Cycle-Saturated Graphs with Minimum Number of Edges

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Received April 6, 2011; Revised March 19, 2012

Published online 7 August 2012 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/jgt.21668

Abstract: A graph G is called H-saturated if it does not contain any copy of H, but for any edge e in the complement of G, the graph $G+e$ contains some H. The minimum size of an n-vertex H-saturated graph is denoted by $\mathrm{sat}(n, H)$. We prove

$$\mathrm{sat}(n, C_k) = n + n/k + O((n/k^2) + k^2)$$

holds for all $n \geq k \geq 3$, where C_k is a cycle with length k. A graph G is H-semisaturated if $G+e$ contains more copies of H than G does for $\forall e \in$

Contract grant sponsors: Hungarian National Science Foundation OTKA; National Science Foundation (to Z. F.); Contract grant number: NFS DMS 09-01276 (to Z. F.).

$E(\overline{G})$. Let $\mathrm{ssat}(n, H)$ be the minimum size of an n-vertex H-semisaturated graph. We have

$$\mathrm{ssat}(n, C_k) = n + n/(2k) + O((n/k^2) + k).$$

We conjecture that our constructions are optimal for $n > n_0(k)$. J. Graph Theory 73: 203–215, 2013

Keywords: *graphs; cycles; extremal graphs; minimal saturated graphs*

1. A SHORT HISTORY

A graph G is said to be H-*saturated* if

(1) it does not contain H as a subgraph, but
(2) the addition of any new edge (from $E(\overline{G})$) creates a copy of H.

Let $\mathrm{sat}(n, H)$ denote the minimum size of an H-saturated graph on n vertices. Given H, it is difficult to determine $\mathrm{sat}(n, H)$ because this function is not necessarily monotone in n, or in H. Recent surveys are by Faudree, Faudree, and Schmitt [11], and by Pikhurko [19] on the hypergraph case. It is known [17] that for every graph H, there exists a constant c_H such that

$$\mathrm{sat}(n, H) < c_H n$$

holds for all n. However, it is not known if the $\lim_{n\to\infty} \mathrm{sat}(n, H)/n$ exists; Pikhurko [19] has an example of a four graph set $\mathcal{H}$ when $\mathrm{sat}(n, \mathcal{H})/n$ oscillates, it does not tend to a limit.

Since the classical theorem of Erdős, Hajnal, and Moon [9] (they determined $\mathrm{sat}(n, K_p)$ for all n and p), and its generalization for hypergraphs by Bollobás [5], there have been many interesting hypergraph results (e.g., Kalai [16], Frankl [14], Alon [1], using Lovász' algebraic method) but here we only discuss the graph case.

Remarkable asymptotics were given by Alon, Erdős, Holzman, and Krivelevich [2], [10] (saturation and degrees). Bohman, Fonoberova, and Pikhurko [4] determined the sat-function asymptotically for a class of complete multipartite graphs. More recently, for multiple copies of K_p, Faudree, Ferrara, Gould, and Jacobson [12] determined $\mathrm{sat}(tK_p, n)$ for $n \geq n_0(p, t)$.

2. CYCLE-SATURATED GRAPHS

What is the saturation number for the k-cycle, C_k? This has been considered by various authors, however, in most cases it has remained unsolved. Here, relatively tight bounds are given.

Theorem 2.1. *For all $k \geq 7$ and $n \geq 2k - 5$*

$$\left(1 + \frac{1}{k+2}\right)n - 1 < \mathrm{sat}(n, C_k) < \left(1 + \frac{1}{k-4}\right)n + \binom{k-4}{2}. \tag{1}$$

The construction giving the upper bound is presented at the end of this section, the proof of the lower bound (which works for all $n, k \geq 5$) is postponed to Section 10.

The case of $\mathrm{sat}(n, C_3) = n - 1$ is trivial; the cases $k = 4$ and $k = 5$ were established by Ollmann [18] in 1972 and by Ya-Chen [7] in 2009, respectively.

$$\mathrm{sat}(n, C_4) = \left\lfloor \frac{3n-5}{2} \right\rfloor \text{ for } n \geq 5.$$
$$\mathrm{sat}(n, C_5) = \left\lceil \frac{10(n-1)}{7} \right\rceil \text{ for } n \geq 21. \tag{2}$$

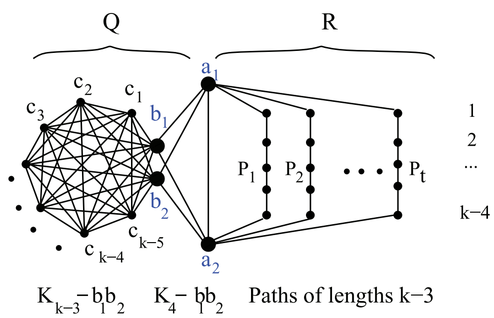
Actually, (2) was conjectured by Fisher, Fraughnaugh, Langley [13]. Later Ya-Chen [8] determined $\mathrm{sat}(n, C_5)$ for all n, as well as all extremal graphs.

The best previously known general lower bound came from Barefoot, Clark, Entringer, Porter, Székely, and Tuza [3], and the best general upper bound (a clever, complicated construction resembling a bicycle wheel) came from Gould, Łuczak, and Schmitt [15]

$$\left(1 + \frac{1}{2k+8}\right) n \leq \mathrm{sat}(n, C_k) \leq \left(1 + \frac{2}{k - \varepsilon(k)}\right) n + O(k^2), \tag{3}$$

where $\varepsilon(k) = 2$ for k even ≥ 10, $\varepsilon(k) = 3$ for k odd ≥ 17. Although there is still a gap, Theorem 2.1 supersedes all earlier results for $k \geq 6$ except the construction giving $\mathrm{sat}(n, C_6) \leq \frac{3}{2} n$ for $n \geq 11$ from [3].

Our new construction for a k-cycle-saturated graph for $n = (k-1) + t(k-4)$, where $k \geq 7$, $t \geq 1$, can be read from the picture below.



To be precise, define the graph $H := H_{k,n}$ on n vertices, for arbitrary $n \geq 2k-5$, $k \geq 7$ as follows. Write n in the form

$$n = (k-1) + r + t(k-4),$$

where $t \geq 1$ is an integer and $0 \leq r \leq k-5$. The vertex set $V(H)$ consists of the pairwise disjoint sets A, B, C, D, and R_i for $1 \leq i \leq t$, $V(H) = A \cup B \cup C \cup D \cup R_1 \cup R_2 \cup \cdots \cup R_t$, where $|A| = |B| = 2$, $|C| = k-5$, $|D| = r$, and $|R_1| = |R_2| = \cdots = |R_t| = k-4$ and $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, $C = \{c_1, c_2, \cdots, c_{k-5}\}$, $D = \{d_1, d_2, \cdots, d_r\}$, $R_\alpha = \{r_{\alpha,1}, r_{\alpha,2}, \ldots, r_{\alpha,k-4}\}$. We also denote $A \cup B \cup C \cup D$ by Q and $R_1 \cup \cdots \cup R_t$ by R.

The edge set of H does not contain $b_1 b_2$ and it consists of an almost complete graph K_{k-3} minus an edge on $C \cup B$, a K_4 minus an edge on $B \cup A$, r pendant edges joining c_i and d_i for $1 \leq i \leq r$, and t paths P_α of length $k-3$ with vertex sets $A \cup R_\alpha$ with endpoints

a_1 and a_2. The number of edges

$$|E(G)| = \binom{k-3}{2} + 4 + r + t(k-3).$$

It is not difficult to check that, indeed, H is C_k-saturated (see details in Section 3). After which, a little calculation yields the upper bound in (1).

We strongly believe that this construction is essentially optimal.

Conjecture 2.2. *There exists a* k_0 *such that* $\operatorname{sat}(n, C_k) = \left(1 + \frac{1}{k-4}\right) n + O(k^2)$ *holds for each* $k > k_0$.

3. THE GRAPH $H_{k,n}$ IS C_k-SATURATED, THE PROOF OF THE UPPER BOUND FOR $\operatorname{sat}(n, C_k)$

First, we check that $H := H_{k,n}$ is C_k-free. If a cycle with vertex set Y is entirely in Q, then it is contained in $A \cup B \cup C$, so $|Y| \leq k-1$. If Y contains a vertex $r_{\alpha,i}$ then $A \cup R_\alpha \subset Y$, the $k-3$ edges of the path P_α are part of the cycle. However, it is impossible to join a_1 and a_2 by a path of length 3, so $|Y| \neq k$. Note that here we used $k \geq 7$, because we needed that $k-3$ (the length of another P_β) is not 3.

The key observation to know that H is C_k-saturated is that a_1 and a_2 are connected inside Q by a path T_ℓ of any lengths ℓ except for 3:

$$\exists \text{ path } T_\ell \subset Q : \ell \in \{1, 2, 4, 5, \ldots, k-3, k-2\} \text{ with endpoints } a_1,\ a_2. \tag{4}$$

For example, $T_1 = a_1a_2$, $T_2 = a_1b_1a_2$, $T_4 = a_1b_1c_1b_2a_2$, etc. Also the vertices a_i $(i = 1, 2)$ and $q \in Q \setminus \{a_i\}$ are connected by a path U_m^i of length m inside Q for $\lceil (k+1)/2 \rceil \leq m \leq k-2$:

$$\exists \text{ path } U_m^i \subset Q : m \in \{\lceil (k+1)/2 \rceil, \ldots, k-3, k-2\} \text{ with endpoints } a_i,\ q \in Q. \tag{5}$$

Note that this is true for any $m \geq 4$ but we will apply (5) only for $\lceil (k+1)/2 \rceil \geq 4$.

Now add an edge e to H from its complement. We distinguish four disjoint cases.

Case 1. If the endpoints of e are in $A \cup R_\alpha$, then we get a path connecting a_1 and a_2 in $A \cup R_\alpha$ of length t, where t is at least two and at most $k-4$. This path together with T_{k-t} forms a k-cycle.

Case 2. If the endpoints of e are $r_{\alpha,i}$ and $r_{\beta,j}$ with $\alpha \neq \beta$ then we may suppose that $1 \leq i \leq j \leq k-4$. The vertex $r_{\alpha,i}$ splits the path P_α into two parts, P_α^1 and P_α^2, where P_α^1 starts at a_1 and has length i, and P_α^2 ends at a_2 and has length $k-3-i$. Consider the path $\pi := P_\alpha^1 e P_\beta^2$, its length is $k-2-j+i$. This length is between 3 and $k-2$, so we can apply (4) to add an appropriate T_{j-i+2} to complete π to a k-cycle unless $j-i+2 = 3$. In the latter, the edge a_1a_2 together with P_β^1, e, and P_α^2 forms a C_k.

Case 3. If the endpoints of e are $r_{\alpha,i}$ and $q \in B \cup C \cup D$, then again by symmetry, we may suppose that $i \leq (k-3)/2$, so the length of P_α^1 is at most $\lfloor (k-3)/2 \rfloor$. Then, by (5) there is an U_m^1 so that P_α^1, e, and U_m^1 form a k-cycle.

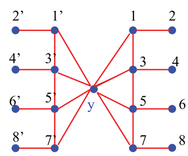
Case 4. Finally, e is contained in Q. Without loss of generality, we may consider only the following six subcases depending on the edge added.

For $e = a_1c_1$, we use P_1 to get the k-cycle $a_1c_1b_1a_2P_1$,
for $e = a_1d_1$, we have the k-cycle $a_1d_1c_1c_2 \ldots c_{k-5}b_2a_2b_1$,
for $e = b_1b_2$, we have to use P_1, (here we need again that $t \geq 1$),
for $e = b_1d_1$, we have the k-cycle $b_1d_1c_1c_2 \ldots c_{k-5}b_2a_2a_1$,
for $e = c_1d_2$, we have the k-cycle $c_1d_2c_2 \ldots c_{k-5}b_2a_2a_1b_1$, finally
for $e = d_1d_2$, we have the k-cycle $d_1d_2c_2 \ldots c_{k-5}b_2a_2b_1c_1$.

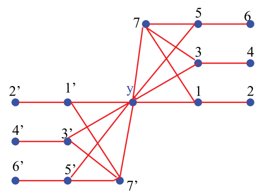
4. SEMISATURATED GRAPHS

A graph G is H-*semisaturated* (formerly called *strongly* saturated) if $G + e$ contains more copies of H than G does for $\forall e \in E(\overline{G})$. Let $\mathrm{ssat}(n, H)$ be the minimum size of an H-semisaturated graph. Obviously, $\mathrm{ssat}(n, H) \leq \mathrm{sat}(n, H)$.

It is known that $\mathrm{ssat}(n, K_p) = \mathrm{sat}(n, K_p)$ (it follows from Frankl/Alon/Kalai generalizations of Bollobás set pair theorem) and $\mathrm{ssat}(n, C_4) = \mathrm{sat}(n, C_4)$ (Tuza [20]). Below on the left we have a C_5-semisaturated graph on $1 + 8t$ vertices and $11t$ edges. Each vertex other than y can be reached by a path of length 2 from y.



n = 1+8t
e = 11t



n = 1+ 7t
e = 10t

Joining one, two, or three triangles to the central vertex y one obtains C_5 semisaturated graphs with $8t + 3$, $8t + 5$, or $8t + 7$ vertices and $11t + 3$, $11t + 6$, or $11t + 9$ edges, respectively. Leaving out a pendant edge, we can extend these constructions for even values of n

$$\mathrm{ssat}(n, C_5) \leq \left\lceil \frac{11}{8}(n - 1) \right\rceil \text{ for all } n \geq 5. \tag{6}$$

The picture on the right is the extremal C_5-saturated graph by (2).

Conjecture 4.1. $\mathrm{ssat}(n, C_5) = \frac{11}{8}n + O(1)$. *Maybe equality holds in 6 for* $n > n_0$.

Since $11/8 = 1.375 < 10/7 = 1.42\ldots$ inequalities (2) and (6) imply that

$$\mathrm{ssat}(n, C_5) < \mathrm{sat}(n, C_5) \text{ for all } n \geq 21.$$

Our next theorem implies that similar statement holds for the k-cycle C_k with $k > 12$ (though probably for $k \in \{6, 7, \ldots, 12\}$, too).

Theorem 4.2. *For all* $n \geq k \geq 6$

$$\left(1 + \frac{1}{2k - 2}\right) n - 2 < \mathrm{ssat}(n, C_k) < \left(1 + \frac{1}{2k - 10}\right) n + k - 1. \tag{7}$$

The proof of the lower bound is postponed to Section 9. The construction yielding the upper bound is presented in the next two sections where we describe a way to improve the $O(k)$ term as well as give better constructions for $k = 6$. We believe that our constructions are essentially optimal.

Conjecture 4.3. *There exists a* k_0 *such that* $\mathrm{ssat}(n, C_k) = \left(1 + \frac{1}{2k - 10}\right) n + O(k)$ *holds for each* $k > k_0$.

5. CONSTRUCTIONS OF SPARSE C_k-SEMISATURATED GRAPHS

In this section, we define an infinite class of C_k-semisaturated graphs, $H^2_{k,n}$ (more precisely $H^2_{k,n}(G)$). Call a graph G *k-suitable* if

(S1) G is C_k-semisaturated,
(S2) $\exists$ a path T_ℓ in G with endpoints a_1 and a_2 and of length ℓ for all $1 \le \ell \le k-2$, and
(S3) for every $q \in V(G) \setminus \{a_1, a_2\}$, and integers m_1 and m_2 with $m_1 + m_2 = k$ and $2 \le m_i \le k-2$ $\exists$ an $i \in \{1, 2\}$ and a path $U(a_i, q, m_i)$ of length m_i and with endpoints a_i and q.

A *k-wheel with r spikes*, W_k^r, is a graph with a $(k + r)$-element vertex set $\{a_1, a_2, \ldots, a_k, d_1, \ldots, d_r\}$ and it has $2k - 2 + r$ edges joining a_1 to all other a_i's, forming a cycle $a_2 a_3 \ldots a_k$ of length $k - 1$, and joining each d_i to a_i. It is easy to see that W_k^r is k-suitable when $k \ge r$ and $k \ge 4$.

Define the graph $H := H^2_{k,n}(G)$ as follows, when n is in the form

$$n = |V(G)| + t(k - 3),$$

where $t \ge 0$ is an integer. The vertex set $V(H)$ consists of the pairwise disjoint sets Q and R_i for $1 \le i \le t$, $V(H) = Q \cup R_1 \cup \cdots \cup R_t$, where $|Q| = |V(G)|$, $|R_1| = |R_2| = \cdots = |R_t| = k - 3$, and $A := \{a_1, a_2\} \subset Q$. The edge set of H consists of a copy of G with vertex set Q, and t paths with endpoints a_1 and a_2 and vertex sets $A \cup R_\alpha$. The number of edges is

$$|E(H)| = |E(G)| + t(k - 2).$$

It is not difficult to check that, indeed, H is C_k-semisaturated, the details are similar (but much simpler) to those in Section 3, so we do not repeat that proof.

Finally, considering $H^2_{k,n}(W_k^r)$ (where now $4 \le r \le k$), we obtain that for all $n \ge k + 4$

$$\mathrm{ssat}(n, C_k) \le n + \left\lfloor \frac{n - 7}{k - 3} \right\rfloor + k - 3. \tag{8}$$

Corollary 5.1. $\mathrm{ssat}(n, C_6) \le \left\lceil \frac{4}{3} n \right\rceil$ *for all* $n \ge 10$.

6. THINNER CONSTRUCTIONS OF SPARSE C_k-SEMISATURATED GRAPHS

In this section, we define another infinite class of C_k-semisaturated graphs, $H^3_{k,n}$ (more precisely $H^3_{k,n}(G)$) yielding the upper bound (7) in Theorem 4.2.

Call a graph G $\{k, k+2\}$-*suitable* with special vertices a_1 and a_2 if (S1) and (S2) hold but (S3) is replaced by the following

(S3)$^+$ for every $q \in V(G) \setminus \{a_1, a_2\}$, and integers m_1, m_2 either there exists a path $U(a_1, q, m_1)$ (of length m_1 and with endpoints a_1 and q) or a path $U(a_2, q, m_2)$ in the following cases

$$m_1 + m_2 = k \text{ and } 3 \le m_i \le k-3,$$
$$m_1 + m_2 = k+2 \text{ and } 4 \le m_i \le k-4.$$

It is easy to see that the wheel W^r_k with r spikes is such a graph, $k \ge r \ge 0$, $k \ge 4$.

Define the graph $H^3_{k,n}(G)$ as follows, when n is in the form

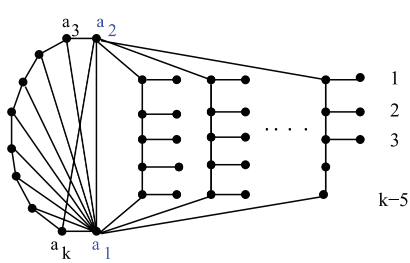
$$n = |V(G)| + t(2k-10) - r, \tag{9}$$

where $t \ge 2$ is an integer and $0 \le r < 2k-10$. The vertex set $V(H)$ consists of the pairwise disjoint sets Q, R_i, and D for $1 \le i \le t$, $V(H) = Q \cup R_1 \cup \cdots \cup R_t \cup D$, where $|Q| = |V(G)|$, $|R_1| = |R_2| = \cdots = |R_t| = k-5$, $|D| = t(k-5) - r$, and $A := \{a_1, a_2\} \subset Q$. The edge set of H consists of a copy of G with vertex set Q, and t paths with endpoints a_1 and a_2 and vertex sets $A \cup R_\alpha$ and finally $|D|$ spikes, a matching with edges from $\cup R_\alpha$ to D.

The number of edges is

$$|E(H)| = |E(G)| + t(2k-9) - r. \tag{10}$$

It is not difficult to check that H is C_k-semisaturated, the details are similar (but simpler) to those in Section 3. As an example, we present one case.



Add the edge qd to H, where $q \in V(G) \setminus \{a_1, a_2\}$ and $d \in D$. Let us denote the (unique) neighbor of d by x, $x \in R_\alpha$. The distance of x to a_1 is denoted by ℓ. Then the length of the $qdx \dots a_1$ path is $\ell + 2 \ge 3$ and the length of the $qdx \dots a_2$ path is $(k-4-\ell) + 2 \ge 3$ and one can find a C_k through qd using property (S3)$^+$.

Considering $H^3_{k,n}(W_k)$ (with $t \ge 2$), we obtain from (9) and (10) that for all $n \ge 3k-9$

$$\mathrm{ssat}(n, C_k) \le \left\lceil \left(1 + \frac{1}{2k-10}\right)(n-k) \right\rceil + 2k - 2. \tag{11}$$

Using $H^2(k, n)$, it is easy to see that (11) holds for all $n \geq k$, leading to the upper bound in (7).

One can slightly improve (8) and (11) if there are special graphs thinner than the wheel W_k.

Problem 6.1. *Determine $s(k)$, the minimum size of a k-vertex k-suitable graph (i.e., one satisfying (S1)–(S3)). Determine $s'(k)$, the minimum size of a k-vertex $\{k, k+2\}$-suitable graph (i.e., one satisfying (S1), (S2), and $(S3)^+$).*

7. DEGREE ONE VERTICES IN (SEMI)SATURATED GRAPHS

Suppose that G is a C_k-semisaturated graph where $k \geq 5$, $|V(G)| = n \geq k$. Obviously, G is connected. Let X be the set of vertices of degree one, $X := \{v \in V(G) : \deg_G(v) = 1\}$, its size is s and its elements are denoted as $X = \{x_1, x_2, \ldots, x_s\}$. Denote the neighbor of x_i by y_i, $Y := \{y_1, \ldots, y_s\}$ and let $Z := V(G) \setminus (X \cup Y)$. We also denote the neighborhood of any vertex v by $N_G(v)$ or briefly by $N(v)$.

Lemma 7.1. *(The neighbors of degree one vertices.)*

(i) $y_i \neq y_j$ for $1 \leq i \neq j \leq s$, so $|Y| = |X|$.
(ii) $\deg(y) \geq 3$ for every $y \in Y$,
(iii) if $\deg_G(x) = 1$, then $G - \{x\}$ is also a C_k-semisaturated graph.

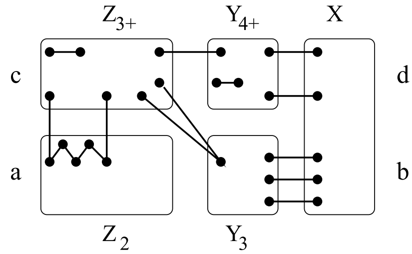
Proof. If $y_i = y_j$, then the addition of $x_i x_j$ to G does not create a new k-cycle. If $\deg(y_i) = 2$ and $N(y_i) = \{x_i, w\}$, the addition of $x_i w$ to G does not create a new k-cycle. Finally, (iii) is obvious. ■

Split Y and Z according to the degrees of their vertices. Thus, divide $V(G)$ into five parts $\{X, Y_3, Y_{4+}, Z_2, Z_{3+}\}$,

$$Y_3 := \{v \in Y : \deg_G(v) = 3\} \text{ and } Y_{4+} := \{v \in Y : \deg_G(v) \geq 4\},$$
$$Z_2 := \{v \in Z : \deg_G(v) = 2\} \text{ and } Z_{3+} := \{v \in Z : \deg_G(v) \geq 3\}.$$

Lemma 7.2. *(The structure of C_k-saturated graphs. See [3]). Suppose that G is a C_k-saturated graph (and $k \geq 5$).*

(iv) If $x_i y_i w$ is a path in G (with $x_i \in X$, $y_i \in Y$), then $\deg(w) \geq 3$. So there are no edges from Z_2 to Y (or to X).
(v) If $y_i y_j$ is an edge of G (with $y_i, y_j \in Y$), then $\deg(y_i) \geq 4$. So there are no edges in Y_3, no edges from Y_3 to Y_4. In other words, every $y \in Y_3$ has one neighbor in X and two in Z_{3+}.
(vi) The induced graph $G[Z_2]$ consists of paths of length at most $k-2$.



8. SEMISATURATED GRAPHS WITHOUT PENDANT EDGES

Claim 8.1. *Suppose that G is a C_k-semisaturated graph on n vertices with minimum degree at least 2, $k \geq 5$. Then every vertex w is contained in some cycle of length at most $k+1$.*

Proof. Consider two arbitrary vertices z_1, z_2 in the neighborhood $N(w)$. If $z_1z_2 \in E(G)$, then w is contained in a triangle. If $z_1z_2 \notin E(G)$, then $G + z_1z_2$ contains a new k-cycle; there is a path P of length $(k-1)$ in G with endpoints z_1 and z_2. If P avoids w, then P together with z_1wz_2 forms a $k+1$ cycle. If w splits P into two paths L_1, L_2, where L_i starts in z_i and ends in w, then either $L_1 + z_1w$, or $L_2 + z_2w$, or both forms a proper cycle of length at most $k-1$. ■

Note that Claim 8.1 itself (and the connectedness of G) immediately imply

$$e(G) \geq (n-1)\frac{k+2}{k+1}.$$

We can do a bit better repeatedly using the semisaturatedness of G.

Lemma 8.2. *Suppose that G is a C_k-semisaturated graph on n vertices with minimum degree at least 2, $k \geq 5$. Then*

$$e(G) \geq \frac{k}{k-1}\,n - \frac{k+1}{k-1}.$$

Proof. We will show that there exists an increasing sequence of subgraphs $G_1, G_2, \ldots, G_t = G$ with vertex sets $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_t = V(G)$ such that G_i is a subgraph of G_{i+1} and

$$|E(G_{i+1}) \setminus E(G_i)| \geq \frac{k}{k-1}\,(|V_{i+1}| - |V_i|) \tag{12}$$

(for $i = 1, 2, \ldots, t-1$). This, together with

$$e(G_1) \geq \frac{k}{k-1}\,|V_1| - \frac{k+1}{k-1}, \tag{13}$$

imply the claim.

G_1 is the shortest cycle in the graph G. Its length is at most $k+1$ so (13) obviously holds.

If G_i is defined and one can find a path P of length at most k with endpoints in V_i but $E(P) \setminus E(G_i) \neq \emptyset$, then we can take $E(G_{i+1}) = E(G_i) \cup E(P)$. From now on, we suppose that such a *short returning* path does not exist. Our procedure stops if $V(G_i) = V(G) =: V$.

In the case of $V \setminus V_i \neq \emptyset$, the connectedness of G implies that there exists an xy edge with $x \in V_i$ and $y \in V \setminus V_i$. Since $|N(y)| \geq 2$, we have another edge $yz \in E(G)$, $z \neq x$.

We have $N(y) \cap V_i = \{x\}$, otherwise one gets a path xyz of length smaller than k with endpoints in V_i but going out of G_i, contradicting our earlier assumption. Similarly, we obtain that $N(y)$ contains no edge, otherwise we can define $E(G_{i+1})$ as either $E(G_i)$ plus the three edges of a triangle xy, yz, xz or we add four edges xy, yz_1, yz_2, and z_1z_2 but only three vertices (namely y, z_1, and z_2). The obtained G_{i+1} obviously satisfies (12) in both cases. Similarly, if there is a cycle C of length at most $k-1$ contaning y, then we can define $E(G_{i+1})$ as $E(G_i)$ plus $E(C)$ and xy. From now on, we suppose that such a *short cycle through* y does not exist.

Fix a neighbor z of y, $z \neq x$. Since $zx \notin E(G)$, G contains a path P of length $k-1$ with endvertices x and z. Since G does not contain a short returning path nor a short cycle through y, we obtain that P avoids y and $V(P) \cap V_i = \{x\}$.

If the cycle $C := P + xy + yz$ of length $k+1$ has any diagonal edge then G_{i+1} is obtained by adding C together with its diagonals. From now on, we suppose that C does not have any diagonals. More generally, if there is any *diagonal path* P of length $\ell \leq k-1$ with edges disjoint from $E(G_i) \cup E(C)$ but with endpoints in $V_i \cup V(C)$ then we can define $E(G_{i+1}) := E(G_i) \cup E(C) \cup E(P)$ and have added $k+\ell-1$ vertices and $k+\ell+1$ edges, obviously satisfying (12).

However, such a diagonal path exists. Let $w \neq y$ be the other neighbor of x in C. Since $wz \notin E(G)$, there is a path P' of length $k-1$ with endpoints w and z. This P' must have edges outside $E(G_i) \cup E(C)$ so it can be shortened to a diagonal path P of length at most $k-1$. This completes the proof of the lemma. ■

9. A LOWER BOUND FOR THE NUMBER OF EDGES OF SEMISATURATED GRAPHS

In this section, we finish the proof of Theorem 4.2. Let G be a C_k-semisaturated graph on n vertices with minimum number of edges, $k \geq 5$. Let X be the set of degree one vertices, $x := |X|$. By Lemma 7.1 $|X| \leq n/2$, and for $G' := G \setminus X$, we have $e(G') = e(G) - x$ and G' is a C_k-semisaturated graph on $n - x$ vertices with minimum degree at least 2. Then Lemma 8.2 can be applied to $e(G')$. We obtain

$$\begin{aligned} \operatorname{ssat}(n, C_k) = e(G) &\geq x + (n-x)\frac{k}{k-1} - \frac{k+1}{k-1} \\ &\geq \frac{n}{2} + \frac{n}{2}\frac{k}{k-1} - \frac{k+1}{k-1} = n\left(1 + \frac{1}{2k-2}\right) - \frac{k+1}{k-1}. \end{aligned}$$

Since $\operatorname{sat}(n, C_k) \geq \operatorname{ssat}(n, C_k)$, this is already a better lower bound than the one in (3) from [3].

10. A LOWER BOUND FOR THE NUMBER OF EDGES OF C_k-SATURATED GRAPHS

In this section, we finish the proof of Theorem 2.1. Let G be a C_k-saturated graph on n vertices, $k \geq 5$. Let us consider the partition of $V(G) = X \cup Y_3 \cup Y_{4+} \cup Z_2 \cup Z_{3+}$ defined in Section 7, where X is the set of degree one vertices, Y is their neighbors. By Lemma 7.1 $|X| = |Y|$. To simplify notation, we use $a := |Z_2|$, $b := |Y_3|$, $c := |Z_{3+}|$, and $d := |Y_{4+}|$. We have

$$n = a + 2b + c + 2d.$$

By definition of the parts, we have the lower bound

$$2e(G) = \sum_{v \in V} \deg(v) \geq |X| + 2|Z_2| + 3|Y_3| + 3|Z_{3+}| + 4|Y_{4+}|.$$

This yields

$$2e \geq 2n + c + d. \tag{14}$$

Now, we estimate the number of edges by considering four disjoint subsets of $E(G)$. The part X is adjacent to $|X|$ edges, and according to Lemma 7.2, Z_2 is adjacent to at least $\frac{k}{k-1}|Z_2|$ edges, Y_3 is adjacent to exactly $3|Y_3|$ edges of which $|Y_3|$ has already been counted at X, and finally Y_{4+} is adjacent to at least another $\frac{3}{2}|Y_{4+}|$ edges. We obtain

$$e(G) \geq |X| + \frac{k}{k-1}|Z_2| + 2|Y_3| + \frac{3}{2}|Y_{4+}|.$$

Therefore, we get

$$e \geq n + \frac{1}{k-1}a + b - c + \frac{1}{2}d. \tag{15}$$

By Lemma 7.1 $G \setminus X$ is also C_k-semisaturated. Apply Lemma 8.2 to estimate $e(G \setminus X) = e - b - d$, multiply by $(k-1)$ and rearrange, we get

$$(k-1)e \geq kn - b - d - (k+1). \tag{16}$$

Adding up the above three inequalities (14), (15), and (16), we obtain

$$(k+2)e \geq (k+3)n + \frac{1}{k-1}a + \frac{1}{2}d - (k+1).$$

This implies the desired lower bound in (1).

Remark. We can do slightly better if we multiply (14), (15), and (16) by k, $k-1$, and $k-3$, respectively, then by adding up and simplifying , we get

$$e(G) > \frac{k^2}{k^2 - k + 2}n - 1. \tag{17}$$

■

ACKNOWLEDGMENTS

Research supported in part by the Hungarian National Science Foundation OTKA, and by the National Science Foundation under grant NFS DMS 09-01276.

REFERENCES

[1] N. Alon, An extremal problem for sets with applications to graph theory, J Combin Theory Ser A 40 (1985), 82–89.

[2] N. Alon, P. Erdős, R. Holzman, and M. Krivelevich, On k-saturated graphs with restrictions on the degrees, J Graph Theory 23 (1996), 1–20.

[3] C. Barefoot, L. Clark, R. Entringer, T. Porter, L. Székely, and Z. Tuza, Cycle-saturated graphs of minimum size, Discrete Math 150 (1996), 31–48.

[4] T. Bohman, M. Fonoberova, and O. Pikhurko, The saturation function of complete partite graphs, J Comb 1 (2010), 149–170.

[5] B. Bollobás, On generalized graphs, Acta Math Acad Sci Hungar 16 (1965), 447–452.

[6] B. Bollobás, Weakly k-saturated graphs, In: Beiträge zur Graphentheorie (Kolloquium, Manebach, 1967), Teubner, Leipzig, 1968, pp. 25–31.

[7] Y.-C. Chen, Minimum C_5-saturated graphs, J Graph Theory 61 (2009), 111–126.

[8] Y.-C. Chen, All minimum C_5-saturated graphs, J Graph Theory 67 (2011), 9–26.

[9] P. Erdős, A. Hajnal, and J. W. Moon, A problem in graph theory, Amer Math Monthly 71 (1964), 1107–1110.

[10] P. Erdős, R. Holzman, On maximal triangle-free graphs, J Graph Theory 18 (1994), 585–594.

[11] J. Faudree, R. Faudree, and J. Schmitt, A survey of minimum saturated graphs and hypergraphs, Electronic J Combinatorics 18 (2011), DS19.

[12] R. Faudree, M. Ferrara, R. Gould, and M. Jacobson, tK_p-saturated graphs of minimum size, Discrete Math 309 (2009) no. 19, 5870–5876.

[13] D. C. Fisher, K. Fraughnaugh, and L. Langley, On C_5-saturated graphs with minimum size, Proceedings of the 26th Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1995), Congr Numer 112 (1995), 45–48.

[14] P. Frankl, An extremal problem for two families of sets, European J Combin 3 (1982), 125–127.

[15] R. Gould, T. Łuczak, and J. Schmitt, Constructive upper bounds for cycle-saturated graphs of minimum size, Electronic J Combinatorics 13 (2006), R29.

[16] G. Kalai, Weakly saturated graphs are rigid, Convexity and Graph Theory (Jerusalem, 1981), volume 87 of North-Holland Mathematics Studies. North-Holland, Amsterdam, 1984, pp. 189–190.

[17] L. Kászonyi and Z. Tuza, Saturated graphs with minimal numbers of edges, J Graph Theory 10 (1986), 203–210.
[18] L. T. Ollmann, $K_{2,2}$ saturated graphs with a minimal number of edges, Proceedings of the 3rd Southeastern Conference on Combinatorics, Graph Theory, and Computing, Florida Atlantic University, Boca Raton, FL, 1972, pp. 367–392.
[19] O. Pikhurko, Results and open problems on minimum saturated hypergraphs, Ars Combin 72 (2004), 111–127.
[20] Z. Tuza, C_4-saturated graphs of minimum size, Acta Univ Carolin Math Phys 30 (1989), 161–167.