Superimposed codes and hypergraphs containing no grids *

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Abstract

A hypergraph is called an $r \times r$ *grid*, $\mathbb{G}_{r\times r}$, if it is isomorphic to a pattern of r horizontal and r vertical lines. Three sets form a *triangle* if they pairwise intersect in three distinct singletons. A hypergraph is *linear* if every pair of edges meet in at most one vertex.

Our aim is to construct a large linear r-hypergraphs which contain no grids. Moreover, a similar construction gives large linear r-hypergraphs which contain neither grids nor triangles. For $r \geq 4$ our constructions are almost optimal. These investigations are also motivated by coding theory: we get new bounds for optimal superimposed codes and designs.

Our main tool is a natural algebraic construction and some properties of pseudoline arrangements.

Motivation, a method of estimating codes

There are many instances in Coding Theory when codewords must be restored from partial information, like defected data (error correcting codes) or some superposition of the strings (these lead to superimposed codes). Generally, the aim of investigating superimposed codes is to construct and to estimate methods where no confusion arises on parallel transmission.

In this paper a *binary code* of *length* n is a family of 0-1 sequences, $\mathcal{C} \subset \{0,1\}^n$. There is one to one correspondence between codes and hypergraphs with vertex set $[n] := \{1, 2, \dots, n\}$. Speaking about a hypergraph $\mathbb{F} = (V, \mathcal{F})$ we frequently identify the vertex set $V = V(\mathbb{F})$ by the set of first integers $[n] := \{1, 2, \dots, n\}$, or elements of a q-element finite field F_q. To shorten notations we

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frequently say 'hypergraph $\mathcal{F}$' (or set system $\mathcal{F}$) thus identifying $\mathbb{F}$ to its edge set $\mathcal{F}$. $\mathbb{F}$ is *linear* if for all $A, B \in \mathcal{F}$, $A \neq B$ we have $|A \cap B| \leq 1$. The *degree*, $\deg_{\mathbb{F}}(x)$, of an element $x \in [n]$ is the number of hyperedges in $\mathcal{F}$ containing x. $\mathbb{F}$ is *regular* if every element $x \in [n]$ has the same degree. It is *uniform* if every edge has the same number of elements, r-uniform means $|F| = r$ for all $F \in \mathcal{F}$.

The aim of this paper is to present a method to estimate the maximum size of some superimposed codes. Namely, first we investigate constant weight codes, these lead to Turan type problems. This case we frequently apply algebraic/geometric methods for constructions. Second, we generalize to the non-uniform case, if we can. We illustrate this by considering two kinds of superimposed codes, cancellative and union-free families.

Cancellative families

A family of sets $\mathcal{F}$ (and the corresponding family of 0-1 vectors) is called **cancellative** if A and $A \cup B$ determine B (in case of $A, B \in \mathcal{F}$ and $A \neq A \cup B$). For a precise definition we require that for all $A, B, C \in \mathcal{F}$, $A \neq B$, $A \neq C$

$$A \cup B = A \cup C \implies B = C.$$

Let $\mathrm{CANC}(n)$ be the size of the largest cancellative family on n elements, and let $\mathrm{CANC}_r(n)$ denote the size of the largest r-uniform family on n elements.

The asymptotics of the maximum size of a cancellative family was given by Tolhuizen [36] (construction) and in [19] (upper bound) showing that there exists a $\gamma > 0$ such that

$$\frac{\gamma}{\sqrt{n}} 1.5^n < \mathrm{CANC}(n) < 1.5^n \tag{1}$$

holds. The problem was proposed by Erdős and Katona [28] who conjectured $\mathrm{CANC}(n) = \Theta(3^{n/3})$ which was disproved by an elegant construction by Shearer [33] showing $\mathrm{CANC}(3k) \geq k3^{k-2}$ leading to $\mathrm{CANC}(n) > 1.46^n$ for $n > n_0$.

A lower bound can be obtained by considering the complete r-partite r-uniform hypergraph with nearly equal parts.

$$\mathrm{CANC}_r(n) \geq \lfloor \frac{n}{r} \rfloor \lfloor \frac{n+1}{r} \rfloor \dots \lfloor \frac{n+r-1}{r} \rfloor. \tag{2}$$

Since a cancellative 2-uniform family is a triangle-free graph Turan theorem implies that equality holds in (2) for $r = 2$. Bollobás [5] showed that equality holds for $r = 3$, too. For the case $r = 4$ equality was proved for all n divisible by 4 by Sidorenko [34] and for all n by Pikhurko [31]. However, Shearer [33] gave larger conctructions for $r \geq 11$, so other cases of $\mathrm{CANC}_r(n)$ are unsolved.

Uniform families and upper bounds

Theorem 1 *(Tolhuizen [36] randomized algebraic construction, and [19] the upper bound for $n \geq 2r$).*

$$\frac{\gamma_0}{2^r}\binom{n}{r} < \mathrm{CANC}_r(n) \leq \frac{2^r}{\binom{2r}{r}}\binom{n}{r}. \tag{3}$$

Here $\gamma_0 := \prod_{k\geq 1}\frac{2^k-1}{2^k} = .2887\dots$

If $\mathcal{C}$ is a cancellative family on $[n]$ and $A \in \mathcal{C}$ then the sets $B \setminus A$ are all distinct ($B \in \mathcal{C}$) yielding

$$|\mathcal{C}| \leq 2^{n-|A|}. \tag{4}$$

We also obtain $\mathrm{CANC}_r(n) \leq 2^{n-r}$. Considering the r-partite hypergraph on $2r$ vertices we get

$$\mathrm{CANC}_r(2r) = 2^r. \tag{5}$$

Proof of the upper bound (3).
Suppose that $\mathcal{F} \subset \binom{[n]}{r}$ is a cancellative family, $n \geq 2r$. Define $\mathcal{F}_S := \{A \in \mathcal{F} : A \subset S\}$ for $|S| = 2r$. Then (5) gives

$$|\mathcal{F}|\binom{n-r}{r} = \sum_{S\subset V}|\mathcal{F}_S| \leq 2^r\binom{n}{2r}.$$ □

Proof of the upper bound (1).
Suppose that $\mathcal{F} \subset 2^{[n]}$ is a cancellative family. If $\max_{F\in\mathcal{F}}|F| \geq n/2$ then $|\mathcal{F}| \leq 2^{n/2}$ by (4). Otherwise, let $\mathcal{F} := \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \dots$ with $\mathcal{F}_r := \{F \in \mathcal{F} : |F| = r\}$, $r < n/2$. The upper bound (3) implies

$$|\mathcal{F}| = \sum_{r<n/2}|\mathcal{F}_r| \leq \sum_{r<n/2}\frac{\sqrt{r+1}}{2^r}\binom{n}{r} < \sqrt{n}\sum_{r}\frac{1}{2^r}\binom{n}{r} = \sqrt{n}\left(\frac{3}{2}\right)^n.$$

Since a kind of product of two cancellative families is again cancellative we have that $\mathrm{CANC}(n_1)\mathrm{CANC}(n_2) \leq \mathrm{CANC}(n_1+n_2)$. This implies that $\mathrm{CANC}(n)^{1/n}$ never exceeds its supremum. □

t-cancellative codes and families

$\mathcal{F}$ is 2-cancellative if for every fourtuple $\{A, B, C, D\}$ with $A, B, C, D \in \mathcal{F}$

$$A \cup B \cup C = A \cup B \cup D \implies C = D.$$

$$t(4) := \limsup_{n\to\infty,\ \mathcal{F}\subset 2^{[n]}} \frac{1}{n}\log_2|\mathcal{F}|.$$

The results of Alon, Fachini, and Körner [3], and Alon, Körner, and Monti [4], imply that $t(4) < 0.4561$. (They were really concerned with so-called *locally thin* families). This was further improved by Fachini, Körner, and Monti [16]. Körner and Sinaimeri [30] showed $t(4) < 0.42$.

One can further generalize this notion. A code $\mathcal{C} \subset \{0,1\}^n$ and its corresponding hypergraph $\mathcal{F} \subset 2^{[n]}$ is called *t-cancellative* if

$$A_1 \cup \cdots \cup A_t \cup B \neq A_1 \cup \cdots \cup A_t \cup C$$

for all $t+2$ distinct members of $\mathcal{F}$. $\text{CANC}(n,t) := \max\{|\mathcal{F}| : \mathcal{F} \subset 2^{[n]}, t\text{-cancellative}\}$.

Problem 2 *What is the rate of t-cancellative families?*

Theorem 3 *(An upper bound on t-cancellative codes, see [23]).*
$\text{CANC}(n,t) \leq \left(1 + \frac{1}{2t} + o(1)\right)^n$, *so* $t(4) := \limsup_{n\to\infty} \frac{1}{n} \log_2 \text{CANC}(n,2) \leq 0.322$.

Our method again is to investigate uniform t-cancellative families (i.e., Turán type problems) and then use/extend the results to estimate $\text{CANC}(n,t)$.

Union-free and sparse designs

Investigating the Rényi's search model Dyachkov and Rykov [9] obtained several sufficient conditions for the existence of regular binary superimposed codes. In [24] we answered their question asymptotically which lead to union-free designs. A Steiner system $S(v,r,2)$ is a collection of r-subsets (blocks) of a v-set which has the property that every pair of distinct elements occurs in one block. Two families of r-sets $\mathcal{A}$ and $\mathcal{B}$ form a **grid**, $\mathbb{G}_{r\times r}$, if $|\mathcal{A}| = |\mathcal{B}| = r$, $\cup\mathcal{A} = \cup\mathcal{B}$ and $|\cup\mathcal{A}| = r^2$, i.e., both $\mathcal{A}$ and $\mathcal{B}$ consists of disjoint sets and every $A \in \mathcal{A}$ meets every $B \in \mathcal{B}$ in exactly one element. Let $\mathbb{I}_{\geq 2}$ be (more precisely $\mathbb{I}^r_{\geq 2}$) the class of hypergraphs of two edges with intersection sizes at least two. This class consists of $r-2$ non-isomorphic hypergraphs. The *Turán number* of the r-uniform hypergraph $\mathcal{H}$, denoted by $\text{ex}(n,\mathcal{H})$, is the size of the largest $\mathcal{H}$-free r-graph on n vertices.

Problem 4 *Given r, construct infinitely many grid-free Steiner systems, $S(v,r,2)$.*

The present author and Ruszinkó [24] showed that there exist a linear, grid-free, r-uniform hypergraph $\mathcal{H}$ on n vertices almost as big as a Steiner system

Theorem 5

$$\frac{n(n-1)}{r(r-1)} - c_r n^{8/5} < \text{ex}_r(n, \{\mathbb{I}_{\geq 2}, \mathbb{G}_{r\times r}\}) \leq \frac{n(n-1)}{r(r-1)}$$

holds for every $n, r \geq 4$.

In the case of $r = 3$ with probabilistic method we only have

$$\Omega(n^{1.8}) \leq \text{ex}_3(n, \{\mathbb{I}_{\geq 2}, \mathbb{G}_{3\times 3}\}) \leq \frac{1}{6} n(n-1).$$

We **conjecture** that here the upper bound holds for infinitely n. On the contrary, Elekes [10] conjectures a $o(n^2)$ upper bound.

The real question is to determine the unavoidable substructures in different classes of designs. Only a few results are know, all in the case $r = 3$. A Steiner triple system is said to be s-*sparse* if it contains no i blocks on $i+2$ elements for any i, $4 \le i \le s$. The question whether s-sparse STS(v) exists was proposed by Erdős [11]. A 4-sparse STS(v) is known to exist for each admissible $v \ne 7, 13$ (Brouwer [6], Grannell, Griggs, and Whitehead [25]). Recently there has been substantial progress towards the goal of establishing that a 5-sparse STS(v) exists for each sufficiently large admissible v (Wolfe [37]). An infinite class of 6-sparse STS(v) is described by Forbes, Grannell, and Griggs [17], but no 7-sparse STS(v) is known for any v.

The basic algebraic construction

Given integers $q \ge r \ge 2$, q prime we define the transversal design mod q, $\mathcal{L}_{r,q}$ as an r-uniform hypergraph on $n := rq$ vertices as follows. Vertex set is $V := \{(j,y):\ 1 \le j \le r,\ y \in F_q\} = [r] \times F_q$, the lattice points on $\mathbb{R}^2$, $\{(j,y):\ y \in F_q\} = j$'th column. For integers $0 \le y, m < q$ define the r-set $L(y,m)$ a *combinatorial line* of slope m

$$L(y,m) = \{(1,y), (2, y+m), \dots, (r, y+(r-1)m)\},$$

where the second coordinates are taken modulo q. Finally, the hypergraph $\mathcal{L}$ is the set of all combinatorial lines

$$\mathcal{L} := \mathcal{L}_{r,F} = \{L(y,m):\ y \in F_q,\ m \in F_q\}.$$

$\mathcal{L}$ has $q^2 = n^2/r^2$ hyperedges (i.e., 'lines', r-tuples). It is easy to see that if q is a prime (and $q \ge r$), then $\mathcal{L}_{r,F}$ is a linear hypergraph. Our main observation is that

Lemma 6 *[24] If q is a prime, $q > r^{4r}$, $r \ge 4$, then $\mathcal{L}_{r,F}$ is $\mathbb{G}_{r\times r}$-free.*

We **conjecture** that this lemma should be true for all $q > r$.

When $r = 3$ there are crossing families of straight (Euclidean) lines. Let $y, m, a, b \in F_q$ (with $a, b > 0$) and consider $L(y+4a+2b, m-3a)$, $L(y-2a+2b, m-3b)$, $L(y-2a-4b, m+3a+3b)$, and $L(y+4a+2b, m-3a-3b)$, $L(y-2a+2b, m+3a)$, $L(y-2a-4b, m+3b)$.

The corresponding crossing system forms a $\mathbb{G}_{3\times 3}$

$$\begin{aligned}
&\{(1, y+4a+2b), (2, y+m+a+2b), (3, y+2m-2a+2b)\}\\
&\{(1, y-2a+2b), (2, y+m-2a-b), (3, y+2m-2a-4b)\}\\
&\{(1, y-2a-4b), (2, y+m+a-b), (3, y+2m+4a+2b)\}
\end{aligned}$$

$$\begin{aligned}
&\{(1, y+4a+2b), (2, y+m+a-b), (3, y+2m-2a-4b)\}\\
&\{(1, y-2a+2b), (2, y+m+a+2b), (3, y+2m+4a+2b)\}\\
&\{(1, y-2a-4b), (2, y+m-2a-b), (3, y+2m-2a+2b)\}.
\end{aligned}$$

More on union-free and cover-free codes

A family $\mathcal{F} \subseteq 2^{[n]}$ is *g-cover-free* if for arbitrary distinct members $A_0, A_1, \dots, A_g$

$$A_0 \not\subseteq \bigcup_{i=1}^{g} A_i.$$

Let $\mathbf{CF}_g(n)$ ($\mathbf{CF}_g(n,r)$) be the maximum size of a g-cover-free n vertex code (r-uniform hypergraph, resp.).

A family $\mathcal{F} \subseteq 2^{[n]}$ is *t-union-free* if for distinct t-multisets $\{A_1, \dots, A_t\}$ and $\{B_1, \dots, B_t\}$ $A_i, B_j \in \mathcal{F}$ we have

$$A_1 \cup A_2 \cup \dots \cup A_t \neq B_1 \cup B_2 \cup \dots \cup B_t.$$

Let $\mathbf{UF}_t(n)$ ($\mathbf{UF}_t(n,r)$) be the maximum size of a t-union-free n vertex code (r-uniform hypergraph, resp.).

If $\mathcal{F}$ is t-**CF** then it is t-**UF**, and if $\mathcal{F}$ is t-**UF** then it is $(t-1)$-**CF**. Hence

$$\mathbf{CF}(n,t) \le \mathbf{UF}(n,t) \le \mathbf{CF}(n,t-1) \le \mathbf{UF}(t-1,n) \le \dots$$

$$\mathbf{CF}_r(n,t) \le \mathbf{UF}_r(n,t) \le \mathbf{CF}_r(n,t-1) \le \mathbf{UF}_r(t-1,n) \le \dots \le \mathbf{UF}_2(n,r).$$

Union free and cover free families were introduced by Kautz and Singleton [29]. They studied binary codes with the property that the disjunctions (bitwise ORs) of distinct at most g-tuples of codewords are all different. In information theory usually these codes are called **superimposed** and they have been investigated in several papers on multiple access communication (see, e.g., Nguyen Quang A and Zeisel [1], D'yachkov and Rykov [8], Johnson [27]). The same problem has been posed – in different terms – by Erdős, Frankl and Füredi [12, 13] in combinatorics, by Sós [35] in combinatorial number theory, and by Hwang and Sós [26] in group testing. For recent generalizations see, e.g., Alon and Asodi [2], and De Bonis and Vaccaro [7]. D'yachkov and Rykov [8] proved that there are positive constants α_1 and α_2 such that

$$\alpha_1 \frac{1}{g^2} < \frac{\log \mathbf{CF}_g(n)}{n} < \alpha_2 \frac{\log g}{g^2}$$

holds for every g and $n > n_0(g)$. One can find short proofs of this upper bound in [22] and in Ruszinkó [32]

Frankl and the present author [21] determined asymptotically the maximum size of an r-uniform g-cover-free family showing that there exists a positive constant $\gamma := \gamma(r,g)$ such that

$$\mathbf{CF}_g(n,r) = (\gamma + o(1)) n^{\lceil r/g \rceil}$$

Problem 7 *Given $r \ge t \ge 1$ find an asymptotic for $\mathbf{UF}_t(n,r)$.*

The order of magnitude of $\mathbf{UF}_r(n,2)$ was determined by Frankl et al. [18, 20]. It is a long-standing **conjecture** of Erdős and Simonovits [14, 15] that

$$\mathbf{UF}_2(n,2) = \left(\frac{1}{2\sqrt{2}} + o(1)\right) n^{3/2} \quad (?)$$

Theorem 8 (Füredi and Ruszinkó [24]) *There exists a $\beta = \beta(r) > 0$ such that for all $n \geq r \geq 4$*

$$n^2 e^{-\beta_r \sqrt{\log n}} < \mathrm{ex}(n, \{\mathbb{I}_{\geq 2}, \mathbb{T}_3, \mathbb{G}_{r\times r}\}) \leq \mathbf{UF}_r(n,r) \leq \frac{n(n-1)}{r(r-1)}.$$

We have only weaker estimates for $r = 3$

$$\Omega(n^{5/3}) \leq \mathbf{UF}_3(n,3) \leq \frac{1}{6} n(n-1).$$

We **conjecture** that here the right hand side should be $o(n^2)$.

The determination of the size of maximal t-union-free families is one of the important and likely solvable Turán type problems.

More problems

Call a code $\mathcal{F}$ t^*-*cancellative* if

$$A_1 \cup \cdots \cup A_t \cup B = A_1 \cup \ldots A_t \cup C \implies B = C \text{ or } \{B, C\} \subset \{A_i, \ldots, A_t\}$$

for every $t+2$ member *sequence* from $\mathcal{F}$, and let $c_t^*(n)$ be the maximum size of such a code $\mathcal{F} \subset 2^{[n]}$. Obviously $\mathbf{CF}(n,t) \leq c_t^*(n) \leq \mathbf{CF}(n, t+1) \leq \mathrm{CANC}(n,t)$. One wonders if equality holds in some of these, and what other relations these functions can have.

The grid cannot be covered by $r-1$ vertices, it has r disjoint edges. So the r-graph having all edges meeting an $(r-1)$-element set is grid free. This gives the lower bound

$$\mathrm{ex}(n, \mathbb{G}_{r\times r}) \geq \binom{n-1}{r-1} + \binom{n-2}{r-1} + \cdots + \binom{n-r+1}{r-1}.$$

The classical result concerning the Turán number of the complete r-partite graph on $r \times r$ vertices by Erdős gives only an upper bound $O(n^{r-\delta})$ with $\delta = r^{-r+1}$. The truth should be much closer to the lower bound.

Problem 9 *Determine the order of magnitude of* $\mathrm{ex}(n, \mathbb{G}_{r\times r})$.

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