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Linear paths and trees in uniform hypergraphs

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Abstract

A linear path $\mathbb{P}_\ell^{(k)}$ is a family of k-sets $\{F_1, \dots, F_\ell\}$ such that $|F_i \cap F_{i+1}| = 1$ and there are no other intersections. We can represent the hyperedges by intervals. With an intensive use of the delta-system method we prove that for $t > 0$, $k > 3$ and sufficiently large n, $(n > n_0(k,t))$, if $\mathcal{F}$ is an n-vertex k-uniform family with

$$|\mathcal{F}| > \binom{n-1}{k-1} + \binom{n-2}{k-1} + \dots + \binom{n-t}{k-1},$$

then it contains a linear path of length $2t+1$. The only extremal family consists of all edges meeting a given t-set. We also determine $\mathbf{ex}_k(n, \mathbb{P}_{2t}^{(k)})$ exactly, and the Turán number of any linear tree asymptotically.

Keywords: extremal uniform hypergraphs, Turán numbers, trees, paths.

1 Introduction

One of the central problems of extremal hypergraph theory is the description of unavoidable subhypergraphs, in other words, the Turán problem.

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Speaking about a hypergraph $\mathbb{F} = (V, \mathcal{F})$ we frequently identify the vertex set $V = V(\mathbb{F})$ by the set of first integers $[n] := \{1, 2, \ldots, n\}$. The set of all k susbsets of $[n]$ is denoted by $\binom{[n]}{k}$. To shorten notations we frequently say "hypergraph $\mathcal{F}$" (or set system $\mathcal{F}$) thus identifying $\mathbb{F}$ to its edge set $\mathcal{F}$. A hypergraph is k-*uniform* if every edge has k elements, $\mathcal{F} \subset \binom{[n]}{k}$. The *Turán number* of a set of k-uniform hypergraphs $\mathbb{H} := \{\mathcal{H}_1, \mathcal{H}_2, \ldots\}$, denoted by $\mathbf{ex}(n, \mathbb{H})$, is the size of the largest $\mathbb{H}$-free k-graph on n vertices. If we want to emphasize k, then we write $\mathbf{ex}_k(n, \mathbb{H})$. $\mathbb{F}$ is *linear* if for all $A, B \in \mathcal{F}$, $A \neq B$ we have $|A \cap B| \leq 1$. The *degree*, $\deg_{\mathbb{F}}(x)$, of an element $x \in [n]$ is the number of hyperedges of $\mathcal{F}$ containing x.

2 Linear paths of k-graphs

Since the paper by G. Y. Katona and Kierstead [12] (1999) there is a renewed interest concerning path and (Hamilton) cycles in uniform hypergraphs. Most of these are Dirac type results (large minimum degree implies the existence of the desired substructure) like in Kühn and Osthus [14], Rödl, Ruciński, and Szemerédi [15] or in Dorbec, Gravier, and Sárközy [2].

There are many ways to define a path in a hypergraph. Here we consider *linear* paths of *length* ℓ. These are formed by ℓ sets $\{F_1, \ldots, F_\ell\}$ such that $|F_i \cap F_{i+1}| = 1$ (for all $1 \leq i < \ell$) and $F_i \cap F_j = \emptyset$ for $i + 1 < j$. We can represent such a path by intervals of consecutive integers. The k-uniform path of length ℓ is denoted by $\mathbb{P}_\ell^{(k)}$. Note that this notation is different from what is usually used in graphs, where, e.g., P_6 denotes a 6-vertex path.

Győri, G. Y. Katona, and Lemons [10] gave estimates of the Turán number of the k-uniform ℓ-path. Their upper bound is about k-times larger than the lower bound. Here we determine the Turán number exactly, but only for large enough n. The family

$$\mathcal{F}^{(k)}(n, t) := \{F \subset [n] : |F| = k,\ F \cap [t] \neq \emptyset\}$$

does not have $t+1$ disjoint edges, so it does not contain a path of length $2t+1$.

Theorem 2.1 (Füredi, Jiang, and Siever [9], Frankl & ZF [5] for $t = 1$) *If $\mathcal{F} \subset \binom{[n]}{k}$ does not contain a linear path of length $2t + 1$, n is sufficiently large, $n > n_{k,t}$, and $k \geq 4$, then $|\mathcal{F}| \leq |\mathcal{F}^{(k)}(n, t)|$. Thus*

$$\mathbf{ex}(n, \mathbb{P}_{2t+1}^{(k)}) = |\mathcal{F}^{(k)}(n, t)| = \binom{n-1}{k-1} + \ldots + \binom{n-t}{k-1}.$$

The only extremal family is $\mathcal{F}^{(k)}(n,t)$. For the case of even length $2t+2$ one can add a few more edges to $\mathcal{F}^{(k)}(n,t)$, namely any family avoiding an intersection size 2. Let

$$\mathcal{F}' := \{F \subset [n] : |F| = k, F \cap [t] = \emptyset, \{t+1, t+2\} \subset F\}.$$

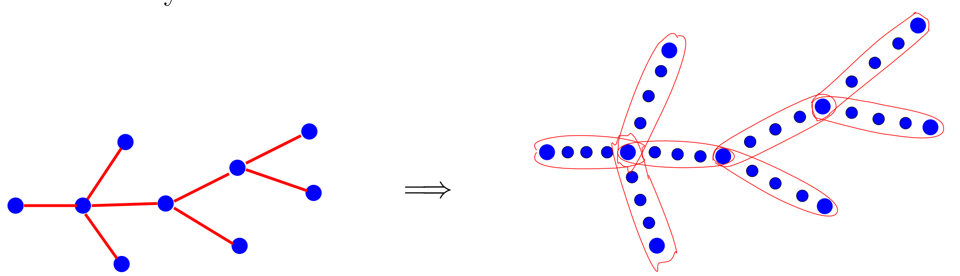
Theorem 2.2 (Füredi, Jiang, and Siever [9], Frankl [4] for $t = 0$) *Suppose $k \geq 4$, $t \geq 0$ integers, n is sufficiently large, $n > n_{k,t}$. Then*

$$\mathbf{ex}(n, \mathbb{P}^{(k)}_{2t+2}) = |\mathcal{F}^{(k)}(n,t) \cup \mathcal{F}'| = \binom{n-1}{k-1} + \ldots + \binom{n-t}{k-1} + \binom{n-t-2}{k-2}.$$

Again, the extremal family is unique. The case $k = 3$ is slightly different.

3 Linear trees

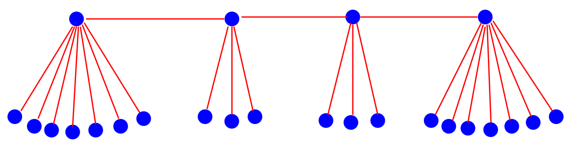
A family of sets $F_1, \ldots, F_\ell$ is called a *linear* tree, if F_i meets $\cup_{j<i} F_j$ in exactly one vertex for all $1 < i \leq \ell$. Any (usual, 2-uniform) tree $\mathbb{T}$ can be blown up in a natural way to a k-uniform linear tree $\mathbb{T}^k$.



Let τ denote the minimum number of vertices to cover all edges of $\mathbb{T}$. We have $\mathbf{ex}_k(n, \mathbb{T}^k) \geq |\mathcal{F}^{(k)}(n, \tau - 1)| = (\tau - 1 + o(1))\binom{n-1}{k-1}$. Define

$$\sigma(\mathbb{T}) := \min\{|A| + e(\mathbb{T} \setminus A) : A \subset V(\mathbb{T}) \text{ is independent}\}.$$

We have $\tau \leq \sigma \leq |V_1| \leq |V_2|$, where $V_1 \cup V_2 = V$ is the unique two-coloring of $\mathbb{T}$. In the next figure $\mathbb{T}$ has $D + c + c + D$ pending edges, $D > c$.



$\tau = 4, \ \sigma = 2c + 3, \ \ |V| = 2 \times (c + D + 2) = 2|V_1|.$

Define $\mathcal{F}_0^{(k)}(n,s) := \{F \in \binom{[n]}{k} : |F \cap \{1, 2, \ldots, s\}| = 1\}$. This hypergraph does not contain $\mathbb{T}^k$.

Theorem 3.1 (see [8])

$$(\sigma-1)\binom{n-\sigma+1}{k-1} \leq \mathbf{ex}(n,\mathbb{T}^k) = (\sigma-1+o(1))\binom{n-1}{k-1}.$$

4 The Erdős–Sós and the Kalai conjecture

A system of k-sets $\mathbb{T} := \{E_1, E_2, \ldots, E_q\}$ is called a **tight** k-tree if for every $2 \leq i \leq q$ we have $|E_i \setminus \cup_{j<i} E_j| = 1$, and there exists an $\alpha = \alpha(i) < i$ such that $|E_\alpha \cap E_i| = k-1$. The case $k = 2$ corresponds to the usual trees in graphs. Suppose that $\mathbb{T}$ is a tight k-tree on v vertices. Consider a $P(n, v-1, k-1)$ packing $P_1, \ldots, P_m$ on the vertex set $[n]$ (i.e., $|P_i| = v-1$ and $|P_i \cap P_j| < k-1$ for $1 \leq i < j \leq m$) and replace each P_i by a complete k-graph. We obtain a $\mathbb{T}$-free hypergraph. Then Rödl's [11] theorem on almost optimal packings gives

$$\mathbf{ex}_k(n,\mathbb{T}) \geq (1-o(1))\frac{\binom{n}{k-1}}{\binom{v-1}{k-1}} \times \binom{v-1}{k} = (1+o(1))\frac{v-k}{k}\binom{n}{k-1}.$$

Conjecture 4.1 (Erdős and Sós for graphs, Kalai 1984 for all k, see in [5])

$$\mathbf{ex}_k(n,\mathbb{T}) \leq \frac{v-k}{k}\binom{n}{k-1}.$$

The Erdős–Sós conjecture has been recently proved by a monumental work of Ajtai, Komlós, Simonovits, and Szemerédi [1], for $v \geq v_0$. Many cases were established earlier, e.g., for paths.

Theorem 4.2 (Erdős-Gallai [3], also see, Kopylov [13]) *Let G be a graph on n vertices containing no path of length ℓ. Then $e(G) \leq \frac{1}{2}(\ell-1)n$. Equality holds iff G is the disjoint union of complete graphs on ℓ vertices.*

The Kalai conjecture has been proved for **star-shaped** trees in [5], i.e., whenever $\mathbb{T}$ contains a central edge which intersects all other edges in $k-1$ vertices. For $k = 2$ these are the diameter 3 trees, 'brooms'.

5 Some tools of the proofs, the delta-system method

A family $\{D_1, D_2, \ldots, D_s\}$ of distinct sets forms a *delta-system* of size s and with center C if $D_i \cap D_j = C$ holds for all $1 \leq i < j \leq s$.

The *intersection structure* of $F \in \mathcal{F}$ in the family $\mathcal{F}$ is defined as

$$\mathcal{I}(F, \mathcal{F}) := \{F \cap F' : F' \in \mathcal{F}, F \neq F'\}.$$

A k-uniform family $\mathcal{F} \subset \binom{[n]}{k}$ is *k-partite* if one can find a k-partition $[n] = X_1 \cup \ldots \cup X_k$ with $|F \cap X_i| = 1$ for all $F \in \mathcal{F}$, $1 \leq i \leq k$. Suppose that $\mathcal{F}$ is k-partite and $S \subset [n]$. Then one can define a *projection* $\pi : 2^{[n]} \to [k]$

$$\pi(S) = \{i : S \cap X_i \neq \emptyset\} \quad \text{és} \quad \pi(\mathcal{I}(F, \mathcal{H})) = \{\pi(S) : S \in \mathcal{I}(F, \mathcal{H})\}.$$

Theorem 5.1 (The intersection semilattice lemma) [7] *For any two positive integers k and s there exists a $c(k, s) > 0$ such that the following holds. Every family $\mathcal{F} \subset \binom{[n]}{k}$ contains a subfamily $\mathcal{F}^* \subset \mathcal{F}$ satisfying*

(1) $|\mathcal{F}^*| \geq c(k, s)|\mathcal{F}|$,

(2) $\mathcal{F}^*$ *is k-***partite***,*

(3) *the intersection structure is* **homogeneous***, i.e., there is a family $\mathcal{J} \subset 2^{[k]} \setminus \{[k]\}$ such that $\Pi(\mathcal{I}(F, \mathcal{F}^*)) = \mathcal{J}$ holds for all $F \in \mathcal{F}^*$,*

(4) *$\mathcal{J}$ is* **closed under intersection***, i.e. $A, B \in \mathcal{J}$ implies $A \cap B \in \mathcal{J}$,*

(5) *every* **pairwise intersection is a center** *of a delta system, $\forall F_1, F_2 \in \mathcal{F}^*$ $\exists F_3, \ldots, F_\ell \in \mathcal{F}^*$ such that $\{F_1, F_2, \ldots, F_s\}$ form a Δ-system of size s.*

In the proof of Theorem 2.1 we use the above tools and a version of the stability method developed by Frankl and the present author in [5].

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References

[1] Ajtai, M., J. Komlós, M. Simonovits, and E. Szemerédi, *The solution of the Erdős-Sós conjecture for large trees*, Manuscripts.

[2] Dorbec, P., S. Gravier, and G. Sárközy, *Monochromatic Hamiltonian t-tight Berge-cycles in hypergraphs*, Journal of Graph Theory **59** (2008), 34–44.

[3] Erdős, P., and T. Gallai, *On maximal paths and circuits of graphs*, Acta Math. Acad. Sci. Hungar. **10** (1959), 337–356.

[4] P. Frankl: *On families of finite sets no two of which intersect in a singleton*, Bull. Austral. Math. Soc. **17** (1977), 125–134.

[5] Frankl, P., and Z. Füredi, *Exact solution of some Turán-type problems*, J. Combinatorial Theory Ser. A **45** (1987), 226–262.

[6] Frankl, P., and G. Y. Katona, *Extremal k-edge Hamiltonian hypergraphs*, Discrete Math. **308** (2008), 1415–1424.

[7] Füredi, Z., *On finite set-systems whose every intersection is a kernel of a star*, Discrete Mathematics **47** (1983), 129–132.

[8] Füredi, Z., *Linear trees in uniform hypergraphs*, 2011, in preparation.

[9] Füredi, Z., T. Jiang, and R. Siever: *Exact solution of the hypergraph Turán problem for k-uniform linear paths*, Combinatorica, 25 pp., submitted.

[10] Győri, E., G. Y. Katona, and N. Lemons, *Hypergraph extensions of the Erdős-Gallai theorem*, Electronic Notes in Discrete Mathematics **36** (2010), 655–662.

[11] Rödl, V., *On a packing and covering problem*, European J. of Combinatorics **6** (1985), 69–78.

[12] G. Y. Katona, and H. A. Kierstead, *Hamiltonian chains in hypergraphs*, J. Graph Theory **30** (1999), 205–212.

[13] Kopylov, G. N., *Maximal paths and cycles in a graph*,(Russian), Dokl. Akad. Nauk SSSR **234** (1977), no. 1, 19–21. (Soviet Math. Dokl. **18** (1977), 593–596.)

[14] Kühn D., and D. Osthus, *Loose Hamilton cycles in 3-uniform hypergraphs of high minimum degree*, J. Combin. Theory Ser. B **96** (2006), 767–821.

[15] Rödl, V., A. Ruciński, and E. Szemerédi, *An approximate Dirac-type theorem for k-uniform hypergaphs*, Combinatorica **28** (2008), 229–260.