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Linear trees in uniform hypergraphs

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ABSTRACT

Given a tree T on v vertices and an integer $k \geq 2$ one can define the k-expansion $T^{(k)}$ as a k-uniform linear hypergraph by enlarging each edge with a new, distinct set of $k-2$ vertices. $T^{(k)}$ has $v + (v-1)(k-2)$ vertices. The aim of this paper is to show that using the delta-system method one can easily determine asymptotically the size of the largest $T^{(k)}$-free n-vertex hypergraph, i.e., the Turán number of $T^{(k)}$.

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1. Definitions: kernel-degree, Turán number

A *hypergraph* $H = (V, \mathcal{F})$ consists of a set V of vertices and a set $\mathcal{F} = E(H)$ of edges, where each edge is a subset of V. We call the edges of H *members* of $\mathcal{F}$. We say that H is a k-*uniform hypergraph* or $\mathcal{F}$ is a k-*uniform set system* if each member of $\mathcal{F}$ is a k-subset of V. To simplify notation we frequently identify the hypergraph H to its edge set $\mathcal{F}$. If $|V| = n$, it is often convenient to just let $V = [n] = \{1, \dots, n\}$. We also write $\mathcal{F} \subseteq \binom{V}{k}$ to indicate that $\mathcal{F}$ is a k-uniform hypergraph, or k-*graph* for short, on vertex set V. So $\binom{V}{k}$ denotes the *complete* k-graph on vertex set V. A set $S \subseteq V$ is a *transversal* (or vertex-cover) of the (hyper)graph $H = (V, \mathcal{E})$ if $S \cap E \neq \emptyset$ for all $E \in \mathcal{E}$. Let $\tau(H)$ denote the minimum number of vertices to cover all edges of H, i.e., the *transversal* number of H. A set of edges $\mathcal{M} \subseteq E(H)$ is called a *matching* if it consists of disjoint members of $E(H)$. $\nu(H)$ denotes the *matching number* of H, i.e., the maximum number of pairwise disjoint edges of H. A family of sets $\{F_1, \dots, F_s\}$ is said to form a Δ-*system* of size s with *kernel* C if $F_i \cap F_j = C$ for all $1 \leq i < j \leq s$.

Given a family $\mathcal{F} \subseteq \binom{[n]}{k}$ and a subset $W \subseteq [n]$, we define the *degree* of W in $\mathcal{F}$ as

$$\deg_{\mathcal{F}}(W) = |\{F : F \in \mathcal{F}, W \subseteq F\}|.$$

The hypergraph $\{F : F \in \mathcal{F}, W \subseteq F\}$ is denoted by $\mathcal{F}[W]$, so $\deg_{\mathcal{F}}(W) = |\mathcal{F}[W]|$ and $\deg_{\mathcal{F}}(\emptyset) = |\mathcal{F}|$.

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We define the *kernel degree* of W, denoted by $\deg^*_{\mathcal{F}}(W)$, as

$$\deg^*_{\mathcal{F}}(W) = \max\{s : \exists \text{ a } \Delta\text{-system of size with kernel } W \text{ in } \mathcal{F}\}.$$

In other words, $\deg^*_{\mathcal{F}}(W)$ is the matching number of $\{E \setminus W : W \subset E \in \mathcal{E}\}$.

Given a family $\mathcal{H} = \{H_1, H_2, \ldots\}$ of hypergraphs, the *k-uniform hypergraph Turán number* of $\mathcal{H}$, denoted by $\mathbf{ex}(n, \mathcal{H})$, is the maximum number of edges in a k-uniform hypergraph $\mathcal{F}$ on n vertices that does not contain a member of $\mathcal{H}$ as a subhypergraph. If we want to emphasize k, then we write $\mathbf{ex}_k(n, \mathcal{H})$. An $\mathcal{H}$-free family $\mathcal{F} \subseteq \binom{[n]}{k}$ is called *extremal* if $|\mathcal{F}| = \mathbf{ex}(n, \mathcal{H})$. If $\mathcal{H}$ consists of a single hypergraph H, we write $\mathbf{ex}(n, H)$ for $\mathbf{ex}(n, \{H\})$. Surveys on Turán problems of graphs and hypergraphs can be found in [17,26].

It is easy to show (see, e.g., Bollobás [2, p. xvii, formula (0.5)]) that any graph $G = (V, \mathcal{E})$ with more than $(\delta - 1)|V|$ edges contains an induced subgraph G' with minimum degree at least δ. Then G' contains every tree of $\delta + 1$ vertices. We have

$$\mathbf{ex}(n, T) \leq (v - 2)n, \tag{1}$$

where T is any v-vertex forest, $v \geq 2$.

For integers $b \geq a \geq 0$, $b \geq t \geq 1$ we have

$$\binom{a}{t} = \frac{a}{t}\binom{a-1}{t-1} \leq \frac{a}{t}\binom{b-1}{t-1} = \frac{a}{b}\binom{b}{t}.$$

This implies the following lemma.

Lemma 1.1. *Suppose that $z_1 \geq z_2 \geq \cdots \geq z_m$ and t are non-negative integers, $z_1 \geq t \geq 1$. Then*

$$\sum_{1 \leq i \leq m} \binom{z_i}{t} \leq \frac{\sum z_i}{z_1}\binom{z_1}{t}. \tag{2}$$

2. Preliminaries: matchings, paths, and stars

The Erdős–Ko–Rado [8] theorem says that for $n \geq 2k$ the maximum size of a k-uniform family on n vertices in which every two members intersect is $\binom{n-1}{k-1}$, with equality achieved by taking all the subsets of $[n]$ containing a fixed element. If we let $M_\nu^{(k)}$ denote the k-uniform hypergraph consisting of ν disjoint k-sets, then the Erdős–Ko–Rado theorem says $\mathbf{ex}_k(n, M_2^{(k)}) = \binom{n-1}{k-1}$ for $n \geq 2k$. More generally, Erdős [5] showed for any positive integers k, ν there exists a number $n(k, \nu)$ such that the following holds. For all $n > n(k, \nu)$, if $\mathcal{F} \subseteq \binom{[n]}{k}$ contains no $\nu + 1$ pairwise disjoint members then

$$|\mathcal{F}| \leq \binom{n}{k} - \binom{n-\nu}{k}. \tag{3}$$

Furthermore, the only extremal family $\mathcal{F}$ consists of all the k-sets of $[n]$ meeting some fixed set S of ν elements of $[n]$.

The value of $n(2, \nu)$ was determined by Erdős and Gallai [7]. Frankl, Rödl, and Rucinśki [15] showed $n(3, \nu) \leq 4\nu$. Finally, $n(3, \nu)$ was determined by Łuczak and Mieczkowska [30] for large ν (for $\nu > 10^5$), and by Frankl [10] for all ν. In general, Huang, Loh, and Sudakov [24] showed $n(k, \nu) < 3\nu k^2$, which was slightly improved in [14] and greatly improved to $n(k, \nu) \leq (2\nu + 1)k - \nu$ by Frankl [11]. Summarizing, for fixed k and ν as $n \to \infty$ we have that

$$\mathbf{ex}_k(n, M_\nu^{(k)}) = (\nu + o(1))\binom{n-1}{k-1}. \tag{4}$$

A *linear path* of length ℓ is a family of sets $\{F_1, \ldots, F_\ell\}$ such that $|F_i \cap F_{i+1}| = 1$ for each i and $F_i \cap F_j = \emptyset$ whenever $|i - j| > 1$. Let $\mathcal{P}_\ell^{(k)}$ denote the k-uniform linear path of length ℓ. It is unique up to isomorphisms. Note that this notation is different from what is usually used, where P_v denotes a v-vertex path. Concerning the graph case ($k = 2$) Erdős and Gallai [7] proved that $\mathbf{ex}_2(n, \mathcal{P}_\ell^{(2)}) \leq \frac{1}{2}(\ell - 1)n$. Here equality holds if G is the disjoint union of complete graphs on ℓ vertices. The value of $\mathbf{ex}_2(n, \mathcal{P}_\ell^{(2)})$ was determined for all n by Woodall [36] and Kopylov [28].

Concerning linear paths of two edges Erdős and Sós [6] proved for triple systems ($k = 3$) that $\mathbf{ex}_3(n, \mathcal{P}_2^{(3)}) = n$ or $n - 1$ (according to n is divisible by 4 or not and $n \geq 4$). They conjectured that

$$\mathbf{ex}_k(n, \mathcal{P}_2^{(k)}) = \binom{n-2}{k-2} \tag{5}$$

for $k \geq 4$ and sufficiently large n with respect to k, and this was proved by Frankl [9]. The case $k = 4$ was finished for all n by Keevash, Mubayi, and Wilson [27].

The case $\ell < k$ was asymptotically determined in [13].

Since the paper of G.Y. Katona and Kierstead [25] (1999), there is a renewed interest concerning paths and (Hamilton) cycles in uniform hypergraphs. Most of these are Dirac type results (large minimum degree implies the existence of the desired substructure) like in Kühn and Osthus [29], Rödl, Ruciński, and Szemerédi [35].

The present author, Tao Jiang, and Robert Seiver [22] determined $\mathbf{ex}_k(n, \mathcal{P}_\ell^{(k)})$ exactly, for *all* fixed k, ℓ, where $k \geq 4$, and sufficiently large n proving

$$\mathbf{ex}_k(n, \mathcal{P}_{2t+1}^{(k)}) = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \cdots + \binom{n-t}{k-1}, \tag{6}$$

where the only extremal family consists of all the k-sets in $[n]$ that meet some fixed set S of t elements, and

$$\mathbf{ex}(n, \mathcal{P}_{2t+2}^{(k)}) = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \cdots + \binom{n-t}{k-1} + \binom{n-t-2}{k-2}, \tag{7}$$

where the only extremal family consists of all the k-sets in $[n]$ that meet some fixed set S of t elements plus all the k-sets in $[n] \setminus S$ that contain some two fixed elements. 'Sufficiently large' n means that (6) and (7) hold when $kt = O(\log \log n)$. It is *conjectured* that they hold for all (or at least almost all) n's. The method in [22] does not quite work for the $k = 3$ case (cf. the remark after Lemma 6.2) but it is *conjectured* that still a similar result holds for $k = 3$.

A (*linear*) *star* of size ℓ with center x is a family of sets $\{F_1, \ldots, F_\ell\}$ such that $x \in F_i$ for all i but the sets $F_i \setminus \{x\}$ are pairwise disjoint. Let $\mathcal{S}_\ell^{(k)}$ denote the k-uniform star of size ℓ. It is obvious that $\mathbf{ex}_2(n, \mathcal{S}_\ell^{(2)}) = \lfloor(\ell - 1)n/2\rfloor$ (for $n \geq \ell$). Chung and Frankl [3] gave an exact formula for $\mathbf{ex}_3(n, \mathcal{S}_\ell^{(3)})$ for $n > 3\ell^3$. The following asymptotic was proved for any fixed $\ell \geq 2$, $k \geq 5$ in [13].

$$\mathbf{ex}_k(n, \mathcal{S}_\ell^{(k)}) = (\varphi(\ell) + o(1))\binom{n-2}{k-2}, \tag{8}$$

where $\varphi(\ell) = \ell^2 - \ell$ for ℓ is odd and it is $\ell^2 - \frac{3}{2}\ell$ when ℓ is even. According to the above mentioned result of Chung and Frankl (8) holds for $k = 3$ too. The order of magnitude $\mathbf{ex}_4(n, \mathcal{S}_\ell^{(4)}) = \Omega(\ell^2 n^2)$ was also proven in [13], and it is *conjectured* that (8) holds for $k = 4$ too.

3. Generalized k-forests, an upper bound

Let us define a generalized k-*forest* in the following inductive way. Every k-graph consisting of a single edge is a k-forest. Suppose that $\mathcal{T} = \{E_1, E_2, \ldots, E_u\} \subseteq \binom{V}{k}$ is a k-forest and suppose that $A := A_{u+1} \subset E_i$ for some $1 \leq i \leq u$, and $B \cap V = \emptyset$, $|A| + |B| = k$, then $\{E_1, E_2, \ldots, E_u, E_{u+1}\}$ is a k-forest with $E_{u+1} := A \cup B$. If it is connected then it is called a generalized k-tree. In that case all defining sets $A_2, \ldots, A_{u+1}$ are nonempty. For graphs ($k = 2$) the above process leads to the usual notions of forests and trees. If each defining set A_i is a singleton or empty then we obtain a *linear* forest, if each defining set is either empty or has $k-1$ elements, then we get a *tight* forest. A forest $\mathcal{T}$ of q edges has at least $q+k-1$ vertices and here equality holds if and only if $\mathcal{T}$ is a tight k-tree.

Consider a k-forest $\mathcal{T} = \{E_1, E_2, \dots, E_q\}$. If a defining set $A_{u+1} \subset E_i$ for some $1 \le i \le u < q$ is smaller than $k-1$, then take an element $x \in (E_i \setminus A_{u+1})$ and another one $y \in (E_{u+1} \setminus A_{u+1})$ and place the new k-set $E := E_i \setminus \{x\} \cup \{y\}$ between E_u and E_{u+1}. The new sequence of k-sets $\{E_1, \dots, E_u, E, E_{u+1}, \dots, E_q\}$ is again a k-forest with the same defining sets except we add $E_i \setminus \{x\} = E \setminus \{y\}$ to the list for E and replace A_{u+1} by $(A_{u+1} \cup \{y\})$ and use the relation $(A_{u+1} \cup \{y\}) \subset E$ for E_{u+1}. Repeating this process we obtain the following statement.

Proposition 3.1. *Suppose that $\mathcal{T}$ is a generalized k-forest of v vertices. Then there is a tight k-tree $\mathcal{T}^+$ on the same vertex set such that $\mathcal{T}$ is a subfamily of $\mathcal{T}^+$.*

We are going to prove the following upper bound for the Turán number of k-forests.

Theorem 3.2. *Suppose that $\mathcal{T}$ is a generalized k-forest of v vertices. Then*

$$\mathbf{ex}_k(n, \mathcal{T}) \le (v-k)\binom{n}{k-1}. \tag{9}$$

Proof. By the previous proposition, it is enough to prove the case when $\mathcal{T}$ is a tight k-forest.

Suppose that $\mathcal{H} \subseteq \binom{[n]}{k}$ avoids the tight k-forest $\mathcal{T} = \{E_1, \dots, E_q\}$, we have $q = v-k+1$. Set $A_i := E_i \cap (E_1 \cup \dots \cup E_{i-1}),\ 2 \le i \le q$. We have that $A_i \subset E_{\alpha(i)}$ for some $1 \le \alpha(i) < i$, $|A_i| = k-1$. Define a list of hypergraphs $\mathcal{H}_0 := \mathcal{H} \supset \mathcal{H}_1 \supset \dots \supset \mathcal{H}_m$ and sets $X_1, \dots, X_m$, as follows.

If $\mathcal{H}_m = \emptyset$ we stop. If one can find a set $X \subset [n]$ such that $|X| = k-1$ and $\deg_{\mathcal{H}_m}(X) \le (v-k)$ then let $X_{m+1} := X$ and $\mathcal{H}_{m+1} := \mathcal{H}_m \setminus \mathcal{H}_m[X]$. If there is no such set X then we stop.

We claim that $\mathcal{H}_m$ should be the empty family. Otherwise, we can embed $\mathcal{T}$ into $\mathcal{H}_m$ as follows. Start with any edge $E_1 \in \mathcal{H}_m$. We define the other edges $E_2, \dots, E_q$ one by one. Observe that for any $(k-1)$-element subset $X, X \subsetneq E \in \mathcal{H}_m$ we have $\deg_{\mathcal{H}_m}(X) \ge v-k+1$. Suppose that $E_1, \dots, E_u$ had already been defined together with $A_2, \dots, A_u$, and $u < q$. Locate A_{u+1} in $E_1 \cup \dots \cup E_u$. Since $\deg_{\mathcal{H}_m}(A_{u+1}) \ge (v-k+1) > |E_1 \cup \dots \cup E_u| - |A_{u+1}|$ there is an $E := E_{u+1} \in \mathcal{H}_m[A_{u+1}]$ such that $E \setminus A_{u+1}$ is disjoint to $E_1 \cup \dots \cup E_u$.

In the sequence $X_1, \dots, X_m$ there is no repetition, so we get

$$|\mathcal{H}| = \sum_i \deg_{\mathcal{H}_i}(X_i) \le (v-k)\binom{n}{k-1}. \quad \square$$

Note that Theorem 3.2 gives the correct order of magnitude if $\cap\mathcal{T} = \emptyset$, since then $\binom{n-1}{k-1}$ is a lower bound. However, the determination of the best coefficient of the binomial term seems to be extremely difficult. Erdős and Sós *conjectured* for graphs (i.e., $k=2$) and Kalai 1984 for all k, see in [13], that for a v-vertex tight tree $\mathcal{T}$

$$\mathbf{ex}_k(n, \mathcal{T}) \le \frac{v-k}{k}\binom{n}{k-1}.$$

For any given tight tree $\mathcal{T}$ a matching lower bound, i.e., $(1-o(1))$ times the conjectured upper bound, can be given for $n \to \infty$ as follows. Consider a $P(n, v-1, k-1)$ packing $P_1, \dots, P_m$ on the vertex set $[n]$ (i.e., $|P_i| = v-1$ and $|P_i \cap P_j| < k-1$ for $1 \le i < j \le m$) and replace each P_i by a complete k-graph. We obtain a $\mathcal{T}$-free hypergraph. Then Rödl's [34] theorem on almost optimal packings gives

$$\mathbf{ex}_k(n, \mathcal{T}) \ge (1-o(1))\frac{\binom{n}{k-1}}{\binom{v-1}{k-1}} \times \binom{v-1}{k} = (1+o(1))\frac{v-k}{k}\binom{n}{k-1}.$$

The Erdős–Sós conjecture has been recently proved by a monumental work of Ajtai, Komlós, Simonovits, and Szemerédi [1], for $v \ge v_0$.

The Kalai conjecture has been proved for *star-shaped* k-trees in [13], i.e., whenever $\mathcal{T}$ contains a central edge which intersects all other edges in $k-1$ vertices. For $k=2$ these are the diameter 3 trees, 'double stars'.

There is only one more class of k-trees where the exact asymptotic is known, namely what is called an *intersection condensed family*. For such a $\mathcal{T}$ we denote $|\cap\mathcal{T}|$ by p_∞, and the number of vertices of degree at least two by p_2 and suppose that $2p_\infty+p_2+2\le k$ (Theorem 5.3 in [13]).

There are many different definitions of a 'path' in a hypergraph. Győri, G.Y. Katona, and Lemons [23] determined the exact value of the Turán number of the so-called *Berge*-paths for infinitely many n's. Mubayi and Verstraëte [32] gave good bounds for the Turán number of k-uniform *loose* paths of length ℓ.

The aim of this paper is to present the best coefficient for a wide class of linear trees, thus generalizing the results in the previous section about matchings (4), paths (5)–(7) and stars (8).

4. The main result, finding expanded forests in k-graphs

Given a graph H, the *k-blowup* (or k-expansion), denoted by $[H]^{(k)}$ (or $H^{(k)}$ for short), is the k-uniform hypergraph obtained from H by replacing each edge xy in H with a k-set E_{xy} that consists of x, y and $k-2$ new vertices such that for distinct edges $xy, x'y'$, $(E_{xy}-\{x,y\})\cap(E_{x'y'}-\{x',y'\})=\emptyset$. If H has p vertices and q edges, then $H^{(k)}$ has $p+q(k-2)$ vertices and q hyperedges. The resulting $H^{(k)}$ is a k-uniform hypergraph whose vertex set contains the vertex set of H.

Given a forest T, define the following

$$\sigma(T):=\min\{|X|+e(T\setminus X): X\subset V(T) \text{ is independent in } T\}. \tag{10}$$

Here $T\setminus X$ is the forest left from T after deleting the vertices of X and the edges incident to them, $e(G)$ stands for the number of edges of the graph G. Since the edges avoiding X can be covered one by one we have that $\tau(T)\le\sigma(T)$ but here equality should not hold. For example, if T consists of a path of four vertices $a_1b_1b_2a_2$ with $2d+2c$ pendant edges such that $d>c\ge 1$ and each a_i has d degree-one neighbors and each b_i has c of those, then one can easily see that $\tau(T)=4$ but $\sigma(T)=2c+3$.

Theorem 4.1. *Given a forest T with at least one edge and an integer $k\ge 4$. Then we have as $n\to\infty$, that*

$$\mathbf{ex}(n,T^{(k)})=(\sigma(T)-1+o(1))\binom{n}{k-1}. \tag{11}$$

Our result, naturally, gives the same asymptotic as Theorem 5.3 in [13] whenever both can be applied to $T^{(k)}$. We *conjecture* that (11) holds for $k=3$, too.

According to (8) (and the remark after that) the above asymptotic holds for stars, since the answer in this case is $o(n^{k-1})$. For every other forest $\sigma\ge\tau\ge 2$.

Let us note that Mubayi [31] and Pikhurko [33] determined precisely (for large n) the Turán number of the k-expansion of some other graphs, namely for the complete graph K_ℓ for $\ell>k\ge 3$. For smaller values of ℓ we know that $\mathbf{ex}_k(n,K_3^{(k)})=\binom{n-1}{k-1}$ for $n>n_0(k)$, $k\ge 3$, a former conjecture of Chvátal and Erdős, established in [13]. A few more related exact results can be found in [19].

5. The product construction

Given two set systems (or hypergraphs) $\mathcal{A}$ and $\mathcal{B}$ their *join* is the family $\{A\cup B: A\in\mathcal{A}, B\in\mathcal{B}\}$. We denote this new hypergraph by $\mathcal{A}\bowtie\mathcal{B}$.

Call a set Y 1-*cross-cut* of a family $\mathcal{C}$ if $|Y\cap E|=1$ holds for each $E\in\mathcal{C}$. Define $\tau_1(\mathcal{C})$ as the minimum size of a 1-*cross-cut* of $\mathcal{C}$ (if such cross-cut exists, otherwise $\tau_1:=\infty$). We claim that for every forest T and $k\ge 3$ the following holds.

$$\sigma(T)=\tau_1(T^{(k)}). \tag{12}$$

Indeed, suppose that $X \subset V(T)$ yields the minimum in (10). Then X is an independent set of $T^{(k)}$ avoiding $e(T \setminus X)$ edges of it. Taking an element $x(E) \in (E \setminus V(T))$ from each such edge and joining them to X one gets a 1-cross-cut of size $\sigma(T)$. We obtain $\tau_1 \leq \sigma$. On the other hand, if S is a 1-cross-cut of $T^{(k)}$ and $|S| = \tau_1$, then $X := S \cap V(T)$ is an independent set in T and it avoids exactly $|S| - |X|$ edges, so $\sigma \leq |S| = \tau_1$.

Thus $\sigma(T)$ is the minimum size of a set Y such that $T^{(k)}$ can be embedded into $\binom{Y}{1} \bowtie \binom{Z}{k-1}$ where Y and Z are disjoint sets. This means that in case of $Y := [\sigma - 1], Z := [n] \setminus Y$ the hypergraph $\binom{Y}{1} \bowtie \binom{Z}{k-1}$ does not contain any copy of $T^{(k)}$. We obtain the lower bound

$$\begin{aligned}\mathbf{ex}(n, T^{(k)}) &\geq \left|\binom{Y}{1} \bowtie \binom{Z}{k-1}\right| = \left|\left\{E : E \in \binom{[n]}{k}, |E \cap [\sigma - 1]| = 1\right\}\right| \\ &= (\sigma - 1)\binom{n - \sigma + 1}{k - 1}. \end{aligned} \tag{13}$$

6. The graph of 2-kernels, starting the proof with the delta-system method

Given a family $\mathcal{F} \subseteq \binom{[n]}{k}$, the *kernel-graph* with *threshold s* is a graph $G := G_{2,s}(\mathcal{F})$ on $[n]$ such that $\forall x, y \in [n], xy \in E(G)$ if and only if $\deg^*_{\mathcal{F}}(\{x, y\}) \geq s$. The following (easy) lemma shows the importance of this definition.

Lemma 6.1 (*See [22]*). *Let H be a graph with q edges, $s = kq$, and let $\mathcal{F} \subseteq \binom{[n]}{k}$. Let G_2 be the kernel graph of $\mathcal{F}$ with threshold s. If $H \subseteq G_2$, then $\mathcal{F}$ contains a copy of $H^{(k)}$.* □

The *delta-system method*, started by Deza, Erdős and Frankl [4], is a powerful tool for solving set system problems. Using a structural lemma from [16] and the method developed in [12,13] the following theorem was obtained in [22] (see Theorem 3.8 and the proof of Lemma 4.3 there).

Lemma 6.2 (*See [22]*). *Let $\mathcal{F} \subseteq \binom{[n]}{k}$, T a forest of v vertices, $s = kv$, $G_2 := G_{2,s}(\mathcal{F})$, and suppose that $\mathcal{F}$ does not contain $T^{(k)}$. Then there is a constant $c := c(k, v)$ and a partition $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ with the following properties.*

- $|\mathcal{F}_1| \leq c\binom{n-2}{k-2}$.
- *Every edge $F \in \mathcal{F}_2$ has a center (not necessarily unique) $x(F) \in F$ such that $G_2|F$ contains a star of size $k - 1$ with center $x(F)$. In other words, $\{x(F), y\} \in E(G_2)$ for all $y \in F \setminus \{x(F)\}$.* □

Actually, the delta-system method describes the intersection structure of $\mathcal{F}$ in a more detailed way, but for our purpose this lemma will be sufficient. The above lemma (and in fact the main result of this paper, Theorem 4.1) preceded (6)–(7), see [18], but since the proof of Lemma 6.2 is now available in [22] we omit the details here.

Note that this is the only point where $k \geq 4$ is used. Lemma 6.2 is not true for $k = 3$. The 3-graph $\mathcal{F}^3$ obtained by joining a matching of size t and t one-element sets has $n = 3t$ vertices, $t^2 = n^2/9 = \Omega(n^{k-1})$ edges, it does not contain any linear tree except stars but $G_{2,s}(\mathcal{F}^3)$ forms a matching for every $s \geq 2$.

7. Proof of the main theorem

Suppose that $\mathcal{F} \subseteq \binom{[n]}{k}$ avoids the k-expansion of the v-vertex forest T, $k \geq 4$. We are going to give an upper bound for $|\mathcal{F}|$. As noted above we may suppose that T is not a star, $\sigma(T) \geq \tau(T) \geq 2$.

Define $s = vk$ and let G_2 be the kernel graph with threshold s with respect to family $\mathcal{F}$ as defined in the previous section. This graph avoids T by Lemma 6.1, so (1) implies

$$e(G_2) \leq (v - 2)n. \tag{14}$$

Consider the degree sequence of G_2 and suppose that

$$\deg(x_1) \geq \deg(x_2) \geq \cdots \geq \deg(x_{n-1}) \geq \deg(x_n).$$

Let $L := \{x_1, \ldots, x_\ell\}$ be the set of highest degrees. We will define ℓ later as n^ε so keep in mind that it is relatively large. Using (14) we obtain

$$z := \deg_{G_2}(x_{\ell+1}) \leq \frac{\deg(x_1) + \cdots + \deg(x_{\ell+1})}{\ell+1} \leq \frac{2e(G_2)}{\ell+1} < \frac{2(v-2)n}{\ell}. \tag{15}$$

Consider the partition $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ given by Lemma 6.2. Let $\mathcal{F}_3$ be the edges of $\mathcal{F}_2$ with center outside L. Using Lemma 6.2 and (2) then (14), a triviality and (15) we get

$$|\mathcal{F}_3| \leq \sum_{\ell+1 \leq i \leq n} \binom{\deg(x_i)}{k-1} \leq \frac{\sum_i \deg(x_i)}{z} \binom{z}{k-1}$$
$$\leq \frac{2(v-2)n}{z} \binom{z}{k-1} < \frac{2(v-2)n}{(k-1)!} z^{k-2} \leq \frac{2^{k-1}(v-2)^{k-1}}{(k-1)!} \frac{n^{k-1}}{\ell^{k-2}}. \tag{16}$$

Every edge of $\mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_3)$ meets L. Let $\mathcal{F}_4$ be the set of members of $\mathcal{F}$ meeting L in at least two vertices. Obviously

$$|\mathcal{F}_4| \leq \binom{\ell}{2}\binom{n-2}{k-2} \leq \frac{1}{2 \times (k-2)!} \ell^2 n^{k-2}. \tag{17}$$

The edges of $\mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_3 \cup \mathcal{F}_4)$ meet L in exactly one element. Let $\mathcal{F}_5$ be the family of edges of $\mathcal{F}$ satisfying $|F \cap L| = 1$ and $\deg_{\mathcal{F}}(F \setminus L) \leq \sigma - 1$. Obviously,

$$|\mathcal{F}_5| \leq (\sigma - 1)\binom{n-\ell}{k-1}. \tag{18}$$

The rest of the edges, i.e., those from $\mathcal{F}_6 := \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5)$ are of the form $F = \{a\} \cup B$ where $a \in L$, $B \cap L = \emptyset$ and $\deg_{\mathcal{F}}(F \setminus L) \geq \sigma$. For every set $A \in \binom{L}{\sigma}$ define $\mathcal{B}_A$ as the $k-1$ uniform family

$$\mathcal{B}_A := \{B : \{a\} \cup B \in \mathcal{F} \text{ for all } a \in A\}.$$

Also set

$$\mathcal{F}_A := \{F \in \mathcal{F} : a \in A, B \in \mathcal{B}_A, \text{ and } \{a\} \cup B = F\}.$$

We have $\mathcal{F}_6 \subseteq \cup_A \mathcal{F}_A$ where $|A| = \sigma$, $A \subseteq L$.

Consider $T^{(k)}$. As noted in Section 5, there is a 1-cross-cut, a set Y of size σ meeting each k-edge of $T^{(k)}$ in a singleton. Let $\mathcal{C}$ be the $(k-1)$-uniform hypergraph obtained by deleting the elements of Y from the edges of $T^{(k)}$, $\mathcal{C} := \{E \setminus Y : E \in E(T^{(k)})\}$. Since $\mathcal{F}_A$ does not contain $T^{(k)}$ we have that $\mathcal{B}_A$ cannot contain $\mathcal{C}$ as a subhypergraph. Also, $\mathcal{C}$ is a generalized forest of at most $v-1$ edges so Theorem 3.2 gives $|\mathcal{B}_A| \leq (v-2)(k-1)\binom{n}{k-2}$. We obtain

$$|\mathcal{F}_6| \leq \sum_{A \in \binom{L}{\sigma}} |\mathcal{F}_A| = \sigma \sum_{A \in \binom{L}{\sigma}} |\mathcal{B}_A| \leq \sigma \binom{\ell}{\sigma} (v-2)(k-1)\binom{n}{k-2}. \tag{19}$$

Finally, since $\mathcal{F}_2 \subseteq \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup \mathcal{F}_6$ we have

$$|\mathcal{F}| \leq |\mathcal{F}_1| + |\mathcal{F}_3| + |\mathcal{F}_4| + |\mathcal{F}_5| + |\mathcal{F}_6|.$$

Using the first part of Lemma 6.2, (16)–(19) we obtain

$$|\mathcal{F}| \leq O(n^{k-2}) + O\left(\frac{n^{k-1}}{\ell^{k-2}}\right) + O(\ell^2 n^{k-2}) + (\sigma-1)\binom{n-\ell}{k-1} + O(\ell^\sigma n^{k-2}). \tag{20}$$

Defining $\ell \sim n^{1/(\sigma+1)}$ we obtain that the sum of the $O()$ terms in (20) is $O(n^{(k-1)-1/(\sigma+1)}) = o(n^{k-1})$ and we are done.

8. Further problems

With a refined version of the above proof one can see that

$$\mathbf{ex}(n, T^{(k)}) = (\sigma - 1)\binom{n}{k-1} + O(n^{k-2}).$$

It seems to be a solvable problem to determine the exact value of this Turán number (for $n > n_0(T, k)$) as it was done for linear paths (for $k \geq 4$) in [22], and for linear cycles (for $k \geq 5$ only) in [20]. The forthcoming manuscript [21] generalizes these to a class of expanded forests, but most of the cases remain unsolved.

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