \mathcal{F}$. Obviously such a family $\mathcal{F}$ is t-intersecting. Conjecture 1.4 claims that an extremal $\mathcal{F} \in \mathcal{M}(n,k,t)$ has a t element kernel, whenever n is large. We will show that the general situation is more complex and determine the size of the maximal t-intersecting families for all $n \ge 2k-t$.

The main idea of our proof is the following: instead of the well-known *left-compression* operation, which is a usual method in the theory of intersecting families, we define (in two different ways) an operation f on $\mathbb{M}(n, k, t)$ which can be called a kind of *down-compression*.

Theorem 1.7. *Let $1 \le t \le k$, $2k - t \le n$ be arbitrary. There exists a function*

$$f : \mathbb{M}(n, k, t) \to \mathbb{M}(n, k, t)$$

satisfying the following properties:
(i) $|\mathcal{F}| = |f(\mathcal{F})|$ *for all* $\mathcal{F} \in \mathbb{M}(n, k, t)$,
(ii) $M(n, 1)$ *is a t-kernel for* $f(\mathcal{F})$.

Using Theorem 1.7 we prove the following theorem which not only verifies Conjecture 1.6, but also gives the maximum cardinality of t-intersecting families of multisets even in the case $2k - t \le n < t(k - t) + 2$.

Theorem 1.8. *Let $1 \le t \le k$ and $2k - t \le n$. If $\mathcal{F} \in \mathbb{M}(n, k, t)$ then*

$$|\mathcal{F}| \le AK(n + k - 1, k, t),$$

where the AK function is defined in Theorem 1.2.

Beside proving Theorem 1.8 our aim is to present the most powerful techniques of extremal hypergraph theory, namely the kernel and the shifting methods.

1.6. A warm up before the proofs

In this subsection we verify Conjecture 1.4 for very large n by applying the kernel method of Hajnal and Rothschild [8]. Let T be a t-multiset. For any family $\mathcal{F}$ let $\mathcal{F}_T := \{F \in \mathcal{F} : T \subseteq F\}$.

Lemma 1.9. *Let $\mathcal{F}$ be a t-intersecting family of k-multisets and T be an arbitrary t-multiset. Then either $\mathcal{F}_T = \mathcal{F}$ or $|\mathcal{F}_T| = O_n(n^{k-t-1})$.*

Proof. If $\mathcal{F}_T \ne \mathcal{F}$, then there is a multiset $F \in \mathcal{F}$ which does not contain T, hence $|F \cap T| \le t-1$. Every member of $\mathcal{F}_T$ contains T, one element of $F \setminus T$, and at most $k - t - 1$ further elements. The element of $F \setminus T$ can be chosen less than k ways, and the other $k - t - 1$ elements have to be chosen out of the nk elements of the rectangle $M(n, k)$. There are at most $k \times \binom{nk}{k-t-1} = O_n(n^{k-t-1})$ ways to do that. □

Corollary 1.10 (Conjecture 1.4)**.** *There is $n_0(k, t)$ such that if $n \ge n_0(k, t)$ and $\mathcal{F} \in \mathbb{M}(n, k, t)$, then*

$$|\mathcal{F}| \le \binom{n + k - t - 1}{k - t}.$$

Furthermore, equality is achieved if and only if each member of $\mathcal{F}$ contains a fixed t element multiset.

Proof. Let $\mathcal{F} \in \mathbb{M}(n, k, t)$ of maximum cardinality. If $\mathcal{F}_T = \mathcal{F}$ for a t-multiset T, the statement follows. If not, then let us fix an $F \in \mathcal{F}$. Every member of $\mathcal{F}$ contains a t-multiset which is also contained in F, hence $\bigcup\{\mathcal{F}_T : T \subset F, |T| = t\} = \mathcal{F}$. Thus $|\mathcal{F}| \le \sum_{T \subset F, |T|=t} |\mathcal{F}_T|$. By Lemma 1.9 $|\mathcal{F}_T| = O_n(n^{k-t-1})$, and there are $\binom{k}{t}$ members of the sum, hence $|\mathcal{F}| \le \binom{k}{t} O_n(n^{k-t-1}) < \binom{n+k-t-1}{k-t}$ if n is large enough. □

To prove Conjecture 1.6 at first we applied a straight-forward generalization of the usual shifting. However, we could not give a threshold below $\Omega(kt \log k)$ using this method. Still we believe it is worth mentioning, as it might be useful solving other related problems.

Suppose that $F \subseteq M(n, \ell)$ is a k-multiset such that $m(i, F) < m(j, F)$ for some $i < j$. Let F' be obtained by exchanging columns i and j, i.e.,

$$F' := (F \setminus \{(j, m(i, F) + 1), \dots, (j, m(j, F))\}) \cup \{(i, m(i, F) + 1), \dots, (i, m(j, F))\}.$$

Let $\mathcal{F} \in \mathbb{M}(n, k, t)$. Define $c_{i,j}(F)$ for each $F \in \mathcal{F}$ as follows.

$$c_{i,j}(F) := \begin{cases} F' & \text{if } m(F,j) - m(F,i) > 0 \quad \text{and} \quad F' \notin \mathcal{F}, \\ F & \text{otherwise.} \end{cases}$$

Let us use the following notation: $c_{i,j}(\mathcal{F}) = \{c_{i,j}(F) : F \in \mathcal{F}\}$. Note that this is the same as the well-known shifting operation on *subsets* of $[n]$.

Lemma 1.11. *$c_{i,j}(\mathcal{F}) \in \mathbb{M}(n, k, t)$ for $\mathcal{F} \in \mathbb{M}(n, k, t)$.*

Proof. If both or neither of $c_{i,j}(F_1)$ and $c_{i,j}(F_2)$ are members of $\mathcal{F}$, then $|c_{i,j}(F_1) \cap c_{i,j}(F_2)| = |F_1 \cap F_2|$ and this intersection has size at least t. From now on, we can assume $c_{i,j}(F_1) = F_1' \notin \mathcal{F}$ and $c_{i,j}(F_2) = F_2 \in \mathcal{F}$.

Case 1: $m(F_2, i) < m(F_2, j)$. Since $c_{i,j}(F_2) = F_2$, this means that $F_2' \in \mathcal{F}$. Then $|F_1' \cap F_2| = |F_2' \cap F_1| \geq t$.

Case 2: $m(F_2, i) \geq m(F_2, j)$. We know that $m(F_1, i) \leq m(F_1, j)$.

Let x be the cardinality of the intersection of F_1 and F_2 in the complement of the union of the ith and jth column. We have

$$|F_1 \cap F_2| = x + \min\{m(F_2, i), m(F_1, i)\} + \min\{m(F_2, j), m(F_1, j)\} \geq t \quad \text{and}$$
$$|c_{i,j}(F_1) \cap c_{i,j}(F_2)| = x + \min\{m(F_2, i), m(F_1, j)\} + \min\{m(F_2, j), m(F_1, i)\}.$$

Apply the following inequality (1) below with $a = m(F_2, i)$, $b = m(F_2, j)$, $c = m(F_1, j)$, and $d = m(F_1, i)$ to obtain $|c_{i,j}(F_1) \cap c_{i,j}(F_2)| \geq |F_1 \cap F_2| \geq t$. This completes the proof of Lemma 1.11. □

If $a \geq b$ and $c \geq d$ reals then

$$\min\{a, c\} + \min\{b, d\} \geq \min\{a, d\} + \min\{b, c\}. \tag{1}$$

This inequality can be easily checked by listing all the 6 possible orderings of $\{a, b, c, d\}$. For example, for $a \geq c \geq d \geq b$ the left hand side is $c + b$ and the right hand side is $d + b$. □

Remark. It is worth mentioning that there is an even more straightforward generalization of shifting, when we just decrease the multiplicity in column j by one and increase it in column i by one. Let $F' := F \cup \{(i, m(i, F)) + 1\} \setminus \{(j, m(j, F))\}$, and

$$c'_{i,j}(F) := \begin{cases} F' & \text{if } m(j, F) > m(i, F) \text{ and } F' \notin \mathcal{F}, \\ F & \text{otherwise.} \end{cases}$$

But this operation does not necessarily preserve the t-intersecting property. However, if we apply our shifting operation to a maximum t-intersecting family and for every pair (i, j), the resulting family will be also shifted according to this second kind of shifting, meaning that applying this second operation does not change the family.

After applying $c_{i,j}$ for every pair i, j, the resulting shifted family has several different t-kernels. For example the union of two rectangles $\lfloor t^{1/2} \rfloor \times (2k - t) \cup t \times \lceil \frac{2k}{t^{1/2}} \rceil$ (more precisely its intersection with $M(n, l)$) is a t-kernel. Another t-kernel is the subset of $M(n, l)$ where the members (x, y) satisfy $yx \leq k,\ x \leq 2k - t,\ y \leq k,\ x, y \geq 0$.

Using these t-kernels and some algebra, we achieved that $n_0(k, t) = O(kt \log k)$. To lower this threshold we had to develop the *down-compression* techniques described in the next section.

2. Proofs of Theorem 1.7

Both methods described below have underlying geometric ideas, it is a kind of discrete, tilted version of symmetrizing a set with respect to the hyperplane $x_i = x_j$.

2.1. A constructive proof

First proof of Theorem 1.7.

We will consider multisets which contain almost exactly the same elements, they differ only in two columns. More precisely, we are interested in multisets whose symmetric difference is a subset of $\{(i,j) : 1 \le j \le k\} \cup \{(i',j) : 1 \le j \le k\}$ with $i \ne i'$. If we are given two such multisets, we can consider the two columns together, one going from k to 1 and the other going from 1 to k. This way the restriction of two multisets to these columns form a subinterval of an interval of length $2k$ (where *interval* means set of consecutive integers). Hence we examine families of intervals.

2.1.1. *A lemma about interval systems*

Let $X := \{1, \ldots, 2k\}$. Let $Y \subset X$ be an interval and p be an integer with $p \le |Y|$, then we define a family of intervals $I(p, Y)$ to be all the p-element subintervals of Y and let $\mathcal{I} := \{I(p, Y)\}$.

We consider a shifted version, where the intervals are pushed to the middle. Let

$$\varphi(Y) := \{k - \lfloor |Y|/2 \rfloor, \ldots, k + \lceil |Y|/2 \rceil - 1\} \quad \text{and} \quad \varphi(I(p, Y)) := I(p, \varphi(Y)).$$

We will show that this operation does not decrease the size of the intersection of two families in $\mathcal{I}$. Let

$$d(I(p, Y), I(q, Y')) := \min\{|J \cap J'| : J \in I(p, Y), J' \in I(q, Y')\}.$$

Lemma 2.1. *$d(I(p, \varphi(Y)), I(q, \varphi(Y'))) \ge d(I(p, Y), I(q, Y'))$ for all possible p, q, Y and Y'.*

Proof. Obviously the smallest intersection is the intersection of the first interval in one of the families and the last interval in the other family. As the length of the intervals are always p and q, the only thing that matters is the difference between the starting and ending points of $\varphi(Y)$ and $\varphi(Y')$. More precisely we want to minimize the largest of $y = \max \varphi(Y) - \min \varphi(Y')$ and $y' = \max \varphi(Y') - \min \varphi(Y)$. As $\max \varphi(Y) - \min \varphi(Y) = |Y| - 1$ and $\max \varphi(Y') - \min \varphi(Y') = |Y'| - 1$, we know that $y + y'$ is constant, hence we get the minimum if y and y' is as close as possible. One can easily see that our shifted system gives this. □

2.1.2. *Interval systems and families of multisets*

Now to apply the method of the previous subsection, we fix $n - 2$ coordinates, i.e., we are given $1 \le i < j \le n$, $g : ([n] \setminus \{i, j\}) \to [1, k]$ and let

$$\mathcal{F}_g := \{F \in \mathcal{F} : m(r, F) = g(r) \text{ for every } r \ne i, j\}.$$

It implies $m(i, F) + m(j, F)$ is the same number $s := s(g)$ for every member $F \in \mathcal{F}_g$.

Let us consider now the case $\mathcal{F}$ is maximal, i.e., no k-multiset can be added to it without violating the t-intersecting property. We show that it implies that the integers $m(i, F)$ are consecutive for $F \in \mathcal{F}_g$. Let $m_i := \min\{m(i, F) : F \in \mathcal{F}_g\}$ and $M_i := \max\{m(i, F) : F \in \mathcal{F}_g\}$. We define m_j and M_j similarly. Let us consider a set $F \notin \mathcal{F}$ which satisfies $m(r, F) = g(r)$ for all $r \ne i, j$ and also $m_i \le m(i, F) \le M_i$, and consequently $m_j \le m(j, F) \le M_j$. It is easy to see that F can be added to $\mathcal{F}$ without violating the t-intersecting property (and then it belongs to $\mathcal{F}_g$).

Now we give a bijection between these type of families and interval systems. We lay down both columns, such that column i starts at its top, and column j start at its bottom. Then move them next to each other to form an interval. More precisely let $\Psi_{i,j}((i, u)) = k - u + 1$ and $\Psi_{i,j}((j, u)) = k + u$. We omit indices i and j for the sake of simplicity, and denote the function $\Psi_{i,j}$ by Ψ. For a multiset F let $\Psi(F) = \{\Psi((i, u)) : (i, u) \in F\} \cup \{\Psi((j, u)) : (j, u) \in F\}$ and for a family of multisets $\mathcal{F}$ let $\Psi(\mathcal{F}) = \{\Psi(F) : F \in \mathcal{F}\}$.

We show that $\Psi(\mathcal{F}_g) \in \mathcal{I}$. It is obvious that $\Psi(F)$ is an interval for any multiset F, and that the length of those intervals is the same number (more precisely s) for every member $F \in \mathcal{F}_g$. We need to show that the intervals $\Psi(F)$ (where $F \in \mathcal{F}_g$) are all the subintervals of an interval Y. It is enough to show that the starting points of these intervals are consecutive integers. The starting points of the intervals $\Psi(F)$ are $\Psi((i, m(i, F)))$, and it is easy to see that they are consecutive if and only if $m(i, F)$ are consecutive.

Since Ψ is a bijection, an interval system also defines a family in the two columns i and j. Let us examine what family we get after applying operation φ from the previous section, i.e., what $\mathcal{F}' = \Psi^{-1}(\varphi(\Psi(\mathcal{F}_g)))$ is. Obviously it is a family of s-multisets with the same cardinality as $\mathcal{F}_g$.

Simple calculations show that they are the s-multisets with $m(i, F) \leq \lfloor (m_i + M_j)/2 \rfloor + 1$ and $m(j, F) \leq \lceil (m_i + M_j)/2 \rceil - 1$.

2.1.3. The construction of f

Let $\psi(\mathcal{F}_g) = \Psi^{-1}(\varphi(\Psi(\mathcal{F}_g)))$, i.e., the family we get from $\mathcal{F}_g$ by keeping everything in the other $n-2$ columns, but making it balanced in the columns i and j in the following sense. It contains all the k-multisets where $m(i, F) \leq \lfloor (m_i + M_j)/2 \rfloor + 1$, $m(j, F) \leq \lceil (m_i + M_j)/2 \rceil - 1$ and the other coordinates are given by g.

Now let us recall that i and j are fixed. Let $G_{i,j}$ be the set of every $g : ([n] \setminus \{i, j\}) \to [1, k]$, i.e., every possible way to fix the other $n-2$ coordinates. Clearly $\mathcal{F} = \cup\{\mathcal{F}_g : g \in G_{i,j}\}$ and they are all disjoint. Let $\psi_{i,j}(\mathcal{F})$ denote the result of applying the appropriate ψ operation for every g at the same time, i.e., $\psi_{i,j}(\mathcal{F}) = \cup\{\psi(\mathcal{F}_g) : g \in G_{i,j}\}$.

Lemma 2.2. *If $\mathcal{F}$ is t-intersecting, then $\psi_{i,j}(\mathcal{F})$ is t-intersecting.*

Proof. Suppose there are $F_1, F_2 \in \psi_{i,j}(\mathcal{F})$ with $|F_1 \cap F_2| < t$. Let $F_1 \in \psi_{i,j}(\mathcal{F}_{g_1})$ and $F_2 \in \psi_{i,j}(\mathcal{F}_{g_2})$. Let $\Psi(\mathcal{F}_{g_1}) = I(p_1, Y_1)$ and $\Psi(\mathcal{F}_{g_2}) = I(p_2, Y_2)$. Then $\Psi(\psi_{i,j}(\mathcal{F}_{g_1})) = \varphi(I(p_1, Y_1))$ and $\Psi(\psi_{i,j}(\mathcal{F}_{g_2})) = \varphi(I(p_2, Y_2))$. It is important to see that Ψ is defined on the elements of $M(n, k)$ such a way that the size of the intersection is the same after applying Ψ.

By Lemma 2.1 $d(I(p_1, \varphi(Y_1)), I(p_2, \varphi(Y_2))) \geq d(I(p_1, Y_1), I(p_2, Y_2))$, which means that there is a member of $\mathcal{F}_{g_1}$ and a member of $\mathcal{F}_{g_2}$ such that their intersection has size at most the size of the smallest intersection between members of $\psi_{i,j}(\mathcal{F}_{g_1})$, which is less than t, a contradiction.

Note that the result is clear if g_1 is equal to g_2. □

In the next lemma we give a non-negative integer function, that decreases when the family of multisets changes after applying $\psi_{i,j}$. Which means that our process will finish in finitely many steps.

Lemma 2.3. *If $\psi_{i,j}(\mathcal{F}) \neq \mathcal{F}$ then*

$$\sum_{F' \in \psi_{i,j}(\mathcal{F})} \left[|\mathcal{F}| nk^2 \sum_{i \in [n]} (m(i, F'))^2 + \sum_{i \in [n]} i(m(i, F')) \right] < \sum_{F \in \mathcal{F}} \left[|\mathcal{F}| nk^2 \sum_{i \in [n]} (m(i, F))^2 + \sum_{i \in [n]} i(m(i, F)) \right]. \quad (2)$$

Sketch of the proof of Lemma 2.3. We know that after applying $\psi_{i,j}$ we have

$$\sum_{F' \in \psi_{i,j}(\mathcal{F})} \sum_{i \in [n]} m(i, F')^2 \leq \sum_{F \in \mathcal{F}} \sum_{i \in [n]} m(i, F)^2 \quad (3)$$

by symmetrization. If the left hand side in (3) is less than the right hand side, then we are done, since the coefficient of these terms is so big in (2), that the left hand side of (2) will be less than the right hand side of it.

If the left hand side and the right hand side of (3) are equal, then using the property of the symmetrization, that if $i < j$, then $\psi_{i,j}(\mathcal{F})$ contains all multisets with $m(i, F) \leq \lfloor (m_i + M_j)/2 \rfloor + 1$ and $m(j, F) \leq \lceil (m_i + M_j)/2 \rceil - 1$ so by $\lfloor (m_i + M_j)/2 \rfloor + 1 > \lceil (m_i + M_j)/2 \rceil - 1$ we have that

$$\sum_{F' \in \psi_{i,j}(\mathcal{F})} \sum_{i \in [n]} i m(i, F') < \sum_{F \in \mathcal{F}} \sum_{i \in [n]} i m(i, F).$$

So we are done with Lemma 2.3. □

Now we are ready to define $f(\mathcal{F})$. If there is a pair (i, j) such that $\psi_{i,j}(\mathcal{F}) \neq \mathcal{F}$, let us replace $\mathcal{F}$ by $\psi_{i,j}(\mathcal{F})$, and repeat this step. Lemma 2.3 implies that it can be done only finitely many times, after that we arrive to a family $\mathcal{F}'$ such that $\psi_{i,j}(\mathcal{F}') = \mathcal{F}'$ for every pair (i, j). This family is denoted by $f(\mathcal{F})$.

We would like to prove that f satisfies Theorem 1.7(ii). This step is the only point we use that $n \geq 2k - t$.

Lemma 2.4. *$|F_1 \cap F_2 \cap M(n, 1)| \geq t$ for all $F_1, F_2 \in f(\mathcal{F})$.*

Proof. We argue by contradiction. Let us choose F_1 and F_2 such a way that $|F_1 \cap F_2 \cap M(n, 1)|$ is the smallest (definitely less than t), and among those $|F_1 \cap F_2|$ is the smallest (definitely at least t). Then there is a coordinate where both F_1 and F_2 have at least 2, and this implies there is an other coordinate, where both have 0, as $2k - t \leq n$. More precisely, there is an $i \leq n$ with $2 \leq \min\{m(i, F_1), m(i, F_2)\}$ and a $j \leq n$ with $m(j, F_1) = m(j, F_2) = 0$. Let F_1' be defined the following way: $m(j, F_1') = 1$, $m(i, F_1') = m(i, F_1) - 1$ and $m(s, F_1') = m(s, F_1)$ for $s \leq n, s \neq i, j$. One can easily see that $F_1' \in \psi_{i,j}(f(\mathcal{F})) = f(\mathcal{F})$. However, $|F_1' \cap F_2| < |F_1 \cap F_2|$ and $|F_1' \cap F_2 \cap M(n, 1)| = |F_1 \cap F_2 \cap M(n, 1)|$, a contradiction. □

To finish the proof of Theorem 1.7 we have to deal with the case $\mathcal{F}$ is not maximal (even though it is not needed in order to prove Theorem 1.8). For sake of brevity here we just give a sketch.

Note that Ψ can be similarly defined in this case. The main difference is that the resulting family of intervals is not in $\mathcal{I}$, as it does not contain all the subintervals of an interval. Also note that $\varphi(I(p, Y))$ is determined by the number and length of the intervals in $I(p, Y)$. Using this we can extend the definition of φ to any family of intervals. This way we can define $\psi_{i,j}$ as well. What happens is that besides being more balanced in the columns i and j, the multisets in $\mathcal{F}_g$ are also pushed closer to each other. Hence one can easily see that the intersections cannot be smaller in this case, which finishes the proof of Theorem 1.7.

So we are done with the first proof of Theorem 1.7. □

2.2. A less constructive, second proof of Theorem 1.7

Proof. For $F \in \mathcal{F} \in \mathbb{M}(n, k, t)$, $1 \leq i, j \leq n$ and $1 \leq s \leq k$, if $s \leq m(i, F)$ then let

$$F' := F \setminus (\cup_{s \leq t \leq m(i,F)}(i, t)) \bigcup (\cup_{1 \leq l \leq m(i,F)-s+1}(j, l)).$$

Using this notation we define a shifting operation.

Definition 2.5. For $F \in \mathcal{F} \in \mathbb{M}(n, k, t)$, $1 \leq i, j \leq n$ and $1 \leq s \leq k$ let

$$S((i, s), j)(F) := \begin{cases} F' & \text{if } (j, 1) \notin F,\ F' \notin \mathcal{F} \text{ and } F' \text{ is defined,} \\ F & \text{otherwise.} \end{cases}$$

For $\mathcal{F} \in \mathbb{M}(n, k, t)$ let $S((i, s), j)(\mathcal{F}) := \{S((i, s), j)(F) : F \in \mathcal{F}\}$.

Now we prove that an operation defined in Definition 2.5 with special choice of s preserves the t-intersection property of $\mathcal{F}$. For $\mathcal{F} \in \mathbb{M}(n, k, t)$ let $\mathcal{K}(\mathcal{F})$ be the set of t-*kernels* of $\mathcal{F}$ which contain $M(n, 1)$ and are multisets.

Lemma 2.6. *Suppose that $1 \leq i, j \leq n$, $\mathcal{F} \in \mathbb{M}(n, k, t)$ and $T \in \mathcal{K}(\mathcal{F})$. Then and $T \in \mathcal{K}(S((i, m(i, T)), j)(\mathcal{F}))$.*

Proof of Lemma 2.6. It is easy to see that the elements of $S((i, m(i, T)), j)(\mathcal{F})$ are multisets of cardinality k. So we only have to prove that $S((i, m(i, T)), j)(\mathcal{F})$ is t-intersecting and $T \in \mathcal{K}(S((i, m(i, T)), j)(\mathcal{F}))$.

Choose two arbitrary members of $F, G \in \mathcal{F}$.

If both $S((i, m(i, T)), j)(F)$ and $S((i, m(i, T)), j)(G)$ are elements of $\mathcal{F}$, then we are easily done.

If neither $S((i, m(i, T)), j)(F)$ nor $S((i, m(i, T)), j)(G)$ are elements of $\mathcal{F}$, then we use the special choice of s in Definition 2.5, i.e., $s = m(i, T)$. By this we have that

$$|(\cup_{m(i,T) \leq t \leq m(i,F)}(i, t)) \cap F \cap G| = 1.$$

However also by Definition 2.5 we have that $(j, 1) \notin F \cup G$ and $(j, 1) \in S((i, m(i, T)), j)(F) \cap S((i, m(i, T)), j)(G)$ and since elements of $\mathcal{K}(\mathcal{F})$ contains $M(n, 1)$, so in this case we are done with Lemma 2.6.

To finish the proof without loss of generality we can assume that $S((i, m(i, T)), j)(F) \notin \mathcal{F}$ and $S((i, m(i, T)), j)(G) \in \mathcal{F}$. By this we know that either $m(i, T) \leq m(i, G)$ and $(j, 1) \in G$ or $m(i, G) < m(i, T)$.

If $m(i,T) \le m(i,G)$ and $(j,1) \in G$, then we are done with Lemma 2.6 similarly as in the previous case.

If $m(i,G) < m(i,T)$, then we have

$$|(\cup_{m(i,T)\le t\le m(i,F)}(i,t)) \cap F \cap G| = 0.$$

So in this case we are trivially done with Lemma 2.6.

We finished the proof of Lemma 2.6. □

For $T \in \mathcal{K}(\mathcal{F})$ let $T_{>1} := T \setminus M(n,1)$. Now we define an operation $(\mathcal{F}')$ on $\mathbb{M}(n,k,t)$ such that $\min\{|T_{>1}| : T \in \mathcal{K}(\mathcal{F})\} > \min\{|T_{>1}| : T \in \mathcal{K}(\mathcal{F}')\}$ for any $\mathcal{F} \in \mathbb{M}(n,k,t)$ with $\min\{|T_{>1}| : T \in \mathcal{K}(\mathcal{F})\} > 0$.

Choose any $\mathcal{F} \in \mathbb{M}(n,k,t), T \in \mathcal{K}(\mathcal{F})$ satisfying $|T_{>1}| > 0$ and let us apply $S((i,m(i,T)),1)$ on $\mathcal{F}$, then $S((i,m(i,T)),2)$ on the resulting family, and so on. Let $\mathcal{F}'$ be the resulting family after applying $S((i,m(i,T)),n)$, i.e.,

$$\mathcal{F}' := S((i,m(i,T)),n)[\ldots[S((i,m(i,T)),2)[S((i,m(i,T)),1)(\mathcal{F})]]\ldots].$$

Lemma 2.7. *Let $\mathcal{F} \in \mathbb{M}(n,k,t), T \in \mathcal{K}(\mathcal{F})$ satisfying $|T_{>1}| > 0$ and let $1 \le i \le n$, $2 \le m(i,T)$. Then we have:*

(i) *$\mathcal{F}' \in \mathbb{M}(n,k,t)$ and $|\mathcal{F}| = |\mathcal{F}'|$,*

(ii) *$(T \setminus (i,m(i,T))) \in \mathcal{K}(\mathcal{F}')$.*

Proof. We start by proving (i).

The facts that $S((i,m(i,T)),n)[\ldots[S((i,m(i,T)),2)[S((i,m(i,T)),1)(F)]]\ldots] \subseteq M(n,k)$, has cardinality k for any $F \in \mathcal{F}$ and that $|\mathcal{F}'| = |\mathcal{F}|$, are trivial. The proof that $\mathcal{F}'$ is t-intersecting is an easy consequence of applying Lemma 2.6 n times and using that $T \in \mathcal{K}(S((i,m(i,T)),n)[\ldots[S((i,m(i,T)),2)[S((i,m(i,T)),1)(F)]]\ldots])$.

It is enough to prove that $S((i,m(i,T)),1)(\mathcal{F})$ is t-intersecting and that T is a t-kernel for the new family, i.e., $T \in \mathcal{K}(S((i,m(i,T)),1)(\mathcal{F}))$, since repeatedly applying this fact we will get the claim.

We are done with the proof of (i) of Lemma 2.7.

Now we prove (ii):

Choose $F, G \in \mathcal{F}$ and let us use the following notation:

$$S(F) := S((i,m(i,T)),n)[\ldots[S((i,m(i,T)),2)[S((i,m(i,T)),1)(F)]]\ldots] \quad \text{and}$$
$$S(G) := S((i,m(i,T)),n)[\ldots[S((i,m(i,T)),2)[S((i,m(i,T)),1)(G)]]\ldots].$$

Now we have to prove that

$$|S(F) \cap S(G) \cap (T \setminus (i,m(i,T)))| \ge t.$$

If $S(F) \ne F$ or $S(G) \ne G$, then we are done similarly as in the previous claim using the fact that $(i,m(i,T)) \notin S(F) \cap S(G)$.

If $S(F) = F$ and $S(G) = G$, then

(a) if $(i,m(i,T)) \notin F \cap G$ we are easily done,

(b) if $(i,m(i,T)) \in F \cap G$ then since $2 \le m(i,T)$ and $2k - t \le n$, there is $j \le n$ with $(j,1) \notin F \cup G$. However as we have $S(F) = F$ and $S(G) = G$ now, $S((i,m(i,T)),j)(F) \in \mathcal{F}$, so

$$t \le |S((i,m(i,T)),j)(F) \cap G \cap T| = |F \cap G \cap (T \setminus (i,m(i,T)))|.$$

We are done with the proof of Lemma 2.7. □

To finish the proof of Theorem 1.7, note that by Lemma 2.7 for any $\mathcal{F} \in \mathbb{M}(n,k,t)$ there will be a smallest natural number ℓ such that after applying the operation before Lemma 2.6 ℓ times on $\mathcal{F}$ repeatedly, we get a family such that $M(n,1)$ is a t-kernel of it. Let us define $f(\mathcal{F})$ to be this family.

We are done with the second proof of Theorem 1.7. □

3. Proof of Theorem 1.8

Let $\mathcal{G}_s := \{F \cap M(n,1) : F \in f(\mathcal{F}),\ |F \cap M(n,1)| = s\}$. Let us consider $G \in \mathcal{G}_s$ and examine the number of multisets $F \in \mathcal{F}$ with $G = F \cap M(n,1)$. Obviously $k - s$ further elements belong to F, and they are in the same s columns, they can be chosen at most $\binom{s+(k-s-1)}{k-s}$ ways. Then we know that

$$|\mathcal{F}| = |f(\mathcal{F})| \leq \sum_{s=t}^{k} |\mathcal{G}_s| \binom{s+(k-s-1)}{k-s} = \sum_{s=t}^{k} |\mathcal{G}_s| \binom{k-1}{k-s}.$$

Now consider a family $\mathcal{F}'$ of sets on an underlying set of size $n+k-1$. Let it be the same on the first n-elements as $f(\mathcal{F})$ in $M(n, 1)$, and extend every s-element set there with all the $(k-s)$-element subsets of the remaining $k-1$ elements of the underlying set. It can happen $\binom{k-1}{k-s}$ ways, thus the cardinality of this family is the right hand side of the above inequality.

Note that $\mathcal{F}'$ is t-intersecting, hence its cardinality is at most $AK(n+k-1, k, t)$, which completes the proof of Theorem 1.8.

4. Concluding remarks

Note that the bound given in Theorem 1.8 is sharp. Using a family $\mathcal{A}_{n+k-1,k,t,i}$ we can define an optimal t-intersecting family of k-multisets in $M(n, k)$. However, we do not know any nontrivial bounds in case of $n < 2k - t$.

After repeated down-shifting we get a following structure theorem. Let $e_i \in \mathbb{R}^n$ denote the standard unit vector with 1 in its ith coordinate.

Lemma 4.1 (*Stable Extremal Families*)**.** *There exists a family $\mathcal{F} \in \mathbb{M}(n, \ell, k, t)$ of maximum cardinality satisfying the following two properties:*

(i) *$\forall i \neq j$ and $F \in \mathcal{F}$, $m(i, F) + 1 < m(j, F)$ imply that $(F - e_j + e_i) \in \mathcal{F}$, too, and*

(ii) *the same holds if $i < j$ and $m(i, F) + 1 \leq m(j, F)$.*

Knowing the structure of $\mathcal{F}$ might help to determine $\max |\mathcal{F}|$ for all n.

The original Erdős–Ko–Rado theorem (Theorem 1.1) concerns the maximum size of an independent set in the Kneser graph. A powerful method to estimate the size of an independent set was developed by Lovász. Indeed, Wilson [13] extended the Erdős–Ko–Rado theorem by determining the Shannon capacity of the generalized Kneser graph. It would be interesting if his ideas were usable for the multiset case, too.

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