Uniform hypergraphs containing no grids

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Abstract

A hypergraph is called an $r \times r$ *grid* if it is isomorphic to a pattern of r horizontal and r vertical lines, i.e., a family of sets $\{A_1, \ldots, A_r, B_1, \ldots, B_r\}$ such that $A_i \cap A_j = B_i \cap B_j = \emptyset$ for $1 \leq i < j \leq r$ and $|A_i \cap B_j| = 1$ for $1 \leq i, j \leq r$. Three sets C_1, C_2, C_3 form a *triangle* if they pairwise intersect in three distinct singletons, $|C_1 \cap C_2| = |C_2 \cap C_3| = |C_3 \cap C_1| = 1$, $C_1 \cap C_2 \neq C_1 \cap C_3$. A hypergraph is *linear*, if $|E \cap F| \leq 1$ holds for every pair of edges $E \neq F$.

In this paper we construct large linear r-hypergraphs which contain no grids. Moreover, a similar construction gives large linear r-hypergraphs which contain neither grids nor triangles. For $r \geq 4$ our constructions are almost optimal. These investigations are motivated by coding theory: we get new bounds for optimal superimposed codes and designs.
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1. Sparse hypergraphs, designs, and codes

In this section we first present some previous investigations in extremal set theory on the topic described in the abstract. Then we state our main theorem. This is followed by motivations

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in coding theory and corollaries where we improve the previously known bounds for so called optimal superimposed codes and designs. To prove the main theorem we are using tools from combinatorial number theory and discrete geometry given in Section 2. In Section 3 we present constructions proving the stated theorems, followed by remarks on union-free and cover-free graphs and triple systems.

1.1. Avoiding grids in linear hypergraphs

Speaking about a hypergraph $\mathbb{F} = (V, \mathcal{F})$ we frequently identify the vertex set $V = V(\mathbb{F})$ by the set of first integers $[n] := \{1, 2, \ldots, n\}$, or points on the plane $\mathbf{R}^2$, or elements of a q-element finite field F_q. To shorten notations we frequently say 'hypergraph $\mathcal{F}$' (or set system $\mathcal{F}$) thus identifying $\mathbb{F}$ to its edge set $\mathcal{F}$. $\mathbb{F}$ is *linear* if for all $A, B \in \mathcal{F}$, $A \neq B$ we have $|A \cap B| \leq 1$. The *degree*, $\deg_{\mathbb{F}}(x)$, of an element $x \in [n]$ is the number of hyperedges in $\mathcal{F}$ containing x. $\mathbb{F}$ is *regular* if every element $x \in [n]$ has the same degree. It is *uniform* if every edge has the same number of elements, r-*uniform* means $|F| = r$ for all $F \in \mathcal{F}$. An $(n, r, 2)$-*packing* is a linear r-uniform hypergraph $\mathcal{P}$ on n vertices. Obviously, $|\mathcal{P}| \leq \binom{n}{2} / \binom{r}{2}$. If here equality holds, then $\mathcal{P}$ is called an $S(n, r, 2)$ *Steiner system*.

Definition 1.1. A set system $\mathcal{F}$ contains an $a \times b$ grid if there exist two disjoint subfamilies $\mathcal{A}, \mathcal{B} \subseteq \mathcal{F}$ such that

- $|\mathcal{A}| = a$, $|\mathcal{B}| = b$, $|\mathcal{A} \cup \mathcal{B}| = a + b$,
- $A \cap A' = B \cap B' = \emptyset$ for all $A, A' \in \mathcal{A}$, $A \neq A'$, $B, B' \in \mathcal{B}$, $B \neq B'$, and
- $|A \cap B| = 1$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$.

Thus an r-uniform $r \times r$ grid, $\mathbb{G}_{r \times r}$, is a disjoint pair $\mathcal{A}, \mathcal{B}$ of the same sizes r such that they cover exactly the same set of r^2 elements.

Theorem 1.2. *For $r \geq 4$ there exists a real $c_r > 0$ such that there are linear r-uniform hypergraphs $\mathcal{F}$ on n vertices containing no grids and*

$$|\mathcal{F}| > \frac{n(n-1)}{r(r-1)} - c_r n^{8/5}.$$

The proof is postponed to Section 3.2.

The *Turán number* of the r-uniform hypergraph $\mathcal{H}$, denoted by $\mathrm{ex}(n, \mathcal{H})$, is the size of the largest $\mathcal{H}$-free r-graph on n vertices. If we want to emphasize r, then we write $\mathrm{ex}_r(n, \mathcal{H})$. Let $\mathbb{I}_{\geq 2}$ be (more precisely $\mathbb{I}^r_{\geq 2}$) the class of r-graphs of two edges and intersection sizes at least two. This class consists of $r - 2$ non-isomorphic hypergraphs, $\mathcal{I}_j$, $2 \leq j < r$, $\mathcal{I}_j := \{A_j, B_j\}$ such that $|A_j| = |B_j| = r$, $|A_j \cap B_j| = j$. Using these notations the above theorem can be restated as follows.

$$\frac{n(n-1)}{r(r-1)} - c_r n^{8/5} < \mathrm{ex}_r(n, \{\mathbb{I}_{\geq 2}, \mathbb{G}_{r \times r}\}) \leq \frac{n(n-1)}{r(r-1)} \tag{1}$$

holds for every $n, r \geq 4$. In the case of $r = 3$ we only have

$$\Omega(n^{1.8}) = \mathrm{ex}_3(n, \{\mathbb{I}_{\geq 2}, \mathbb{G}_{3 \times 3}\}) \leq \frac{1}{6} n(n-1), \tag{2}$$

see in Section 1.5. The case of graphs, $r = 2$, is different, see later in Section 3.5.

Conjecture 1.3. *The asymptotic* (1) *holds for* $r = 3$, *too.*

- *Even more, for any given $r \geq 3$ there are infinitely many Steiner systems avoiding $\mathbb{G}_{r \times r}$.*
- *Probably there exists an $n(r)$ such that, for every admissible $n > n(r)$ (this means that $(n-1)/(r-1)$ and $\binom{n}{2} / \binom{r}{2}$ are both integers) there exists a grid-free $S(n, r, 2)$.*

1.2. Sparse Steiner systems

There are many problems and results concerning subfamilies of block designs, see, e.g., Colbourn and Rosa [14]. A Steiner triple system $\mathrm{STS}(n) := S(n, 3, 2)$ is called *e-sparse* if it contains no set of e distinct triples spanning at most $e + 2$ points. Every Steiner triple system is 3-sparse. A longstanding conjecture of Erdős [21] is that for every $e \geq 4$ there exists an $n_0(e)$ such that if $n > n_0(e)$ and n is admissible (i.e., $n \equiv 1$ or $3 \pmod 6$), then there exists an *e-sparse* $\mathrm{STS}(n)$. Systems that are 4-sparse are those without a Pasch configuration $\mathbb{P}_4$ (4 blocks spanning 6 points, $\{a, b, c\}, \{a, d, e\}, \{b, d, f\}, \{c, e, f\}$). Completing the works of Brouwer [8], Murphy [39,40], Ling and Colbourn [52] and others, finally Grannell, Griggs and Whitehead [38] proved that 4-sparse $\mathrm{STS}(n)$'s exist for all admissible n except 7 and 13.

A 5-sparse system is precisely one lacking Pasch, $\mathbb{P}_4$, and mitre configurations, $\mathbb{M}_5$, the latter comprising five blocks of the form $\{a, b, c\}, \{a, d, e\}, \{a, f, g\}, \{b, d, f\}, \{c, e, g\}$. In a sequence of papers (e.g., Colbourn, Mendelsohn, Rosa, and Širáň [13]) culminating in Fujiwara [34] and Wolfe [69] it was established that systems having no mitres exist for all admissible orders, except for $n = 9$.

Concerning the even more difficult problem of constructing 5-sparse systems (see Ling [51]) Wolfe [68,70] proved that such systems exist for almost all admissible n. More precisely, let $A(x) := \{n : n \equiv 1$ or $3 \pmod 6, n \leq x\}$ and $S(x) := \{n :$ there exists a 5-sparse $\mathrm{STS}(n)$ with $n \leq x\}$, then $\lim_{x \to \infty} (|S(x)| / |A(x)|) = 1$.

Forbes, Grannell and Griggs [29,30] constructed infinite classes of 6-sparse $\mathrm{STS}(n)$'s. As Teirlinck [66] writes in his 2009 review of [70] "currently no non-trivial example of a 7-sparse Steiner triple system is known".

Our Conjecture 1.3 is related to but not a consequence of Erdős' problem. Colbourn [15] has checked (using a computer) all the 80 different $\mathrm{STS}(15)$'s and each contained at least 11 copies of $\mathbb{G}_{3 \times 3}$. Blokhuis [7] reformulated (a weaker version of) Conjecture 1.3 as follows: Are there Latin squares without the following subconfiguration?

$$\begin{pmatrix} * & a & b \\ a & * & c \\ b & c & * \end{pmatrix}.$$

Although the evidence is scarce one is tempted to generalize. An $S(n, r, 2)$ is *e-sparse* if the union of any e blocks exceeds $e(r - 2) + 2$.

Conjecture 1.4. *For every $e \geq 4$ there exists an $n_0(e, r)$ such that if $n > n_0(e, r)$ and n is admissible, then there exists an e-sparse $S(n, r, 2)$.*

1.3. Sparse hypergraphs

Brown, Erdős and Sós [19,10,11] introduced the function $f_r(n, v, e)$ to denote the maximum number of edges in an r-uniform hypergraph on n vertices which does not contain e edges spanned by v vertices. Such hypergraphs are called $\mathbb{G}(v, e)$-*free* (more precisely $\mathbb{G}_r(v, e)$-*free*).

They showed that $f_r(n, e(r - k) + k, e) = \Theta(n^k)$ for every $2 \le k < r$ and $e \ge 2$, in particular $f_r(n, e(r - 2) + 2, e) = \Theta(n^2)$. The upper bound $\binom{n}{2} / \binom{r}{2}$ is easy, and this was the source of Erdős' conjecture concerning sparse Steiner systems. On the other hand, if we forbid e edges spanned by one more vertex this problem becomes much more difficult. Brown, Erdős and Sós conjectured that

$$f_r(n, e(r - k) + k + 1, e) = o(n^k). \tag{3}$$

One of the most famous results of this type is the $(6, 3)$-theorem of Ruzsa and Szemerédi [59], the case $(e, k, r) = (3, 2, 3)$, saying that if no six points contain three triples then the size of the triple system is $o(n^2)$, on the other hand $n^{2-o(1)} < f_3(n, 6, 3)$. This was extended by Erdős, Frankl, and Rödl [24] for arbitrary fixed $r \ge 3$,

$$n^{2-o(1)} < f_r(n, 3(r - 2) + 3, 3) = o(n^2). \tag{4}$$

The case $e = 3$ was further extended by Alon and Shapira [3]

$$n^{k-o(1)} < f_r(n, 3(r - k) + k + 1, 3) = o(n^k).$$

Even the case $k = 2$, $f_r(n, e(r - 2) + 3, e) = o(n^2)$, is still open (for $e \ge 4$). Nearly tight upper bounds were established by Sárközy and Selkow [60,61]:

$$f_r(n, e(r - k) + k + \lfloor \log_2 e \rfloor, e) = o(n^k) \quad \forall r > k \ge 2 \text{ and } e \ge 3,$$

and for the case $e = 4$, $r > k \ge 3$

$$f_r(n, 4(r - k) + k + 1, 4) = o(n^k).$$

1.4. Neither grids nor triangles

Definition 1.5. Three sets C_1, C_2, C_3 form a *triangle*, $\mathbb{T}_3$, if they pairwise intersect in three distinct singletons, $|C_1 \cap C_2| = |C_2 \cap C_3| = |C_3 \cap C_1| = 1$, $C_1 \cap C_2 \ne C_1 \cap C_3$. An r-uniform triangle is denoted by $\mathbb{T}_3^r$.

A *perfect matching* is a subfamily $\mathcal{M}$ of the set system $\mathcal{F}$ such that the members of $\mathcal{M}$ cover every element of $V(\mathcal{F})$ exactly once.

The main result of this paper is a construction.

Theorem 1.6. *For $r \ge 4$ there exist an $n_0(r)$ and $\beta_r > 0$ such that*

$$ex_r(n, \{\mathbb{I}_{\ge 2}, \mathbb{T}_3, \mathbb{G}_{r \times r}\}) > n^2 e^{-\beta_r \sqrt{\log n}} \tag{5}$$

holds for $n \ge n_0(r)$. In other words, there exists a linear r-uniform hypergraph $\mathcal{F}$ which contains neither grids nor triangles and $|\mathcal{F}| \ge n^2 \exp[-\beta_r \sqrt{\log n}]$. In addition, if r divides n, then $\mathcal{F}$ can be decomposed into perfect matchings, in particular it is regular.

For the case $r = 3$ we have the same with a much weaker lower bound

$$ex_3(n, \{\mathbb{I}_{\ge 2}, \mathbb{T}_3, \mathbb{G}_{3 \times 3}\}) > n^{1.6} e^{-\beta_3 \sqrt{\log n}}. \tag{6}$$

Again, the proof is postponed to Section 3.3.

Note that $|\mathcal{F}| = o(n^2)$ by (4) so the lower bound (5) is almost optimal. This result slightly improves the Erdős–Frankl–Rödl (4) construction in two ways. We make the hypergraph regular, and avoid not only triangles but grids, too.

1.5. A probabilistic lower bound

Almost all of the problems discussed in this paper can be formulated as a forbidden substructure question, i.e., as a Turán type problem. Here we present the standard probabilistic lower bound for the Turán number due to Erdős, in a slightly stronger form as usual. An r-uniform hypergraph $(V, \mathcal{F})$ is called *r-partite* if there exists an r-partition of V, $V = V_1 \cup \cdots \cup V_r$, such that $|F \cap V_i| = 1$ for all $F \in \mathcal{F}$, $i \in [r]$.

Lemma 1.7 (*Erdős' Lower Bound on the Turán Number*)**.** *Suppose that $\mathcal{H}$ is a (finite) family of r-graphs each of them having at least two edges, and let*

$$h := \min \left\{ \frac{re - v}{e - 1} : \mathbf{H} \in \mathcal{H} \text{ is } r\text{-partite with } e \text{ edges and } v \text{ vertices} \right\}.$$

Then there exists a $c := c(\mathcal{H}) > 0$ such that one can find an n-vertex r-partite r-graph $(n \geq r)$ of size at least cn^h avoiding each member of $\mathcal{H}$. In other words

$$ex(n, \mathcal{H}) = \Omega(n^h). \tag{7}$$

Sketch of the proof. Choose independently each of the $(n/r)^r$ edges of the complete r-partite hypergraph on n vertices with probability p. Leave out an edge from this random selection of each copy of $\mathbf{H} \in \mathcal{H}$. The expected size of the remaining edges is at least

$$p(n/r)^r - \sum p^e n^v.$$

The rest is an easy calculation. $\quad\square$

The value of $h = \min \frac{re - v}{e - 1}$ for the family $\mathbb{I}^r_{\geq 2}$ is 2, $h = r^2/(2r - 1)$ for $\mathbb{G}_{r \times r}$ and it is $3/2$ for $\mathbb{T}_3$. So Lemma 1.7 implies (2), i.e., $\mathrm{ex}_3(\mathbb{I}_{\geq 2}, \mathbb{G}_{3 \times 3}) = \Omega(n^{9/5})$. However, if $\mathbb{T}_3$ is among the forbidden substructures then the probabilistic lower bound fails miserably, it gives only $\Omega(n^{3/2})$ which is very far from the truth. To prove the slightly better lower bound (6) we are going to use a version of the original Ruzsa–Szemerédi method in Section 3.3.

1.6. Tallying up the Turán type problems

The three forbidden configurations, $\mathbb{I}^r_{\geq 2}$, $\mathbb{G}_{r \times r}$, $\mathbb{T}^r_3$, have 7 non-empty combinations. The cases $\mathrm{ex}(n, \{\mathbb{I}_{\geq 2}, \mathbb{G}_{r \times r}\})$ and $\mathrm{ex}(n, \{\mathbb{I}_{\geq 2}, \mathbb{T}_3, \bar{\mathbb{G}}_{r \times r}\})$ were discussed above in Theorems 1.2 and 1.6, respectively. It is easy to see that

$$\mathrm{ex}_r(n, \{\mathbb{I}^r_{\geq 2}, \mathbb{T}_3\}) = f_r(n, 3(r - 2) + 3, 3) + O(n),$$

so the Ruzsa and Szemerédi [59] and the Erdős, Frankl, and Rödl [24] theorems, see (4), determine the right order of magnitude, $O(n^{2-o(1)})$.

It was conjectured by Chvátal and Erdős and proved by Frankl et al. [33] that

$$\mathrm{ex}_r(n, \mathbb{T}_3) = \binom{n - 1}{r - 1}$$

for $r \geq 5$ and $n > n_0(r)$. The only extremal r-graph consists of all r-tuples sharing a common element. This hypergraph has no grid either, so we have

$$\mathrm{ex}(n, \{\mathbb{T}_3, \mathbb{G}_{r \times r}\}) = \binom{n - 1}{r - 1}$$

for the same range of r and n.

We have $\mathrm{ex}_r(n, \mathbb{I}_{\geq 2}^r) = \binom{n}{2} / \binom{r}{2}$ if and only if a Steiner system $S(n, r, 2)$ exists, which problem was solved for $n > n_0(r)$ by Wilson [67] and the exact packing number was determined for all large enough n by Caro and Yuster [12].

The grid cannot be covered by $r - 1$ vertices, it has r disjoint edges. So the r-graph having all edges meeting an $(r - 1)$-element set is grid free. This gives the lower bound for the last case out of the seven.

$$\mathrm{ex}(n, \mathbb{G}_{r \times r}) \geq \binom{n-1}{r-1} + \binom{n-2}{r-1} + \cdots + \binom{n-r+1}{r-1}. \tag{8}$$

The classical result concerning the Turán number of the complete r-partite graph on $r \times r$ vertices by Erdős [20] gives only an upper bound $O(n^{r-\delta})$ with $\delta = r^{-r+1}$. The truth should be much closer to the lower bound.

Problem 1.8. Determine the order of magnitude of $\mathrm{ex}(n, \mathbb{G}_{r \times r})$.

1.7. Union-free and cover-free hypergraphs

Union free families were introduced by Kautz and Singleton [48]. They studied binary codes with the property that the disjunctions (bitwise ORs) of distinct at most r-tuples of codewords are all different. In information theory usually these codes are called *superimposed* and they have been investigated in several papers on multiple access communication (see, e.g., Nguyen Quang A and Zeisel [55], D'yachkov and Rykov [17], Johnson [45,47,46]). Alon and Asodi [1,2], and De Bonis and Vaccaro [16] studied this problem in a more general setup. Small values of generalized superimposed codes and their relation to designs were considered by Kim, Lebedev and Oh [49,50].

The same problem has been posed – in different terms – by Erdős, Frankl and Füredi [22,23] in combinatorics, by Sós [63] in combinatorial number theory, and by Hwang and Sós [42,43] in group testing. One can find short proofs of the best known upper bounds of these codes in the papers [35,57] by the present authors. In [36] the connection of these codes to the big distance ones is shown. A geometric version has been posed by Ericson and Györfi [28] and later investigated in [37].

A family $\mathcal{F} \subseteq 2^{[n]}$ is *e-union-free* if for arbitrary two distinct subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{F}$ with $0 < |\mathcal{A}|, |\mathcal{B}| \leq e$

$$\bigcup_{A \in \mathcal{A}} A \neq \bigcup_{B \in \mathcal{B}} B.$$

Let $U(n, e)$ ($U_r(n, e)$) be the maximum size of an e-union-free n vertex hypergraph (r-uniform hypergraph, resp.). The order of magnitude of $U_r(n, 2)$ was determined by Frankl et al. [31,32].

A family $\mathcal{F} \subseteq 2^{[n]}$ is *e-cover-free* if for arbitrary distinct members $A_0, A_1, \ldots, A_e \in \mathcal{F}$

$$A_0 \not\subseteq \bigcup_{i=1}^{e} A_i.$$

Let $C(n, e)$ ($C_r(n, e)$) be the maximum size of an e-cover-free n vertex hypergraph (r-uniform hypergraph, resp.).

An e-cover-free hypergraph is e-union-free and an e-union-free is $(e-1)$-cover-free. (Indeed, the existence of an $(e-1)$-cover $A_0 \subseteq A_1 \cup \cdots \cup A_{e-1}$ gives $\bigcup_{0 \leq i \leq e-1} A_i = \bigcup_{1 \leq i \leq e-1} A_i$.)

Therefore,

$$C(n, e) \leq U(n, e) \leq C(n, e - 1) \leq U(n, e - 1) \cdots \tag{9}$$

and

$$C_r(n, e) \leq U_r(n, e) \leq C_r(n, e - 1) \leq U_r(n, e - 1) \cdots. \tag{10}$$

We have $C_r(n, r) = n - r + 1$ (for $n \geq r$) because every member of an r-uniform r-cover-free family has a vertex of degree one. In this section, based on Theorem 1.6, we determine the next two terms of the sequence (10). First, observe that

$$C_r(n, r - 1) \leq \frac{n(n - 1)}{r(r - 1)}. \tag{11}$$

Indeed, an r-uniform, $(r - 1)$-cover-free family either has a vertex of degree one (and then we use induction on n), or it is a linear hypergraph.

Corollary 1.9. *For all $r \geq 4$ there exists $\beta = \beta_r > 0$ such that*

$$n^2 e^{-\beta_r \sqrt{\log n}} < U_r(n, r) \leq \frac{n(n - 1)}{r(r - 1)}.$$

In addition, if r divides n, then our n-vertex, r-uniform, r-union-free family yielding the lower bound can be decomposed into perfect matchings, in particular it is regular.

Proof. The upper bound follows from (10) and (11), i.e.,

$$U_r(n, r) \leq C_r(n, r - 1) \leq \frac{n(n - 1)}{r(r - 1)}.$$

On the other hand, we claim that

$$\mathrm{ex}(n, \{\mathbb{I}^r_{\geq 2}, \mathbb{T}_3, \mathbb{G}_{r \times r}\}) \leq U_r(n, r), \tag{12}$$

hence the lower bound for $U_r(n, r)$ follows from (5).

   We have to show that a linear r-graph without triangles and grids is r-union-free. Suppose, on the contrary, that $\mathcal{A} \neq \mathcal{B}$, $|\mathcal{B}| \leq |\mathcal{A}| \leq r$, $\cup_{A \in \mathcal{A}} A = \cup_{B \in \mathcal{B}} B$ and $\mathcal{A} \cup \mathcal{B}$ form a linear r-uniform hypergraph. Then $\exists A_1 \in \mathcal{A} \setminus \mathcal{B}$. Since $|A_1 \cap B| \leq 1$, to cover the elements of A_1 the family $\mathcal{B}$ must contain r sets, i.e., $|\mathcal{B}| = |\mathcal{A}| = r$. Moreover, the sets $B_1, \ldots, B_r \in \mathcal{B}$ meet A_1 in distinct elements. If $\mathcal{B}$ consists of disjoint sets only, then $|\cup_{B \in \mathcal{B}} B| = r^2$, and to cover these r^2 elements $\mathcal{A}$ must consist of disjoint sets, too, and $\mathcal{A} \cup \mathcal{B}$ form a grid $\mathbb{G}_{r \times r}$. Otherwise, $\exists B_i, B_j \in \mathcal{B}$, such that $B_i \cap B_j = \{x\} \notin A_1$. Then A_1, B_i, and B_j form a triangle. $\quad \square$

Proposition 1.10. *In the case $r = 3$ the probabilistic lower bound (7) implies*

$$\Omega(n^{5/3}) = U_3(n, 3). \tag{13}$$

The details are postponed to Section 3.4.

   A straightforward argument (similar to the proof of (12)) shows that $U_r(n, r) = O(f_r(n, r^2 - r + 1, r + 1))$. Thus the case $k = 2$, $e = r + 1$ of Erdős' conjecture (3) implies the following more modest conjecture.

Conjecture 1.11.

$$U_r(n, r) = o(n^2). \tag{14}$$

Proposition 1.12. *Suppose that $r \geq 2$, $n \equiv r \pmod{r^2 - r}$, $1 \leq k \leq (n-1)/(r-1)$ and $n > n_0(r)$. Then there exists a k-regular, $(r-1)$-cover-free r-graph. Thus, in this case* (11) *gives*

$$C_r(n, r-1) = \frac{n(n-1)}{r(r-1)}.$$

Proof. To obtain the k-regular construction one can apply a classical theorem of Ray-Chaudhuri and Wilson [56]: For any given $r \geq 2$ there exists an $n_0(r)$ such that, if $n > n_0(r)$ and $n \equiv r \pmod{r^2 - r}$, then there exists a *resolvable*, r-uniform, n-vertex Steiner system $\mathcal{S}$. This means that $\mathcal{S}$ can be decomposed into $K := (n-1)/(r-1)$ perfect matchings, (also called *parallel classes*) $\mathcal{S} = \cup_{1 \leq i \leq K} \mathcal{S}_i$, where $|\mathcal{S}_i| = n/r$ and $|\cup \mathcal{S}_i| = n$. Taking k of these parallel classes give the desired $(r-1)$-cover-free family.   □

1.8. Optimal superimposed codes

D'yachkov and Rykov [18] introduced the concept of *optimal superimposed codes and designs*. Recall an easy observation.

Proposition 1.13 (*D'yachkov, Rykov [18]*). *If $\mathcal{F} \subseteq 2^{[n]}$ is $(r-1)$-cover-free, $(r \geq 2)$, it has maximum degree k and $|\mathcal{F}| = t \geq n$ then $\lceil nk/r \rceil \geq t$ holds.*

By (9) a similar statement holds for r-union-free families, too. Note that, from coding theory point of view, it is reasonable to assume that $t \geq n$ since a collection of singletons $\mathcal{F}$ is $(r-1)$-cover-free for arbitrary $2 \leq r \leq n$ with $|\mathcal{F}| = n$ and usually the goal is to get a code as large as possible.

Proof of 1.13. Let $\mathcal{F}_0 = \{A \in \mathcal{F} : \exists x \in A, \deg_{\mathcal{F}}(x) = 1\}$, $|\mathcal{F}_0| = t_0$. Clearly, $|A| \geq r$ for every $A \in \mathcal{F} \setminus \mathcal{F}_0$ otherwise the union of some other $(r-1)$ members of $\mathcal{F}$ cover A. We obtain

$$r(t - t_0) + t_0 \leq \sum_{A \in \mathcal{F}} |A| = \sum_{x \in [n]} \deg(x) \leq k(n - t_0) + t_0. \quad □$$

It follows that in case of $nk = rt$ the family $\mathcal{F}$ should be k-regular and r-uniform.

Definition 1.14 (*See [18]*). The n-vertex family $\mathcal{F}$ is called an *optimal $(r-1)$-superimposed code* if it is an $(r-1)$-cover-free, r-uniform, k-regular, linear hypergraph. It is called an *optimal r-superimposed design* if in addition it is r-union-free, too. In both cases $nk = rt$ holds.

Let $k(r-1, n)(k'(r, n))$ denote the maximum k that such a k-regular optimal $(r-1)$-superimposed code (optimal r-superimposed design) exits. They have shown for every $r \geq 2$ and n that

$$k'(r, n) \leq k(r-1, n) \leq (n-1)/(r-1)$$
$$\log_2 n - O(1) \leq k'(2, n), \qquad n/2 \leq k(1, n) \leq n-1$$
$$4 \leq k'(3, n), \qquad (n/3) - 1 \leq k(2, n) \leq (n-1)/2.$$

They and Macula [53] gave a lower bound for every $r \geq 3$ for a few special but infinitely many values of n.

$$\left(\frac{n}{r}\right)^{1/(r-1)} \leq k'(r, n). \tag{15}$$

Our results, Corollary 1.9 and Proposition 1.12, imply that

Corollary 1.15. *If $r \geq 4$, $r | n$ and $n \geq n_0(r)$, then there exists an optimal k-regular, r-superimposed design for every $1 \leq k \leq rne^{-\beta_r \sqrt{\log n}}$, hence*

$$rne^{-\beta_r \sqrt{\log n}} \leq k'(r, n). \quad \square$$

Proposition 1.16. *For the case $r = 3$ we have the same statement with a weaker lower bound*

$$\frac{1}{25} n^{2/3} \leq k'(3, n). \tag{16}$$

The details are postponed to Section 3.4.

Corollary 1.17. *Suppose that $r \geq 2$, $n \equiv r \pmod{r^2 - r}$ and $n > n_0(r)$. Then there exists an optimal $(r - 1)$-superimposed code for every $1 \leq k \leq (n - 1)/(r - 1)$, hence*

$$k(r - 1, n) = (n - 1)/(r - 1). \quad \square$$

2. Tools from combinatorial number theory and discrete geometry

2.1. Three lemmata from combinatorial number theory

Lemma 2.1 (*Minkowski's Theorem of Simultaneous Approximation* [54]). *Let q be a prime and $(n_1, \ldots, n_d) \in \mathbf{R}^d$ an integer point. Then there exist an integer $0 < \alpha < q$ and residues r_i such that $r_i \equiv \alpha n_i \pmod{q}$ and $|r_i| \leq q^{1-1/d}$ for all $1 \leq i \leq d$.*

Sketch of the proof. Consider all vectors of the form $a\mathbf{n} \bmod q$, $a = 0, 1, 2, \ldots, q - 1$. There will be two of them $a_1\mathbf{n}$ and $a_2\mathbf{n}$ 'close' to each other. Take $\alpha = a_1 - a_2$. $\quad \square$

Let $r_k(q)$ be the maximum number of integers which can be selected from $\{1, \ldots, q\}$ containing no k-term arithmetic progression. This function has been extensively studied in the last six decades by leading mathematicians see, e.g., Ruzsa [58]. The major important known bounds (apart from some recent minor improvements) are due to Behrend [6], Heath-Brown [41] and Szemerédi [64,65]: there are positive constants α and β such that

$$qe^{-\beta\sqrt{\log q}} < r_3(q) < q(\log q)^{-\alpha} \quad \text{and for all } k \quad r_k(n) = o(n). \tag{17}$$

Call a set $M \subset [q]$ r-*sum-free* if the equation

$$c_1 m_1 + c_2 m_2 = (c_1 + c_2) m_3$$

has no solutions with $m_1, m_2, m_3 \in M$ and c_1, c_2 are positive integers with $c_1 + c_2 \leq r$ except the one with $m_1 = m_2 = m_3$. We will need the following lower bound used by Erdős, Frankl and Rödl [24], also see Ruzsa [58]. Its proof requires only a slight modification of Behrend's [6] argument.

Lemma 2.2 (*Generalized Behrend*). *For arbitrary positive integer r there exists a $\gamma_r > 0$ such that for any integer q one can find an r-sum-free subset $M \subseteq \{0, 1, \ldots, q\}$ such that $|M| > qe^{-\gamma_r \sqrt{\log q}}$.*

The case $r = 2$ (and $c_1 = c_2 = 1$) is the original statement of Behrend [6]. Ruzsa also notes that an upper bound $O(q/(\log q)^{\alpha_r})$ for the general case can be proved by the methods of [41] and [65].

Call a set of numbers A_6-*free* if it does not contain a subset of the form

$$\{x - a - b, x - b, x - a, x + a, x + b, x + a + b\}$$

for some $a, b > 0$, $a \neq b$. Call it A_4-*free* if it does not contain a fourtuple of the form $\{x - 2a, x - a, x + a, x + 2a\}$ for some $a > 0$, and call it AP_k-*free* if it contains no k-term arithmetic progression. Let $r(n, P_1, P_2, \ldots)$ denote the maximum number of integers which can be selected from $\{1, \ldots, n\}$ avoiding the patterns $P_1, P_2, \ldots$. With this notation $r_3(n) := r(n, AP_3)$.

Since an A_4-free set has no 5-term arithmetic progression we get $r(n, A_4) \leq r_5(n) = o(n)$ by Szemerédi's [64] theorem (17). A 4-sum-free sequence is A_4-free as well (one has, e.g., $1 \times (x - 2a) + 3 \times (x + 2a) = 4 \times (x + a)$), thus Lemma 2.2 gives a lower bound showing

$$r(n, A_4) = n^{1-o(1)}.$$

Lemma 2.3.

$$\frac{2}{5} r_3(n)^{3/5} < r(n, A_6, A_4, AP_3).$$

Sketch of the proof. Similar to the proof of Lemma 1.7. Take an AP_3-free set $M \subset \{1, 2, \ldots, n\}$ of maximum size. Choose independently each element of M with probability p, and leave out an element from this random selection of each copy of the arising configurations we want to avoid. The expected size of the remaining elements is at least

$$p|M| - p^6|M|^3 - p^4|M|^2.$$

Define p as $\frac{1}{2}|M|^{-2/5}$. $\quad \square$

Starting with $M = [n]$, the same process gives

$$\frac{2}{5} n^{3/5} < r(n, A_6, A_4) \leq r(n, A_6).$$

Concerning the upper bounds we only have $r(n, A_6) \leq r_7(n) = o(n)$.

The random method notoriously gives a weak lower bound on Sidon type problems (for definitions, see, e.g., Babai and Sós [4]), e.g., the above argument gives only $r(n, \text{Sidon}) = \Omega(n^{1/3})$, although the truth is $\Theta(n^{1/2})$ (Erdős and Turán [27]) and it gives $r_3(n, AP_3) = \Omega(n^{1/2})$ although the truth is $n^{1-o(1)}$. So we believe that there are much better lower bounds.

Conjecture 2.4. *There is an $\varepsilon > 0$ such that*

$$n^{3/5+\varepsilon} < r(n, A_6)$$

holds for large enough n. Possibly the order of magnitude of this function is $n^{1-o(1)}$.

Unfortunately, the recent general construction of Shapira [62] does not seem to be applicable here. We are thankful for I. Ruzsa (Budapest) and J. Wolf (Paris) for this observation.

There is a connection between sets of integers that are A_6-free and another (seemingly very difficult) open problem of Ruzsa; finding the maximum size, $r(n, \star)$, of a set $M \subseteq [n]$ with no

non-trivial solution to the equation

$$2x + 2y = 3u + v. \tag{$\star$}$$

Ruzsa [58] proved that $cn^{1/2} \le r(n, \star)$, remarked that $r(n, \star) = o(n)$ and asked whether there is a set $M \subseteq [n]$ of size $n^{1-o(1)}$ with no non-trivial solution to $(\star)$. Notice, that if M has no non-trivial solution to $(\star)$ then it must be A_6-free since

$$2(x - a) + 2(x - b) = 3(x - a - b) + (x + a + b).$$

Hence $r(n, \star) \le r(n, A_6) = o(n)$. We are thankful for the referee pointing out this connection.

2.2. Grids of pseudolines

A set of *pseudolines* is a set of (infinite) planar curves pairwise meeting in at most one point with crossing, no two pseudolines are tangent. The main result of this subsection in fact deals with pseudoline arrangements but we formulate it in a simpler way.

Let $\ell_1, \dots, \ell_r$ be parallel vertical lines on the plane, $\ell_j := \{(x, y) : x = j\}$, $r \ge 2$. Let V_j be an r-set of points on the line ℓ_j, $V_j := \{Q_{1,j}, \dots, Q_{r,j}\}$, $Q_{i,j} = (j, y_{i,j})$, such that $y_{1,j} > y_{2,j} > \dots > y_{r,j}$ for every $1 \le j \le r$. The points of $\Pi := \cup V_j$ can be arranged in a matrix form

$$\begin{pmatrix} Q_{1,1} & Q_{1,2} & \dots & Q_{1,r} \\ Q_{2,1} & Q_{2,2} & \dots & Q_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{r,1} & Q_{r,2} & \dots & Q_{r,r} \end{pmatrix}$$

where the elements of the jth column lie on ℓ_j and are ordered from the top to the bottom. The vertical distances $y_{i,j} - y_{i+1,j}$ are positive, but can be distinct from each other.

A Π-*polygon* π consists of $r - 1$ segments of the form $\pi = \cup_j [Q_j Q_{j+1}]$ $(1 \le j \le r - 1)$ with $Q_j \in V_j$. There are r^r such polygonal arcs. Such a π can be considered as a piecewise linear continuous function $\pi : [1, r] \to \mathbf{R}$. Two sets of Π-polygons $\mathcal{P}$ and $\mathcal{R}$ are called *crossing* if

(C1) $|\mathcal{P}| = |\mathcal{R}| = r$,
(C2) $\mathcal{P}$ covers all vertices of Π, i.e., $\{\pi \cap \ell_j : \pi \in \mathcal{P}\} = V_j$ for $1 \le j \le r$ and the same holds for $\mathcal{R}$,
(C3) $\mathcal{P} \cup \mathcal{R}$ is almost disjoint, i.e., $\pi, \rho \in \mathcal{P} \cup \mathcal{R}$ (and $\pi \ne \rho$) imply that $|\pi \cap \rho| \le 1$, and
(C4) $\mathcal{P} \cup \mathcal{R}$ behaves like pseudolines, i.e., $\pi, \rho \in \mathcal{P} \cup \mathcal{R}$, $|\pi \cap \rho| = 1$ imply that they are truly crossing, i.e., if $\pi \cap \rho = (x_0, y_0)$ then
either $\pi(x) < \rho(x)$ for all $1 \le x < x_0$ together with $\pi(x) > \rho(x)$ for $x_0 < x \le r$,
or $\pi(x) > \rho(x)$ for all $1 \le x < x_0$ together with $\pi(x) < \rho(x)$ for $x_0 < x \le r$.

Note that (C1) and (C2) imply that every member of Π belongs to a unique $\pi \in \mathcal{P}$ and also to a unique $\rho \in \mathcal{R}$. Then (C3) yields that each $\pi \in \mathcal{P}$ meets each $\rho \in \mathcal{R}$ at an element of Π, and only there. The members of $\mathcal{P}$ (and $\mathcal{R}$) can be disjoint among themselves.

The Lemma 2.5 below claims that a crossing system has a simple structure what we will call *very special* and its properties are listed in (VS1)–(VS7). That long list looks complicated when in fact it is very easy to draw. A very special crossing system is depicted below in (18) using thin and thick lines to indicate the segments of $\mathcal{P}$ and $\mathcal{R}$. Informally, there are two types of paths that

define $\mathcal{P} \cup \mathcal{R}$, *top paths* and *bottom paths*. A top path begins at the point $Q_{i,1}$ and makes a right-up step to get to $Q_{i-1,2}$ and continues making right-up steps until it reaches $Q_{1,i'}$ where it takes one right step to $Q_{1,i'+1}$ and then begins taking right-down steps first moving to $Q_{2,i'+2}$ then $Q_{3,i'+3}$ and so on. A bottom path is defined in a very similar way, we just interchange the words up and down in our description of top paths and make the appropriate changes to the indices of the $Q_{i,j}$'s.

$$(18)$$
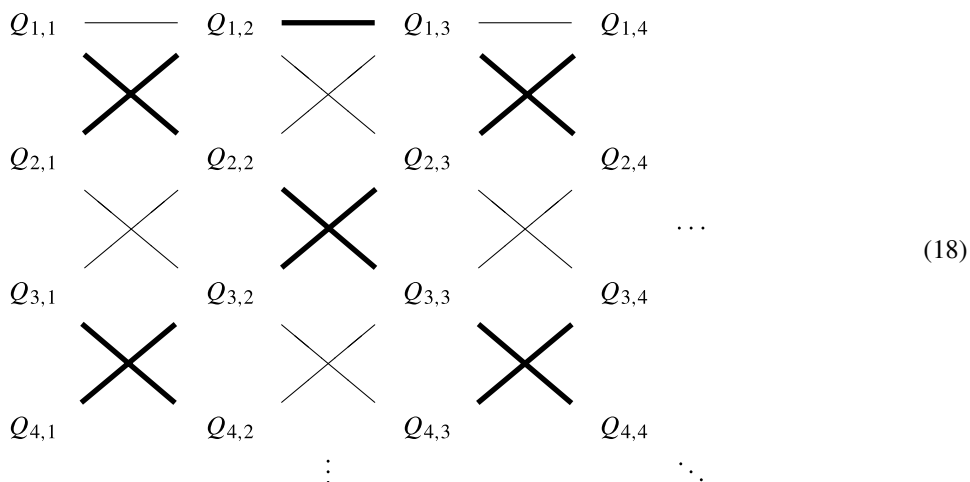
There are two types of segments defining $\mathcal{P} \cup \mathcal{R}$,

(VS1) all edges of the upper and lower envelops $[Q_{1,j}, Q_{1,j+1}]$, $[Q_{r,j}, Q_{r,j+1}]$, $1 \le j \le r-1$, and

(VS2) all diagonal edges $Q_{i,j}Q_{i\pm1,j+1}$.

(VS3) The edges on the upper (lower) envelops alternate between $\mathcal{P}$ and $\mathcal{R}$.

(VS4) The crossing diagonal edges $[Q_{i,j}, Q_{i+1,j+1}]$ and $[Q_{i+1,j}, Q_{i,j+1}]$ simultaneously belong to either $\mathcal{P}$ or $\mathcal{R}$.

$\mathcal{P} \cup \mathcal{R}$ consists of the following three types of polygonal paths:

(VS5) the two diagonals $Q_{1,1}, Q_{2,2}, \ldots, Q_{r,r}$, and $Q_{r,1}, Q_{r-1,2}, \ldots, Q_{1,r}$,

(VS6) for $1 \le j \le r-1$ a *top* path consisting of an increasing part (see figure (18)) $Q_{j,1}, Q_{j-1,2}, \ldots, Q_{1,j}$ a horizontal edge $Q_{1,j}, Q_{1,j+1}$ and a decreasing part $Q_{1,j+1}$, $Q_{2,j+2}, \ldots, Q_{r-j,r}$,

(VS7) for each $2 \le j \le r$ a *bottom* path starting at $Q_{j,1}$ and consisting of a decreasing part having vertices $Q_{j+x,1+x}$ ($x = 0, 1, 2, \ldots, r-j$), a horizontal edge $[Q_{r,r-j+1}, Q_{r,r-j+2}]$ and an increasing part $Q_{r-x,r-j+2+x}$ ($x = 0, 1, 2, \ldots, j-2$).

Lemma 2.5. *Suppose that the two sets of Π-polygons $\mathcal{P}$ and $\mathcal{R}$ form a crossing Π-polygon system (i.e., satisfy (C1)–(C4)). Then they have the unique structure described by (VS1)–(VS7).*

The proof is postponed to the next subsection. From this Lemma we can read out the intersection structure. We obtain
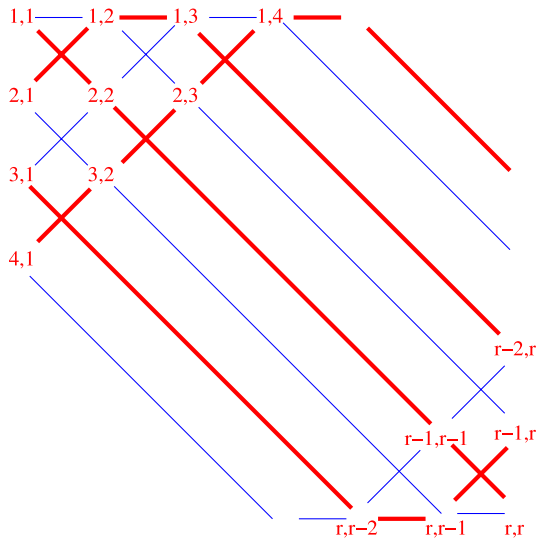
$$(19)$$

Corollary 2.6. *Suppose that the two sets of Π-polygons $\mathcal{P} := \{\pi_1, \ldots, \pi_r\}$ and $\mathcal{R} := \{\rho_1, \ldots, \rho_r\}$ form a crossing Π-polygon system with $Q_{i,1} \in \pi_i, \rho_i (1 \le i \le r)$ and with $[Q_{1,1}, Q_{1,2}] \subset \pi_1$. Then*

- *the vertices of π_1 are $Q_{1,1}, Q_{1,2}, Q_{2,3}, \ldots, Q_{r-1,r}$,*
- *the vertices of π_2 are $Q_{2,1}, Q_{3,2}, \ldots, Q_{r,r-1}, Q_{r,r}$,*
- *the vertices of π_3 are $Q_{3,1}, Q_{2,2}, Q_{1,3}, Q_{1,4}, \ldots, Q_{r-3,r}$,*
- *the vertices of π_4 are $Q_{4,1}, \ldots, Q_{r,r-3}, Q_{r,r-2}, Q_{r-1,r-1}, Q_{r-2,r}$,*
- *the vertices of ρ_1 are $Q_{1,1}, Q_{2,2}, \ldots, Q_{r-1,r-1}, Q_{r,r}$,*
- *the vertices of ρ_2 are $Q_{2,1}, Q_{1,2}, Q_{1,3}, \ldots, Q_{r-2,r}$,*
- *the vertices of ρ_3 are $Q_{3,1}, \ldots, Q_{r,r-2}, Q_{r,r-1}, Q_{r-1,r}$,*
- *the vertices of ρ_4 are $Q_{4,1}, Q_{3,2}, Q_{2,3}, Q_{1,4}, Q_{1,5}, \ldots, Q_{r-4,r}$.* □

2.3. The proof of the uniqueness of the crossing structure

Here we prove Lemma 2.5 with a series of propositions. Assume $Q_{i,1} = \pi_i \cap \rho_i$ for $1 \le i \le r$. For $1 \le j \le r-1$ let $G_j^{\mathcal{P}}$ be the bipartite graph (a matching) with parts V_j and V_{j+1} and with edges defined by the corresponding parts of the polygons from $\mathcal{P}$, i.e., $[Q_{i,j}, Q_{k,j+1}] \in E(G_j^{\mathcal{P}})$ if and only if there is a $\pi \in \mathcal{P}$ with $\pi(j) = y_{i,j}$ and $\pi(j+1) = y_{k,j+1}$. (Thus, to simplify notations, we identify the graph $G_j^{\mathcal{P}}$ with its geometric representation.) $G_j^{\mathcal{R}}$ is defined similarly. Finally, $G^{\mathcal{P}}$ is the union $\cup_j G_j^{\mathcal{P}}$ and the graph G has all the edges of $G^{\mathcal{P}}$ and $G^{\mathcal{R}}$.

Proposition 2.7. *Suppose $\pi \cap \ell_j = Q_{i,j}$ for some $\pi \in \mathcal{P}$. Then $\pi \cap \ell_{j+1} \in \{Q_{i-1,j+1}, Q_{i,j+1}, Q_{i+1,j+1}\}$. Similarly, $\rho \in \mathcal{R}$, $1 \le j \le r-1$, $\rho \cap \ell_j = Q_{i,j}$ and $\rho \cap \ell_{j+1} = Q_{k,j+1}$ imply $|i - k| \le 1$.*

Proof. We prove the second statement. Assume to the contrary that $k \le i - 2$ (the case $k \ge i + 2$ is similar). Consider the $i - 1$ edges of $G_j^{\mathcal{P}}$ with vertices $Q_{1,j}, \ldots, Q_{i-1,j}$. Since there is not enough room to match these vertices to $Q_{h,j+1}(1 \le h \le k)$ there exits a $[Q_{u,j}, Q_{v,j+1}] \in$

$E(G_j^{\mathcal{P}})$ with $u < i$ and $v > k$. Then this segment intersects $[Q_{i,j}, Q_{k,j+1}] \in E(G_j^{\mathcal{R}})$ inside the open strip $\{(x, y) : j < x < j + 1\}$. This contradicts to the fact that a $\pi \in \mathcal{P}$ and a $\rho \in \mathcal{R}$ meet only in the points of Π. $\quad \square$

In the same way we obtain the following.

Proposition 2.8. *Suppose that* $\gamma \in \mathcal{P} \cup \mathcal{R}$*, the first point of* γ *is* $Q_{a,1}$*, the last one is* $Q_{b,r}$*. Then* $b \in \{r - a, r - a + 1, r - a + 2\}$*.*

Proof. Suppose $\gamma = \pi_a$. Every $\rho_1, \ldots, \rho_{a-1}$ starts above π_a on ℓ_1, they meet π_a in a point of Π, so their endpoints on ℓ_r lie below or on the endpoint of γ, $Q_{b,r}$. Hence $b \leq r - (a - 2)$. Considering $\rho_{a+1}, \ldots, \rho_r$ the same argument gives $b \geq r - a$. $\quad \square$

Now we prove Lemma 2.5 by checking (VS1)–(VS7).

Proof of (VS5). The paths π_1 and ρ_1 end at either $Q_{r-1,r}$ or at $Q_{r,r}$ by the previous Proposition. They cannot meet in a second point, so one of them finishes at $Q_{r,r}$. Then Proposition 2.7 implies that this path is the diagonal of the $r \times r$ array, $Q_{1,1}, Q_{2,2}, Q_{3,3}, \ldots, Q_{r-1,r-1}, Q_{r,r}$. By symmetry, the other diagonal $Q_{r,1}, Q_{r-1,2}, \ldots, Q_{1,r}$ also belongs to $\mathcal{P} \cup \mathcal{R}$. $\quad \square$

Proof of (VS1). We show that each top edge, $[Q_{1,j}, Q_{1,j+1}]$, $1 \leq j \leq r - 1$ belongs to $E(G)$. According to Proposition 2.7 the neighbor of $Q_{1,j}$ in $G_j^{\mathcal{P}}$ (in $G_j^{\mathcal{R}}$) is either $Q_{1,j+1}$ or $Q_{2,j+1}$. The $\mathcal{P}$ and $\mathcal{R}$ edges are distinct, so both $[Q_{1,j}, Q_{1,j+1}]$ and $[Q_{1,j}, Q_{2,j+1}]$ must belong to $E(G)$. Both $[Q_{1,j}, Q_{1,j+1}]$ and $[Q_{2,j}, Q_{1,j+1}]$ belong to $E(G)$ and we got the edges of the top layer of (18) with $[Q_{1,j}, Q_{1,j+1}] \in E(G)$ and below it $[Q_{1,j}, Q_{2,j+1}]$ and $[Q_{2,j}, Q_{1,j+1}]$ are crossing.

A similar argument gives that $[Q_{r,j}, Q_{r,j+1}]$ belongs to $E(G)$. $\quad \square$

Proof of (VS3). We show that the top edges alternate between $G^{\mathcal{P}}$ and $G^{\mathcal{R}}$. If two consecutive of them, $[Q_{1,j}, Q_{1,j+1}]$ and $[Q_{1,j+1}, Q_{1,j+2}]$, both belong to $G^{\mathcal{P}}$ then they are part of the same $\pi \in \mathcal{P}$ and the diagonal edges $[Q_{1,j}, Q_{2,j+1}]$ and $[Q_{2,j+1}, Q_{1,j+2}]$ are necessarily $G^{\mathcal{R}}$ edges, belonging to the same $\rho \in \mathcal{R}$. Then π and ρ cross twice, violating (C3). $\quad \square$

Proof of (VS6). Call a path $\gamma \in \mathcal{P} \cup \mathcal{R}$ a *top* (*bottom*) path if it contains a top edge $[Q_{1,j}, Q_{1,j+1}]$ (bottom edge $[Q_{r,j}, Q_{r,j+1}]$). The pseudoline structure implies that these paths are all distinct, so $\mathcal{P} \cup \mathcal{R}$ consists of the two diagonals and $r - 1$ top and $r - 1$ bottom paths. Again Proposition 2.7 implies that if $[Q_{1,j}, Q_{1,j+1}] \subset \gamma$ and the first point of γ is $Q_{a,1}$, the last is $Q_{b,r}$ then $a \leq j$ and $b \leq r - j$ so $a + b \leq r$. This and Proposition 2.8 give that $b = r - a$ and γ must have the shape as described in (VS6). By symmetry, we have the same for the bottom paths, property (VS7). $\quad \square$

Properties (VS6–7) imply that all diagonal edges $[Q_{i,j}, Q_{i\pm 1,j+1}]$ are in $E(G)$, this is property (VS2). Property (VS4) follows from (VS2) and from the fact that a $\mathcal{P}$ and an $\mathcal{R}$ edge can meet only at the vertices of Π. This completes the proof of Lemma 2.5. $\quad \square$

2.4. Grids of Euclidean lines

Here we apply the results of the previous subsections when the sets of crossing polygons $\mathcal{P}$ and $\mathcal{R}$ are actually straight lines. Let $\mathbf{y}, \mathbf{m}, \mathbf{m}' \in \mathbf{R}^r$, $\mathbf{y} = (y_1, \ldots, y_r)$ with $y_1 > y_2 > \cdots > y_r$, $\pi_i := [(1, y_i), (r, y_i + (r - 1)m_i)]$, $\rho_i := [(1, y_i), (r, y_i + (r - 1)m_i')]$ be $2r$ segments on the plane such that $\Pi := \{\ell_i \cap \pi_j : i, j \in [r]\} = \{\ell_i \cap \rho_j : i, j \in [r]\}$, $|\Pi| = r^2$.

Lemma 2.9. *If $\mathcal{P}$ and $\mathcal{R}$ form a crossing pair of straight lines, then $r \leq 3$.*

Proof. $\mathcal{P}$ and $\mathcal{R}$ satisfy (C1)–(C4) so Lemma 2.5 can be applied. Consider the first four lines $\pi_1, \ldots, \pi_4 \in \mathcal{P}$ and also $\rho_1, \ldots, \rho_4 \in \mathcal{R}$. According to Corollary 2.6 these lines meet at $Q_{1,1}, Q_{2,1}, Q_{3,1},$ and $Q_{4,1}$ and at 12 further points of Π (see (19)). These 12 points yield 12 equations for $y_1, \ldots, y_4, m_1, \ldots, m_4, m_1', \ldots, m_4'$, e.g., considering $Q_{1,2}$ we get the equation $y_1 + m_1 = y_2 + m_2'$. So the points $Q_{1,2}, Q_{1,3}, Q_{1,4}, Q_{2,2}, Q_{2,3}, Q_{3,2},$ and $Q_{r-2,r}, Q_{r-1,r-1},$ $Q_{r-1,r}, Q_{r,r-2}, Q_{r,r-1},$ and $Q_{r,r}$ give the following 12 equations.

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & -3 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & r-1 & 0 & -r+1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & r-2 & -r+2 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & r-1 & 0 & 0 & 0 & 0 & 0 & -r+1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & r-3 & 0 & 0 & -r+3 & 0 \\ 0 & 1 & -1 & 0 & 0 & r-2 & 0 & 0 & 0 & 0 & -r+2 & 0 \\ -1 & 1 & 0 & 0 & 0 & r-1 & 0 & 0 & -r+1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_1' \\ m_2' \\ m_3' \\ m_4' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (20)$$

The solution set of this homogeneous linear equation system has dimension at least two because every vector $(\mathbf{y}, \mathbf{m}, \mathbf{m}')$ generated by $(1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0)$ and $(0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1)$ is a solution. We claim that these are all the solutions. This implies $y_1 = y_2 = y_3 = y_4$ and $m_1 = \cdots = m_4'$, contradicting the fact that the lines $\pi_1, \ldots, \pi_4$ are distinct.

Let $M(r)$ be the 12×12 coefficient matrix of (20). Let us denote the characteristic polynomial of $M(r)$ by $f(r, \lambda)$. We obtain

$$\begin{aligned} f(r, \lambda) &:= \det(M(r) - \lambda I) \\ &= \lambda^{12} - 4\lambda^{11} + \lambda^{10}(-r^2 + 5r) + \lambda^9(2r^2 - 19r + 21) + \lambda^8(-6r^2 + 40r - 68) \\ &\quad + \lambda^7(r^4 - 3r^3 + 15r^2 - 61r + 113) + \lambda^6(2r^4 - 30r^3 + 107r^2 - 99r - 94) \\ &\quad + \lambda^5(2r^5 - 26r^4 + 120r^3 - 241r^2 + 123r + 132) \\ &\quad + \lambda^4(r^5 - 21r^4 + 103r^3 - 210r^2 + 216r - 152) \\ &\quad + \lambda^3(r^5 - 19r^4 + 143r^3 - 418r^2 + 412r) + \lambda^2(6r^4 - 38r^3 + 60r^2). \end{aligned}$$

Here the coefficient of λ^2 is not 0 for $r \geq 4$ (since $6r^4 - 38r^3 + 60r^2 = 2r^2(r-3)(3r-10)$). Thus the rank of $M(r)$ is 10 and the solution set of (20) has dimension 2, as stated. The calculations have been verified by both the *Maple* and the *Mathematica* programs. $\quad \square$

2.5. 3×3 Euclidean grids

When $r = 3$ there are crossing families of straight lines. For example, for any y, m, a, b (with $a, b > 0$) we have such a system with values

$$\mathbf{y} = (y + 4a + 2b, y - 2a + 2b, y - 2a - 4b),$$
$$\mathbf{m} = (m - 3a, m - 3b, m + 3a + 3b), \qquad \mathbf{m}' = (m - 3a - 3b, m + 3a, m + 3b). \quad (21)$$

The corresponding crossing systems are

$$\pi_1 = \{y + 4a + 2b, y + m + a + 2b, y + 2m - 2a + 2b\}$$

$$\pi_2 = \{y - 2a + 2b, \ y + m - 2a - b, \ y + 2m - 2a - 4b\}$$
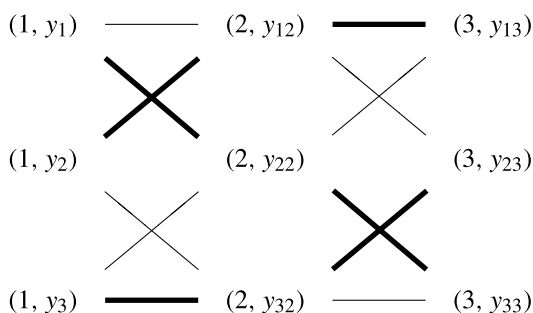$$\pi_3 = \{y - 2a - 4b, \ y + m + a - b, \ y + 2m + 4a + 2b\}$$

and

$$\rho_1 = \{y + 4a + 2b, \ y + m + a - b, \ y + 2m - 2a - 4b\}$$
$$\rho_2 = \{y - 2a + 2b, \ y + m + a + 2b, \ y + 2m + 4a + 2b\}$$
$$\rho_3 = \{y - 2a - 4b, \ y + m - 2a - b, \ y + 2m - 2a + 2b\}.$$

Lemma 2.10. *If the set of slopes $M := \{m_1, m_2, m_3\} \cup \{m'_1, m'_2, m'_3\}$ is A_4-free and A_6-free, then $\mathcal{P}$ and $\mathcal{R}$ cannot be a set of 3×3 crossing Euclidean lines.*

Proof. From Lemma 2.5 we know that a grid for $r = 3$ has the following intersection pattern:

Considering the intersection points $Q_{1,2}$, $Q_{1,3}$, $Q_{2,2}$, $Q_{2,3}$, $Q_{3,2}$, and $Q_{3,3}$, we derive the following system of linear equations.

$$\begin{aligned}
y_1 + m_1 &= y_2 + m'_2 \\
y_3 + 2m_3 &= y_2 + 2m'_2 \\
y_3 + m_3 &= y_1 + m'_1 \\
y_1 + 2m_1 &= y_3 + 2m'_3 \\
y_2 + m_2 &= y_3 + m'_3 \\
y_2 + 2m_2 &= y_1 + 2m'_1.
\end{aligned} \tag{22}$$

It is easy to see that all solutions of (22) are of the form of (20). □

3. Constructions

3.1. Grid-free systems mod q

In this section we prove our main results, Theorems 1.2 and 1.6. Given integers $q \geq r \geq 2$, $M \subset \{0, 1, \dots, q - 1\}$ define the hypergraph $\mathcal{F}_M$ as follows. Define the vertex set on the Euclidean plane

$$V := \{(j, y) : \ 1 \leq j \leq r, \ y \in Z_q\}.$$

Thus $|V| = rq$, and let $V_j = \{(j, y) : \ y \in Z_q\}$. For integers $0 \leq y, m < q$ define the r-set

$$A(y, m) = \{(1, y), (2, y + m), \dots, (r, y + (r - 1)m)\},$$

where the second coordinates are taken modulo q. For $|m| < q/(4r)$ and $q/4 < y < 3q/4$ or for $0 \leq y \leq y + (r-1)m < q$ the points of $A(y, m)$ are collinear. For a subset of *slopes* $M \subseteq \{0, \ldots, q-1\}$ let

$$\mathcal{F}_M := \mathcal{F}^r_{M,q} = \{A(y, m) : y \in Z_q, \ m \in M\}. \tag{23}$$

Obviously, this hypergraph is r-uniform, $|M|$-regular. Even more, it can be decomposed into $|M|$ perfect matchings.

Lemma 3.1. *Suppose that $q \geq r \geq 4$, and for all slopes $m \in M$ we have $|m| < q/(4r)$ (taken modulo q). Then $\mathcal{F}_M$ is $\mathbb{G}_{r \times r}$-free.*

Proof. Suppose for a contradiction that we have a grid $\mathcal{P} = \{A(y_i, m_i)\}^r_{i=1}$, $\mathcal{R} = \{A(y'_i, m'_i)\}^r_{i=1} \subset \mathcal{F}$. The hypergraph $\mathcal{F}_M$ is shift invariant, so we may assume that $\exists \pi_1 \in \mathcal{P}$ and $\exists \rho_1 \in \mathcal{R}$ which start in $(1, \lfloor q/2 \rfloor)$. In other words, replace each $A(y, m)$ by $A(y + \lfloor q/2 \rfloor - y_1, m)$ (the second coordinates are always taken modulo q). The shifted system of hyperedges have the same intersection structure as $\mathcal{P}$ and $\mathcal{R}$. So from now on, we may suppose that $y_1 = \lfloor q/2 \rfloor$. Since the slopes m_1 and m'_1 are small (i.e., $|m_1|, |m'_1| < q/(4r)$) the points of $\pi_1 := A(\lfloor q/2 \rfloor, m_1)$ and $\rho_1 := A(\lfloor q/2 \rfloor, m'_1)$ form Euclidean lines. All other sets of $\mathcal{P}$ and $\mathcal{R}$ meet either π_1 or ρ_1, and all other slopes are small (at most $q/(4r)$) so all other members of $\mathcal{P} \cup \mathcal{R}$ form Euclidean lines, too.

Finally, Lemma 2.9 completes the proof that r should be at most 3. $\quad\square$

Lemma 3.2. *If q is a prime (and $q \geq r$), then $\mathcal{F}_M$ is a linear hypergraph.*

Proof. Well-known and easy. $|A(y, m) \cap A(y', m')| \geq 2$ implies that there exist $1 \leq i \neq j \leq r$

$$y + im \equiv y' + im' \pmod{q}$$
$$y + jm \equiv y' + jm' \pmod{q},$$

implying $(j - i)(m' - m) \equiv 0 \pmod{q}$, a contradiction. $\quad\square$

Lemma 3.3. *If q is a prime, $q > r^{4r}$, $r \geq 4$, then the whole $\mathcal{F}_M$ with $M = Z_q$ is $\mathbb{G}_{r \times r}$-free.*

Proof. Suppose that the families $\mathcal{P} = \{A(y_i, m_i)\}^r_{i=1}$ and $\mathcal{R} = \{A(y'_i, m'_i)\}^r_{i=1} \subset \mathcal{F}$ form a grid $\mathbb{G}_{r \times r}$. Note that $r^{4r} \geq (4r)^{2r}$ so we can apply Minkowski's theorem of simultaneous approximation (Lemma 2.1) for the vector $\mathbf{n} := (m_1, \ldots, m_r, m'_1, \ldots, m'_r) \in \mathbf{R}^{2r}$. There exists an $0 < \alpha < q$ such that for all $i \in [r]$ we have $|\alpha m_i|$ and $|\alpha m'_i| \leq q^{1-1/2r} < q/(4r) \pmod{q}$. The collections $A(\alpha y_i, \alpha m_i)$ and $A(\alpha y'_i, \alpha m'_i)$, $i = 1, \ldots, r$ have the same intersection pattern, i.e., they form a grid, too. Then Lemma 3.1 implies that $r \leq 3$. $\quad\square$

3.2. Grid-free systems for every n, the proof of Theorem 1.2

We use induction on n to show that $\frac{n(n-1)}{r(r-1)} - c_r n^{8/5} < \text{ex}(n, \{\mathbb{I}_{\geq 2}, \mathbb{G}_{r \times r}\})$ holds for $r \geq 4$ with appropriate $c_r > 0$. Let q be the largest prime $q \leq n/r$. It is well-known [44] that $q > n/r - Cn^{3/5}$, where C is an absolute constant. Let $V_1, \ldots, V_r$ be disjoint q-sets, and let $\mathcal{F}_M$ be the grid-free hypergraph of size q^2 on the vertex set $V_1 \cup \cdots \cup V_r$ given by Lemma 3.3. Consider a grid-free, linear hypergraph $\mathcal{H}$ on q vertices. By induction hypothesis, there is such a hypergraph of size $|\mathcal{H}| > \frac{q(q-1)}{r(r-1)} - c_r q^{8/5}$. Put a copy of $\mathcal{H}$, $\mathcal{H}_j$, into each V_j after randomly permuting their vertices, $(1 \leq j \leq r)$. The union of these hypergraphs, $\mathcal{F} := \mathcal{F}_M \cup \mathcal{H}_1 \cup \ldots, \cup \mathcal{H}_r$, is obviously linear.

Suppose that $\mathcal{P} = \{A_1, \ldots, A_r\}$ and $\mathcal{R} = \{B_1, \ldots, B_r\}$ form a grid in $\mathcal{F}$. Since $\mathcal{F}_M$ is grid-free there must be an edge, say $A_j \in \mathcal{P}$ such that $A_j \in \mathcal{H}_j$. The vertices of A_j are covered by the edges of $\mathcal{R}$, each meeting A_j in a distinct singleton, so $\mathcal{R} \subset \mathcal{F}_M \cup \mathcal{H}_j$. If all $B_i \in \mathcal{F}_M$ then, since $\cup \mathcal{P} = \cup \mathcal{R}$, we obtain that $A_i := (\cup \mathcal{R}) \cap V_i$ belongs to $\mathcal{H}_i$ for each $i \in [r]$. Call such a grid of *type 0* and their number is denoted by $g_0 := g_0(\mathcal{F})$.

Otherwise, there is an edge $B_j \in \mathcal{R} \cap \mathcal{H}_j$. We claim that this edge B_j is unique. If B_j and $B_j' \in \mathcal{R} \cap \mathcal{H}_j$, then all A_i meet V_j in at least two vertices, hence all $A_i \in \mathcal{H}_j$, hence $\mathcal{P}$ and then $\mathcal{R}$ are subfamilies of $\mathcal{H}_j$, but $\mathcal{H}_j$ is grid-free. It follows that $\mathcal{P} \setminus \{A_j\}$ and $\mathcal{R} \setminus \{B_j\}$ are parts of $\mathcal{F}_M$ and they form an $(r-1) \times (r-1)$ grid (on $V \setminus V_j$). Call this $\mathcal{P} \cup \mathcal{R}$ a grid of *type j* and denote the number by type j grids by $g_j := g_j(\mathcal{F})$.

We obtain a grid-free family of size at least $|\mathcal{F}| - g_0 - g_1 - \cdots - g_r$ if we leave out an edge from each grid in $\mathcal{F}$. Next we estimate the expected size of $g_j(\mathcal{F})$ when the vertices of each $\mathcal{H}_i$ are permuted randomly and independently.

Concerning type 0, there are at most $\binom{|\mathcal{F}|}{r}$ choices of the vertex disjoint $B_1, \ldots, B_r \in \mathcal{F}_M$. Given $B_1, \ldots, B_r$ and j the probability that $A_j := V_j \cap (\cup B_i)$ indeed belongs to $\mathcal{H}_j$ is exactly $|\mathcal{H}_j| / \binom{q}{r}$, which is at most $1 / \binom{q-2}{r-2}$. These events are independent, and the number of ways to choose $B_1, \ldots, B_r$ is at most $\binom{q^2}{r}$, so the expected number of type 0 grids

$$\mathbb{E}(g_0) \leq \binom{q^2}{r} \binom{q-2}{r-2}^{-r} = O(q^{4r-r^2}).$$

Given $\mathcal{F}_M$ one can count type 1 grids as follows (the cases $j > 1$ are similar). Choose the edges $A_2, A_3 \in \mathcal{F}_M$. Each of $B_2, \ldots, B_r$ intersect both of them. For $i = 2, \ldots, r$ let e_i be the pair $\{B_i \cap A_2, B_i \cap A_3\}$. Clearly, the edges e_i form a matching between the vertices $A_2 \setminus V_1$ and $A_3 \setminus V_1$ and they determine $B_2, \ldots, B_r$. Similarly, a matching between $B_2 \setminus (V_1 \cup A_2 \cup A_3)$ and $B_3 \setminus (V_1 \cup A_2 \cup A_3)$ determines $A_4, \ldots, A_r$. Finally, choosing a vertex $c \in V_1$ the number of possible choices so far is

$$\binom{|\mathcal{F}_M|}{2} (r-1)!(r-3)! \times q = O(q^5).$$

The vertices $A_i \cap V_1$ and $B_i \cap V_1$ are called a_i and b_i, resp., $2 \leq i \leq r$. The probability that $\{c, a_2, \ldots, a_r\}$ and $\{c, b_2, \ldots, b_r\}$ both belong to $\mathcal{H}_1$ is at most

$$\frac{|\mathcal{H}_1|}{\binom{q}{r}} \times \frac{(q-r)/(r-1)}{\binom{q-r}{r-1}} = O(q^{4-2r}),$$

yielding $\mathbb{E}(g_1) = O(q^{9-2r})$. Thus, for $r \geq 4$ the expected number of grids in $\mathcal{F}$ is

$$\mathbb{E}(g_0(\mathcal{F}) + \cdots + g_r(\mathcal{F})) = O(q).$$

So there is a choice of permutations to make $\mathcal{F}$ grid-free deleting only $O(q)$ edges. This gives

$$\mathrm{ex}(n, \{\mathbb{I}_{\geq 2}, \mathbb{G}_{r \times r}\}) \geq |\mathcal{F}| - O(q) \geq q^2 + r \frac{q(q-1)}{r(r-1)} - c_r q^{8/5} - O(q).$$

A short calculation shows that the right hand side is at least $\frac{n(n-1)}{r(r-1)} - c_r n^{8/5}$ with some c_r. $\square$

3.3. Triangle-free systems, the proof of Theorem 1.6

Since the Turán function is monotone we have to consider only the case when r divides n, $n = qr$. Let $M \subset \{0, 1, \ldots, \lfloor q/(4r) \rfloor\}$ be an r-sum-free set of size $|M| > qe^{-\gamma_r \sqrt{\log q}}/(4r)$ provided by Lemma 2.2 and let $\mathcal{F}_M$ be the family defined by (23) in Section 3.1.

By Lemma 3.1 for $r \geq 4$ $\mathcal{F}_M$ is a linear hypergraph containing no grid. Since the set of slopes M is an r-sum-free set we cannot have three lines with slopes $m_1 < m_3 < m_2$ forming a triangle, either. Otherwise, we get $c_1 m_1 + c_2 m_2 = (c_1 + c_2) m_3 \pmod{q}$ for some $c_1 + c_2 \leq r - 1$. Finally, $|\mathcal{F}_M| = q|M| \geq n^2 e^{-\beta_r \sqrt{\log n}}$ for some β_r, as stated.   □

Proof of Theorem 1.6 for $r = 3$. Here we establish the lower bound (6) for $\mathrm{ex}(n, \{\mathbb{I}_{\geq 2}^3, \mathbb{T}_3, \mathbb{G}_{3 \times 3}\})$ when $n = 3q$, and q is a prime.

First, recall that for $r = 3$, $\mathcal{F}_M$ may contain a grid, see (20). Therefore, to avoid grids and triangles at the same time we need to choose the slopes in M more restrictively. For example, M could be a set in $\{0, 1, \ldots, \lfloor q/12 \rfloor\}$ which is A_6-free, A_4-free and AP_3-free simultaneously. Then Lemma 2.10 gives that $\mathcal{F}_M$ is linear and grid-free. Also the AP_3-free property implies that $\mathcal{F}_M$ has no triangles either. Lemma 2.3 implies

$$\frac{2}{5} r_3(q/12)^{3/5} \times q \leq |M|q = |\mathcal{F}_M| \leq \mathrm{ex}(n, \{\mathbb{I}_{\geq 2}^3, \mathbb{T}_3, \mathbb{G}_{3 \times 3}\})$$

and then (17) completes the proof of the lower bound (6).   □

Since $\mathcal{F}_M$ contains neither grids nor triangles it is 3-union free, too, by (12). This implies $n^{8/5 - o(1)} < U_3(n, 3)$. Since $\mathcal{F}_M$ is regular, (and can be split into matchings), it is an optimal 3-superimposed design. The bound $|\mathcal{F}_M|/n = \Omega(n^{3/5 - o(1)})$ exceeds the bound (15) for $k'(3, n)$ by D'yachkov and Rykov [18] for $r = 3$. Below we further improve both lower bounds with a different construction.

3.4. Union-free triple systems

Define the hypergraphs $\mathbb{G}_6$ and $\mathbb{G}_7$ as follows on 6 and 7 vertices.

$E(\mathbb{G}_6) := \{123, 156, 426, 453\}$,
$E(\mathbb{G}_7) := \{123, 456, 726, 753\}$.

Note that both are three-partite and the 3-partition of their vertices is unique.

Lemma 3.4. *Suppose that $\mathcal{F}$ is a three-partite, linear hypergraph. It is 3-union-free if and only if it avoids $\mathbb{G}_{3 \times 3}$, $\mathbb{G}_6$, and $\mathbb{G}_7$.*

Proof. We start like in the proof of Corollary 1.9. Suppose, that $\mathcal{A} \neq \mathcal{B}$, $|\mathcal{B}| \leq |\mathcal{A}| \leq 3$, $\cup_{A \in \mathcal{A}} A = \cup_{B \in \mathcal{B}} B$ and $\mathbb{G} := \mathcal{A} \cup \mathcal{B}$ form a linear 3-uniform hypergraph. Then $\exists A_1 \in \mathcal{A} \setminus \mathcal{B}$. Since $|A_1 \cap B| \leq 1$, to cover the elements of A_1 the family $\mathcal{B}$ must contain 3 sets. We obtain $|\mathcal{B}| = |\mathcal{A}| = 3$. Moreover, the sets $B_1, B_2, B_3 \in \mathcal{B}$ meet A_1 in distinct elements.

In the case of $\mathcal{A} \cap \mathcal{B} = \emptyset$ the latest property implies that every $a \in \cup\mathcal{A}$ is covered by a unique $B \in \mathcal{B}$, and every $b \in \cup\mathcal{B}$ is covered by a unique $A \in \mathcal{A}$, so $\mathbb{G}$ is 2-regular, on 9 vertices, we obtain the grid $\mathbb{G}_{3 \times 3}$.

In the case of $|\mathcal{A} \cap \mathcal{B}| = 2$, say $A_2 = B_2$ and $A_3 = B_3$ we have that $A_1 \setminus (B_2 \cup B_3)$ is a singleton and it must be the same element as $B_1 \setminus (A_2 \cup A_3)$. Taking into the account that $\mathbb{G}$ is

3-partite we obtain that it is isomorphic to $\mathbb{G}_7$ when A_2 and A_3 are disjoint, and it is isomorphic to $\mathbb{G}_6$ when A_2 and A_3 meet.

Finally, in the case $|\mathcal{A} \cap \mathcal{B}| = 1$, say $A_3 = B_3$, the other four sets meet A_3 in exactly one element, so there is a vertex v of A_3 of degree at least 3. If $v \in A_1 \cap A_2 \cap A_3$ then B_1 and B_2 covers $A_1 \cup A_2 \setminus v$ and we could not finish because $\mathbb{G}$ is 3-partite. Similarly, if $v \in A_2 \cap B_2 \cap A_3$ then $A_2 \setminus v$ must be covered by B_1, a contradiction. So if the three configurations are avoided then $\mathcal{F}$ is 3-union-free. □

Proof of Proposition 1.10. The probabilistic lower bound from Lemma 1.7 and the previous Lemma imply the lower bound (13)

$$\Omega(n^{5/3}) = \text{ex}(n, \{\mathbb{G}_{3\times3}, \mathbb{G}_6, \mathbb{G}_7\}) \leq U_3(n, 3). \quad \Box$$

Proof of Proposition 1.16. Here we prove (16) claiming $\Omega(n^{2/3}) = k'(3, 3n)$. The proof is a refined version of the previous proof.

Let $\mathcal{F}$ be the three-partite complete hypergraph with parts V_1, V_2 and V_3 where $V_i := \{(i, j) : j \in Z_n\}$ as before. Split it into n^2 perfect matchings

$$M(\alpha, \beta) := \{\{(1, y), (2, y + \alpha), (3, y + \beta)\} : y \in Z_n\}$$

where the second coordinates are taken modulo n. We also call these *parallel classes*.

Choose independently each of the n^2 matchings with probability p, p will be defined as $n^{-4/3}/2$. Call the obtained random hypergraph $\mathcal{H}$. Count the expected number of the arising configurations $\mathbb{I}_{\geq2}^3$, $\mathbb{G}_6$, $\mathbb{G}_7$, and $\mathbb{G}_{3\times3}$, those we want to avoid. Here we have to be more careful, because although each edge belongs to $\mathcal{H}$ with probability p (so its expected size is pn^3) the choices of edges are not independent. More precisely, the probability that a subhypergraph $\mathcal{A}$ appears in $\mathcal{H}$ is exactly p^i, where i is the number of different parallel classes in $\mathcal{A}$.

Intersecting triples always belong to different classes, so they are independent, so the expected number of $\mathbb{I}_{\geq2}$'s is $p^2 \times 3n^2\binom{n}{2}$ and the expected number of $\mathbb{G}_6$'s is $p^4 \times 2\binom{n}{2}^3$.

The expected number of grids in $\mathcal{H}$ with independent hyperedges is $O(p^6 n^9)$ and the expected number of $\mathbb{G}_7$'s with independent edges is $O(p^4 n^7)$.

A configuration $\mathbb{G}_7$ might have only one pair of dependent triples, namely the disjoint pair 123, 246. The number of these configurations is at most $3n^5$ (first we chose the triple corresponding to 123 in n^3 ways, then its parallel edge at most n ways, and finally the 7th vertex at most $3n$ ways). So the expected number of these is at most $3p^3n^5$.

Consider, finally, the grids $\mathcal{A} \cup \mathcal{B}$ containing parallel triples. Note that the parallel classes of $\mathcal{A}$ and $\mathcal{B}$ are distinct, because every $A \in \mathcal{A}$ meets every $B \in \mathcal{B}$. The number of these configurations is $O(n^7)$ (like before, first we chose a triple in n^3 ways, then its parallel edge at most n ways, and finally the three remaining vertices at most n^3 ways). So if the number of independent classes $i \geq 4$, then the expected number of these grids is bounded by $O(p^4n^7)$.

If $i \leq 3$, then one of the three edges of the grid, say $\mathcal{A}$, consists of three parallel edges. The number of these configurations is at most n^5, so we have an upper bound $O(p^3n^5)$ for the expected number of these whenever $i = 3$.

Finally, if $i = 2$, then both $\mathcal{A}$ and $\mathcal{B}$ consists of parallel edges. The number of those systems is $O(n^3)$ so we have an upper bound $O(p^2n^3)$.

Altogether the total expected number of configurations we want to avoid is bounded by a constant multiple of

$$p^2n^4 + p^4n^6 + p^6n^9 + p^4n^7 + p^3n^5 + p^4n^7 + p^3n^5 + p^2n^3.$$

This is less than half of $E(|\mathcal{F}|) = pn^3$ with our choice of p. So we still have $\frac{1}{2}pn^3$ of the edges if we leave out an edge from each of the configurations we want to avoid, and thus we make the rest 3-union-free.

Moreover, conveniently, we can erase the unwanted edges together with all its parallels (if $\mathcal{A}$ is in $\mathcal{F}$ then all of its shifted copy belongs to $\mathcal{F}$), so there is a random choice of $\mathcal{F}$ where the remaining 3-union-free part is still large and union of matchings.   □

3.5. Union free and cover-free graphs

D'yachkov and Rykov [18] showed that there exists an optimal 1-superimposed code (i.e., a regular graph) with $\leq n^2/4$ edges. It is easy to see, that there are k-regular n-vertex graphs for all $k < n$ (for nk even). Indeed, if n is even, and $n > k$ then, e.g., by Baranyai's Theorem [5] the edge set of K_n can be decomposed into $n - 1$ perfect matchings. Take k of them. For odd n (and k even) K_n can be decomposed into $(n-1)/2$ 2-factors by a theorem of Tutte. Take $k/2$ of them.

D'yachkov and Rykov [18] showed that there exist optimal 2-superimposed designs (i.e., a 2-union-free, k-regular graphs) for $k \leq \log(n + 2) - 2$. This can be improved to

$$k'(2, n) = \Theta(n^{1/2}). \tag{24}$$

Indeed, such an optimal design is just a k-regular, triangle and C_4-free graph. Such a graph can be constructed using (23) by taking a mod q Sidon set $M \subset Z_q$. The defined $\mathcal{F}_M$ is the desired bipartite graph. We have an obvious upper bound $(n/2)k'(2, n) \leq \mathrm{ex}(n, \{C_3, C_4\})$. Since

$$\mathrm{ex}(n, \{C_3, C_4\}) \leq \mathrm{ex}(n, C_4) = (1 + o(1))\frac{1}{2}n^{3/2}$$

by [9,25] we have an upper bound $k'(2, n) \leq (1 + o(1))\sqrt{n}$ giving the right order of magnitude. To determine the coefficient of the $\sqrt{n}$ seems to be a very difficult question. Erdős and Simonovits [26] conjectured that

$$(1 + o(1))\frac{n^{3/2}}{2\sqrt{2}} = \mathrm{ex}(n; \{C_4, C_3\}). \ (?) \tag{25}$$

They showed $\mathrm{ex}(n; \{C_4, C_5\}) \sim n^{3/2}/(2\sqrt{2})$, i.e., forbidding C_5 in magnitude is the same as forbidding all non-bipartite graphs.

4. Conclusion

Our main result is that the widely investigated transversal design $\mathcal{F}_M$ (see (23)) with $M = Z_q$ is $\mathbb{G}_{r \times r}$-free for $r \geq 4$. If M is r-sum-free then in addition $\mathcal{F}_M$ has no triangles. It is natural to ask what other small substructures can be avoided this way.

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