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The structure of the typical graphs of given diameter

Zoltán Füredi [a], Younjin Kim [b,*]

[a] Rényi Institute of the Hungarian Academy, Budapest, P.O.Box 127, H-1364, Hungary

[b] Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

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ABSTRACT

In this paper it is proved that there are constants $0 < c_2 < c_1$ such that the number of (labeled) n-vertex graphs of diameter d is

$$(1+o(1))\frac{d-2}{2}n_{(d-1)}3^{n-d+1}2^{\binom{n-d+1}{2}}$$

whenever $n \to \infty$ and $3 \le d \le n - c_1 \log n$, where $n_{(d-1)} = n(n-1)\dots(n-d+2)$. A typical graph of diameter d consists of a combination of an induced path of length d and a highly connected block of size $n-d+3$. In the case $d > n - c_2 \log n$ the typical graph has a completely different snakelike structure. The number of n-vertex graphs of diameter d is $(1+o(1))\frac{1}{2}n_{(d+1)}3^{n-d-1}d^{n-d-1}$ whenever $n \to \infty$ and $d > n - c_2 \log n$.

1. Introduction, notations

Let $\mathcal{G}(n, \text{diam} = d)$ be the class of graphs of diameter d on n labeled vertices. We usually identify the vertex sets with the set of the first n integers, $[n] = \{1, 2, \dots, n\}$. It is well known [1] that almost all graphs have diameter 2, $|\mathcal{G}(n, \text{diam} = 2)| = (1-o(1))2^{\binom{n}{2}}$. Tomescu [4] proved that $|\mathcal{G}(n, \text{diam} = d)| = 2^{\binom{n}{2}}(6 \cdot 2^{-d} + o(1))^n$ for any fixed $d \ge 3$ as $n \to \infty$. Our aim is to give an exact asymptotic and to extend his result for almost all d and n.

For a graph G and vertex v we use the notation $N(v)$ (or $N_G(v)$) for the neighborhood of v. For positive integers n and k we use $n_{(k)}$ for the k-term product $n(n-1)\dots(n-k+1)$. $\exp_2[x]$ stands for 2^x and $\binom{n}{a,b,\dots,z}$ is the multinomial coefficient $n!/(a!b!\dots z!)$.

2. Two classes of diameter d graphs

Let $S \cup \{a, b\}$ be an $s+2$-element set, $|S| = s > 1$. Define $\mathcal{H}(S, a, b)$ as the class of graphs, G, with underlying set $S \cup \{a, b\}$ such that the distance between every pair of vertices is at most 2 except for a and b, their distance is 3. We have

$$2^{\binom{s}{2}}3^s\left(1 - c_3 0.9^s\right) < |\mathcal{H}(S, a, b)| < 3^s 2^{\binom{s}{2}}, \tag{1}$$

where $c_3 > 0$ is an absolute constant, independent of s. Indeed, the neighborhoods of a and b are disjoint, there are at most 3^s possibilities for $(N(a), N(b))$. This gives the upper bound. On the other hand, we can get the lower bound by counting the number of graphs on $S \cup \{a, b\}$ with the property that $N(a) \cap N(b) = \emptyset$ and $N(x) \cap N(y) = \emptyset$ for some $(x, y) \ne (a, b)$; e.g., see Tomescu [3].

* Corresponding author.
E-mail addresses: furedi.zoltan@renyi.mta.hu, z-furedi@illinois.edu (Z. Füredi), ykim36@illinois.edu (Y. Kim).

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Example 1 (*A Block Plus a Path*)**.** Suppose $3 \le d < n$. Let $\mathcal{H}_1(n, d)$ be a class of graphs of diameter d with vertex sets $V := [n]$ obtained as follows. Split V into three disjoint non-empty parts A, S, B with $|A| = i$, $|S| = n - d + 1$, $|B| = d - 1 - i (1 \le i \le d - 2)$. Put a path $(v_0, v_1, \dots, v_{i-1})$ to A, a path $(v_{i+2}, \dots, v_{d-1}, v_d)$ to B and a copy of $\mathcal{H}(S, v_{i-1}, v_{i+2})$.

As the reversed sequences $A' := \{v_d, v_{d-1}, \dots, v_{i+2}\}$, $B' := \{v_{i-1}, \dots, v_0\}$ yield the very same graphs, we have that the number of graphs in the above class is

$$h_1(n, d)\left(1 - c_3 0.9^{n-d}\right) \le |\mathcal{H}_1(n, d)| \le \frac{d-2}{2} n_{(d-1)} 3^{n-d+1} 2^{\binom{n-d+1}{2}} =: h_1(n, d). \tag{2}$$

Example 2 (*Snake-like Graphs*)**.** Suppose $\frac{2}{3}n < d < n$. Let $(V_0, V_1, \dots, V_d)$ be a partition of $[n]$ into 1 and 2 element parts such that $|V_0| = |V_1| = |V_2| = |V_{d-2}| = |V_{d-1}| = |V_d| = 1$ and there are no two consecutive 2-element sets (i.e., $|V_i| = 2$ implies $|V_{i+1}| = 1$). Let us connect each vertex of V_i to at least one vertex of V_{i-1}, and add edges inside the V_i's arbitrarily. The class of graphs obtained this way is denoted by $\mathcal{H}_2(n, d)$. Every $G \in \mathcal{H}_2(n, d)$ is of diameter d, and the only pair of vertices of distance d is $\{V_0, V_d\}$. Let N_i be the set of vertices of G of distance i from V_d. We have $N_d = V_0$. If the sequence $N_0, N_1, \dots, N_d$ also satisfies $|N_0| = |N_1| = |N_2| = 1$, $|N_{d-2}| = |N_{d-1}| = |N_d| = 1$, and $|N_i| \le 2$ then G appears twice in $\mathcal{H}_2(n, d)$. Denote the class of these graphs by $\mathcal{H}_2^2(n, d)$, and let $\mathcal{H}_2^1(n, d) = \mathcal{H}_2(n, d) \setminus \mathcal{H}_2^2(n, d)$.

Every partition gives $2^{n-d-1}3^{n-d-1}$ graphs, and the number of partitions is

$$\binom{n}{2}\binom{n-2}{2}\cdots\binom{n-2(n-d-2)}{2} \times [n - 2(n-d-1)]! \times \binom{d-5-(n-d-1)+1}{n-d-1}.$$

So this procedure produces $n_{(d+1)}(2d-3-n)_{(n-d-1)}3^{n-d-1}$ graphs and the members of $\mathcal{H}_2^2(n, d)$ are counted twice. Hence

$$2|\mathcal{H}_2^2(n, d)| + |\mathcal{H}_2^1(n, d)| = n_{(d+1)}(2d-3-n)_{(n-d-1)}3^{n-d-1}.$$

We have $|N_1| = |N_2| = |N_3| = 1$ and $|N_d| = 1$. One can see that $\max\{|N_{d-1}|, |N_{d-2}|\} > 1$ is possible only if $\max\{|V_{d-3}|, |V_{d-4}|\} = 2$. Similarly, $|N_i| \ge 3$ implies that $|V_{d-i}| = |V_{d-i+2}| = 2$. The number of such partitions $(V_0, V_1, \dots, V_d)$ is at most

$$\frac{n!}{2^{n-d-1}} \times \left(2\binom{d-7-(n-d-2)+1}{n-d-2} + (n-d-2)\binom{d-7-(n-d-2)+1}{n-d-2}\right).$$

The sum in the parentheses is at most

$$(n-d)\binom{d-6-(n-d-1)+1}{n-d-2} = (n-d) \times \frac{(n-d-1)}{d-5-(n-d-1)+1}\binom{d-5-(n-d-1)+1}{n-d-1}.$$

We obtain

$$2|\mathcal{H}_2(n, d)| \le n_{(d+1)}(2d-3-n)_{(n-d-1)}3^{n-d-1}\left(1 + \frac{(n-d)(n-d-1)}{2d-n-3}\right).$$

Since

$$d^{n-d-1}\left(1 - \frac{2(n-d+1)}{d}\right)^{n-d-1} < (2d-3-n)_{(n-d-1)}\left(1 + \frac{(n-d)(n-d-1)}{2d-n-3}\right) \le d^{n-d-1},$$

we get for some $c_4 > 0$

$$\left(1 - c_4\frac{(n-d-1)^2}{n}\right)h_2(n, d) < |\mathcal{H}_2(n, d)| < \frac{1}{2}n_{(d+1)}d^{n-d-1}3^{n-d-1} =: h_2(n, d). \tag{3}$$

The estimates (2) and (3) give the lower bounds for the next two theorems.

3. Results

Theorem 1. *There is a constant $c_1 > 0$ such that the following holds. If $3 \le d < n - c_1 \log n$ and $n \to \infty$ then almost all n-vertex graphs of diameter at least d belong to $\mathcal{H}_1(n, d)$, hence*

$$|\mathcal{G}(n, \text{diam} = d)| = (1 + o(1))\frac{d-2}{2}n_{(d-1)}3^{n-d+1}2^{\binom{n-d+1}{2}}.$$

Theorem 2. *There exists a constant $c_2 > 0$ such that for $n - c_2 \log n < d < n$, $n \to \infty$ almost all n-vertex graphs of diameter at least d belong to $\mathcal{H}_2(n, d)$, hence*

$$|\mathcal{G}(n, \text{diam} = d)| = (1 + o(1))\frac{1}{2}n_{(d+1)}d^{n-d-1}3^{n-d-1}.$$

Corollary 1. *For $2 \le d < n - c_1 \log n$ or $n > d > n - c_2 \log n$*

$$\lim_{n\to\infty} \frac{|\mathcal{G}(n, \text{diam} \ge d + 1)|}{|\mathcal{G}(n, \text{diam} = d)|} = 0. \tag{4}$$

Eq. (4) was proved by Tomescu [4] for every fixed $d \ge 2$ and by Grable [2] for all $2 \le d \ll \sqrt{n}/\log n$. The main ideas of our proofs are rather straightforward, but one needs very careful estimates and calculations.

4. Lemmas for the upper bound

Let V be an n-element set, $x_0 \in V$, and let $P := (N_0, N_1, \dots, N_d)$ be an ordered partition of V into $d + 1$ non-empty parts, $N_0 = \{x_0\}$, $n_i := |N_i|$. Let $\mathcal{G}(x_0, N_1, \dots, N_d)$ be the class of graphs G with vertex set V such that N_i is the i'th neighborhood of x_0, $N_i = \{y \in V : d_G(x_0, y) = i\}$. The number of graphs in each set of the partition is $2^{\binom{n_i}{2}}$ and the number of bipartite graphs between N_i and N_{i+1} with no isolated vertex in N_{i+1} is $(2^{n_i} - 1)^{n_{i+1}}$. We obtained

$$|\mathcal{G}(x_0, N_1, \dots, N_d)| = 2^{\sum_{i=1}^{d}\binom{n_i}{2}} \prod_{i=1}^{d-1}(2^{n_i} - 1)^{n_{i+1}}. \tag{5}$$

Taking all possible $(d + 1)$-partitions $(x_0, N_1, \dots, N_d)$ we count each graph from $\mathcal{G}(n, \text{diam} = d)$ at least twice. We have

$$2|\mathcal{G}(n, \text{diam} = d)| \le \sum_{\substack{n_1+n_2+\cdots+n_d=n-1\\ n_1,n_2,\dots,n_d\ge 1}} \binom{n}{1, n_1, n_2, \dots, n_d} 2^{\sum_{i=1}^{d}\binom{n_i}{2}} \prod_{i=1}^{d-1}(2^{n_i} - 1)^{n_{i+1}}. \tag{6}$$

In the rest of the proof we give sharp upper bounds for the right hand side of (6). We will use the following estimate.

$$\begin{aligned}
&\binom{n}{1, n_1, n_2, \dots, n_d} 2^{\sum_{i=1}^{d}\binom{n_i}{2}} \prod_{i=1}^{d-1}(2^{n_i} - 1)^{n_{i+1}} \\
&= n_{(d+1)} \binom{n - d - 1}{n_1 - 1, n_2 - 1, \dots, n_d - 1} 2^{\sum_{i=1}^{d}\binom{n_i}{2}} \prod_{i=1}^{d-1}\frac{1}{n_i}(2^{n_i} - 1)^{n_{i+1}} \\
&\le n_{(d+1)} \binom{n - d - 1}{n_1 - 1, n_2 - 1, \dots, n_d - 1} \times \exp_2\left[\sum_{1\le i\le d}\binom{n_i}{2} + \sum_{1\le i\le d-1}(n_i n_{i+1} - 1)\right].
\end{aligned} \tag{7}$$

Define

$$f(x_1, \dots, x_d) := \sum_{1\le i\le d}\frac{1}{2}x_i^2 + \sum_{1\le i\le d-1}x_i x_{i+1}.$$

Lemma 1. *Let $x_1, \dots, x_d \ge 0$ be real numbers, $\sum_i x_i = s$, $m := \max_{1<i<d}(x_{i-1} + x_i + x_{i+1})$. Then*

$$f(\mathbf{x}) \le \frac{1}{2}m^2 + \frac{1}{2}(s - m)^2, \tag{8}$$

and

$$f(\mathbf{x}) \le \frac{3}{4}ms. \tag{9}$$

Proof. Suppose that $m = x_{k-1} + x_k + x_{k+1}$, then $x_{k-2} \le x_{k+1}$ and $x_{k-1} \ge x_{k+2}$. We have

$$\begin{aligned}
f(\mathbf{x}) &\le \frac{1}{2}\left(\left(\sum x_i\right) - (x_{k-1} + x_k + x_{k+1})\right)^2 + \frac{1}{2}(x_{k-1} + x_k + x_{k+1})^2 + x_{k-2}x_{k-1} + x_{k+1}x_{k+2} - x_{k-1}x_{k+1} - x_{k-2}x_{k+2} \\
&= \frac{1}{2}(s - m)^2 + \frac{1}{2}m^2 + (x_{k-2} - x_{k+1})(x_{k-1} - x_{k+2}).
\end{aligned}$$

Here the last term is non-positive and we get (8).

To show (9) consider

$$\begin{aligned} 4f(\mathbf{x}) + \sum x_i^2 &= x_1^2 + (x_1+x_2)^2 + (x_1+x_2+x_3)^2 + \cdots + (x_{i-1}+x_i+x_{i+1})^2 + \cdots \\ &\quad + (x_{d-2}+x_{d-1}+x_d)^2 + (x_{d-1}+x_d)^2 + x_d^2 - 2\sum x_i x_{i+2} \\ &\le m\,(x_1 + (x_1+x_2) + \cdots + (x_{i-1}+x_i+x_{i+1}) + \cdots + (x_{d-1}+x_d) + x_d) \\ &= 3ms. \quad \square \end{aligned}$$

We will use this lemma to bound in the following form

$$\sum \binom{n_i}{2} + \sum (n_i n_{i+1} - 1) = f(x_1, \ldots, x_d) + \frac{5s}{2} - x_1 - x_d \tag{10}$$

where $x_i := n_i - 1$ $(1 \le i \le d)$, $s = \sum x_i = n - d - 1$.

5. Proof of the upper bound for Theorem 1

From now on, we suppose that $3 \le d < n - c \log n$, where c is a sufficiently large constant. We put the terms of the right hand side of (6) into four groups according to the relation of $s := n - d - 1$ and $m := \max_{1<i<d}(n_{i-1} + n_i + n_{i+1} - 3)$.

- Case 1: $m < 0.6s$,
- Case 2: $0.6s \le m < s - 1$,
- Case 3: $m = s - 1$.
  This means that for some $1 < i < d$ one has $n_{i-1} + n_i + n_{i+1} = s + 2$, there is an $n_t = 2$ $(t \ne i-1, i, i+1)$ and all other $n_j = 1$. We consider three subcases
  - Case 3.1: $t \ne i-2, i+2$,
  - Case 3.2: $t = i-2$, $n_{i+1} \ge 2$,
  - Case 3.3: $t = i+2$, $n_{i-1} \ge 3$,
- Case 4: $m = s$.
  We have $n_{i-1} + n_i + n_{i+1} = s + 3$, all other $n_j = 1$. Again we handle three subcases separately
  - Case 4.1: $n_{i-1} \ge 2$, $n_{i+1} \ge 2$,
  - Case 4.2: $n_0 = n_1 = \cdots n_{d-2} = 1$, $n_{d-1} + n_d = s + 2$,
  - Case 4.3: $n_{i-1} + n_i = s + 2$, $1 < i < d$, all other $n_j = 1$.

These exhaust all possibilities. We will show that the sum in each of the above groups is $o(h_1(n, d))$, except in Case 4.3. We denote by $\Sigma_1, \Sigma_2, \Sigma_{31}, \ldots$ the sum of the right hand side of (5) corresponding to the above cases.

Case 1. To get an upper bound we use (7), rearrange, and then apply (10) and finally (9). We have

$$\begin{aligned} \Sigma_1 &:= \sum_{\substack{n_1+n_2+\cdots+n_d=n-1 \\ n_1,n_2,\ldots,n_d \ge 1 \\ m<0.6s}} \binom{n}{1, n_1, n_2, \ldots, n_d} 2^{\sum_{i=1}^d \binom{n_i}{2}} \prod_{i=1}^{d-1} (2^{n_i} - 1)^{n_{i+1}} \\ &\le n_{(d+1)} \sum_{m<0.6s} \left(\binom{n-d-1}{n_1-1, n_2-1, \ldots, n_d-1} \times \exp_2\left[\sum \binom{n_i}{2} + \sum (n_i n_{i+1} - 1) \right] \right) \\ &\le n_{(d+1)} \left(\sum \binom{n-d-1}{n_1-1, n_2-1, \ldots, n_d-1} \right) \times \exp_2\left[\max_{m<0.6s} \left\{ \sum \binom{n_i}{2} + \sum (n_i n_{i+1} - 1) \right\} \right] \\ &\le n_{(d+1)} d^{n-d-1} \times \exp_2\left[\max_{m<0.6s} f(\mathbf{x}) + \frac{5s}{2} \right] \\ &= n_{(d+1)} d^{n-d-1} \exp_2[(3/4)(0.6s)s + 5s/2]. \end{aligned} \tag{11}$$

This implies

$$\log_2 \Sigma_1 \le \log(n_{(d+1)}) + s \log d + 0.45s^2 + 2.5s.$$

On the other hand (2) gives

$$\log_2 h_1(n, d) = -1 + \log(d-2) + \log(n_{(d-1)}) + (s+2)\log 3 + \binom{s+2}{2}. \tag{12}$$

A little algebra gives $\log h_1(n,d) - \log \Sigma_1 > s^2/20 - s\log d$ (for $n-d-1 > 100$) and this goes to infinity as $s \to \infty$ because $d < n - 41\log n$ implies $n-d-1 > 40\log n > 40\log d$. Thus in this range $\Sigma_1 = o(h_1(n,d))$.

Case 2. To get an upper bound we use (7) but rearrange more carefully. We have

$$\begin{aligned}
\Sigma_2 &:= \sum_{\substack{n_1+n_2+\cdots+n_d=n-1\\ n_1,n_2,\ldots,n_d\ge 1\\ 0.6s\le m\le s-2}} \binom{n}{1,n_1,n_2,\ldots,n_d} 2^{\sum_{i=1}^d \binom{n_i}{2}} \prod_{i=1}^{d-1}(2^{n_i}-1)^{n_{i+1}}\\
&\le n_{(d+1)} \sum_{0.6s\le m\le s-2}\left(\binom{n-d-1}{n_1-1,n_2-1,\ldots,n_d-1} \times \exp_2\left[\sum\binom{n_i}{2} + \sum(n_in_{i+1}-1)\right]\right)\\
&\le n_{(d+1)} \sum_{0.6s\le m\le s-2}\left(\left(\sum_{m\text{ is fixed}}\binom{n-d-1}{n_1-1,n_2-1,\ldots,n_d-1}\right)\right.\\
&\qquad \left.\times \exp_2\left[\max_{m\text{ is fixed}}\left\{\sum\binom{n_i}{2} + \sum(n_in_{i+1}-1)\right\}\right]\right). \qquad (13)
\end{aligned}$$

The total sum of all of the d-nomial coefficients of order s is d^s, the number of d-colorings of an s-element set. In the sum (13) we add up only those where $m = n_{i-1}-1+n_i-1+n_{i+1}-1$ for some $2\le i\le d-1$. Choose first an i, then m elements from the s-set, then color those with 3 colors (namely colors $i-1$, i and $i+1$), and color the rest by the remaining $d-3$ colors. We obtain

$$\sum_{m\text{ is fixed}}\binom{n-d-1}{n_1-1,n_2-1,\ldots,n_d-1} \le (d-2)\binom{s}{m}3^m(d-3)^{s-m} < (d-2)s^{s-m}3^s(d-3)^{s-m}.$$

Using again (10) and then (8) we have

$$\max_{m\text{ is fixed}}\left\{\sum\binom{n_i}{2}+\sum(n_in_{i+1}-1)\right\} \le \max_{m\text{ is given}} f(\mathbf{x}) + \frac{5s}{2} \le \frac{1}{2}m^2 + \frac{1}{2}(m-s)^2 + \frac{5s}{2}.$$

So (13) gives

$$\Sigma_2 \le n_{(d+1)}\sum_{0.6s\le m\le s-2}(d-2)s^{s-m}3^s(d-3)^{s-m}\exp_2\left[\frac{1}{2}m^2+\frac{1}{2}(m-s)^2+\frac{5s}{2}\right].$$

Hence

$$\frac{\Sigma_2}{h_1(n,d)} \le \frac{(s+1)(s+2)}{9}2^s \sum_{0.6s\le m\le s-2}\left(s(d-3)2^{-m}\right)^{s-m}.$$

One can see that in the given range this sum is dominated by the term $m = s-2$, when it is $O(s^2d^2)2^{-2s+4}$, hence $\Sigma_2 = O(s^4d^22^{-s}) = o(h_1(n,d))$ follows.

Case 3.1. $n_{i-1}+n_i+n_{i+1} = s+2$, $(1<i<d)$, $n_t = 2$ where $t \ne i-2, i+2$, and $n_j = 1$ for $0\le j\le d$, $j\notin\{i-1,i,i+1,t\}$.

There are $d-2$ ways to choose i then at most $d-3$ possibilities were left to t, then $n_{(d-3)}$ possibilities to fix N_j, $j \ne i-1, i, i+1, t$. Then one can select N_t in $\binom{s+4}{2}$ ways and distribute the remaining $s+2$ elements among N_{i-1}, N_i and N_{i+1}. Then (5) gives

$$\begin{aligned}
\Sigma_{31} &\le n_{(d-3)}(d-2)(d-3)\binom{s+4}{2}\sum_{\substack{a+b+c=s+2\\ a,b,c\ge 1}}\binom{s+2}{a,b,c}2^{\binom{a}{2}+\binom{b}{2}+\binom{c}{2}+\binom{2}{2}}(2^a-1)^b(2^b-1)^c(2^c-1)^1(2^2-1)^1\\
&\le 12n_{(d-3)}\binom{d-2}{2}\binom{s+4}{2}2^{\binom{s+2}{2}}\sum_{\substack{a+b+c=s+2\\ a,b,c\ge 1}}\binom{s+2}{a}\binom{s+2-a}{c}2^{-ac+c}. \qquad (14)
\end{aligned}$$

Using standard binomial identities we get

$$\begin{aligned}
&\sum_{\substack{a+b+c=s+2\\ a,b,c\ge 1}}\binom{s+2}{a}\binom{s+2-a}{c}2^{-ac+c}\\
&= \sum_{a=1,\,1\le c<s+1}\binom{s+2}{1}\binom{s+1}{c} + \sum_{a\ge 2}\binom{s+2}{a}\sum_{1\le c<s+2-a}\binom{s+2-a}{c}(2^{-a+1})^c
\end{aligned}$$

$$\leq (s+2)2^{s+1} + \sum_{a\geq 2} \binom{s+2}{a} (1+2^{-a+1})^{s+2-a}$$

$$\leq (s+2)2^{s+1} + \sum_{a\geq 2} \binom{s+2}{a} (3/2)^{s+2-a} \leq (s+2)2^{s+1} + (5/2)^{s+2}. \tag{15}$$

This is $o(3^s/d)$ so (14) gives $\Sigma_{31} = o(h_1(n,d))$.

The rest of the cases are quite similar.

Case 3.2. $n_{i-1}+n_i+n_{i+1} = s+2$, $n_{i-2} = 2$, $(2 < i < d)$, $n_{i+1} \geq 2$, and $n_j = 1$ for $0 \leq j \leq d, j \notin \{i-2, i-1, i, i+1\}$.

There are $d-3$ ways to choose i, then $n_{(d-3)}$ possibilities to fix $N_j, j \neq i-2, i-1, i, i+1$. Then (5) gives

$$\Sigma_{32} \leq n_{(d-3)}(d-3) \times \sum_{\substack{a+b+c=s+2\\ a,b\geq 1,\, c\geq 2}} \binom{s+4}{2, a, b, c} 2^{\binom{2}{2}+\binom{a}{2}+\binom{b}{2}+\binom{c}{2}} (2^2-1)^a(2^a-1)^b(2^b-1)^c(2^c-1)$$

$$\leq 2n_{(d-3)}(d-3)\binom{s+4}{2} 2^{\binom{s+2}{2}} \sum_{\substack{a+b+c=s+2\\ a,b\geq 1,\, c\geq 2}} \binom{s+2}{a}\binom{s+2-a}{c} 3^a 2^{-ac+c}. \tag{16}$$

We have

$$\sum_{\substack{a+b+c=s+2\\ a,b\geq 1,\, c\geq 2}} \binom{s+2}{a}\binom{s+2-a}{c} 3^a 2^{-ac+c}$$

$$\leq \sum_{a\geq 1,\, 2\leq c<s+2-a} \binom{s+2}{a}\binom{s+2-a}{c} 3^a 2^{-2a+2} \leq \sum_{a\geq 1} \binom{s+2}{a} 2^{s+2-a} 3^a 2^{-2a+2}$$

$$= 2^{s+4} \sum_{a\geq 1} \binom{s+2}{a} (3/8)^a \leq 2^{s+4}(11/8)^{s+2}. \tag{17}$$

This is $o(3^s)$ so (16) gives $\Sigma_{32} = o(h_1(n,d))$.

Case 3.3. $n_{i-1}+n_i+n_{i+1} = s+2$, $n_{i+2} = 2$, $(1 < i < d-1)$, $n_{i-1} \geq 3$, and $n_j = 1$ for $0 \leq j \leq d, j \notin \{i-1, i, i+1, i+2\}$.

There are $d-3$ ways to choose i, then $n_{(d-3)}$ possibilities to fix $N_j, j \neq i-1, i, i+1, i+2$. Then (5) gives

$$\Sigma_{33} \leq n_{(d-3)}(d-3) \times \sum_{\substack{a+b+c=s+2\\ a\geq 3,\, b,c\geq 1}} \binom{s+4}{a, b, c, 2} 2^{\binom{a}{2}+\binom{b}{2}+\binom{c}{2}+\binom{2}{2}} (2^a-1)^b(2^b-1)^c(2^c-1)^2(2^2-1)$$

$$\leq 6n_{(d-3)}(d-3)\binom{s+4}{2} 2^{\binom{s+2}{2}} \sum_{\substack{a+b+c=s+2\\ a\geq 3,\, b,c\geq 1}} \binom{s+2}{a}\binom{s+2-a}{c} 2^{-ac+2c}. \tag{18}$$

We have

$$\sum_{\substack{a+b+c=s+2\\ a\geq 3,\, b,c\geq 1}} \binom{s+2}{a}\binom{s+2-a}{c} 2^{-ac+2c} = \sum_{a\geq 3} \binom{s+2}{a} \left(\sum_{1\leq c<s+2-a} \binom{s+2-a}{c} (2^{-a+2})^c \right)$$

$$\leq \sum_{a\geq 3} \binom{s+2}{a} (1+2^{-a+2})^{s+2-a}$$

$$\leq \sum_{a\geq 3} \binom{s+2}{a} (3/2)^{s+2-a} \leq (5/2)^{s+2}. \tag{19}$$

This is $o(3^s)$ so (18) gives $\Sigma_{33} = o(h_1(n,d))$.

Case 4.1. $n_{i-1}+n_i+n_{i+1} = s+3$, $n_{i-1} \geq 2$, $n_{i+1} \geq 2$, and $n_j = 1$ for $0 \leq j \leq d, j \notin \{i-1, i, i+1\}$.

There are $d-2$ ways to choose i, then $n_{(d-2)}$ possibilities to fix $N_j, j \neq i-1, i, i+1$. Then (5) gives

$$\Sigma_{41} \leq n_{(d-2)}(d-2) \times S, \tag{20}$$

where

$$S := \sum_{\substack{a+b+c=s+3\\ a\geq 2,\, b\geq 1,\, c\geq 2}} \binom{s+3}{a, b, c} 2^{\binom{a}{2}+\binom{b}{2}+\binom{c}{2}} (2^a-1)^b(2^b-1)^c(2^c-1).$$

We separate the case $a = 2$ and use obvious upper bounds

$$S \le \sum_{\substack{b+c=s+1\\2\le c\le s}} \binom{s+3}{2}\binom{s+1}{c} 2^{1+\binom{b}{2}+\binom{c}{2}} 3^b 2^{bc+c} + \sum_{\substack{a+b+c=s+3\\a\ge 3,\, b\ge 1,\, c\ge 2}} \binom{s+3}{a,b,c} 2^{\binom{a}{2}+\binom{b}{2}+\binom{c}{2}+ab+bc+c}$$

$$= 2\binom{s+3}{2} 2^{\binom{s+1}{2}} \sum_{2\le c\le s} \binom{s+1}{c} 3^{s+1-c} 2^c \tag{21}$$

$$+ 2^{\binom{s+3}{2}} \sum_{1\le b\le s-2} \binom{s+3}{b} \left(\sum_{\substack{a+c=s+3-b\\a\ge 3,\, c\ge 2}} \binom{s+3-b}{a} 2^{-ac+c} \right). \tag{22}$$

In the row (22), for a given b, the terms in the last sum form a unimodal sequence, the two terms at the ends are the largest ones. More precisely, for $a, c \ge 2$ integers

$$\frac{\binom{a+c}{a} 2^{-ac+c}}{\binom{a+c}{a+1} 2^{-(a+1)(c-1)+(c-1)}} = \frac{(a+1)2^{-a}}{c2^{-c}} > 1 \iff a \le c.$$

Thus we can upper estimate these terms by the (sum of the) extreme ends, when $(a, c) = (3, s-b)$ and when $(a, c) = (s-b+1, 2)$.

$$\sum_{\substack{a+c=s+3-b\\a\ge 3,\, c\ge 2}} \binom{s+3-b}{a} 2^{-ac+c} \le (s-1-b)\left(\binom{s+3-b}{3} 2^{-2s+2b} + \binom{s+3-b}{2} 2^{-2s+2b} \right)$$
$$\le s^4 4^{-s+b}.$$

In the row (21) the sum is at most $(3+2)^{s+1}$. We obtain

$$S \le (s+3)(s+2)2^{\binom{s+1}{2}} 5^{s+1} + 2^{\binom{s+3}{2}} s^4 4^{-s} \sum_{1\le b\le s-2} \binom{s+3}{b} 4^b$$
$$\le O(s^4) 2^{\binom{s+1}{2}} 5^s.$$

This is $o(2^{\binom{s+2}{2}} 3^s)$ so (20) gives $\Sigma_{41} = o(h_1(n, d))$.

Case 4.2. $n_{d-1} + n_d = s + 2$, and $n_j = 1$ for $0 \le j \le d - 2$.

There are $n_{(d-1)}$ possibilities to fix N_j, $j = 0, 1, \dots, d - 2$. Then (5) gives

$$\Sigma_{42} \le n_{(d-1)} \sum_{a+b=s+2} \binom{s+2}{a} 2^{\binom{a}{2}+\binom{b}{2}} (2^a - 1)^b$$
$$\le n_{(d-1)} \sum \binom{s+2}{a} 2^{\binom{s+2}{2}} = n_{(d-1)} 2^{\binom{s+2}{2}} 2^{s+2} = o(h_1(n, d)).$$

Case 4.3. $n_{i-1} + n_i = s + 2$, $1 < i < d$, and $n_j = 1$ for $0 \le j \le d$, $j \notin \{i - 1, i\}$.

There are $d - 2$ choices for i and $n_{(d-1)}$ possibilities to fix N_j, $j = 0, 1, \dots, d$, $j \ne i - 1, i$. Then (5) gives

$$\Sigma_{43} \le n_{(d-1)}(d-2) \sum_{a+b=s+2} \binom{s+2}{a} 2^{\binom{a}{2}+\binom{b}{2}} (2^a - 1)^b (2^b - 1)$$
$$\le n_{(d-1)}(d-2) \sum \binom{s+2}{a} 2^{\binom{s+2}{2}} 2^b = n_{(d-1)}(d-2) 2^{\binom{s+2}{2}} 3^{s+2} = 2h_1(n, d).$$

Adding up the above eight cases, we get that the right hand side of (6) is at most $(2 + o(1))h_1(n, d)$, completing the proof of the upper bound. Together with the lower bound (2) we have the asymptotic.

We also obtained that almost all members of $\mathcal{G}(n, \mathrm{diam} = d)$ belong to the group of Case 4.3. One can see that almost all members of the group 4.3 belong to $\mathcal{H}_1(n, d)$, thus finishing the proof of Theorem 1.

6. Upper bound for Theorem 2

In this section we suppose that $n - c \log n < d$, where c is a sufficiently small constant. Again we are going to use (6). We put the terms of the right hand side of (6) into four groups according to t, the number of non-singleton classes

$$t := |\{i : |N_i| > 1\}|.$$

We have $t \le n - d - 1$. If $t = n - d - 1$, then we have t pairs and $d + 1 - t$ singletons, i.e., all $n_i \le 2$.

- Case 1: $t < n - d - 1$,
- Case 2: $t = n - d - 1$ and $\max\{n_1, n_2, n_{d-2}, n_{d-1}, n_d\} = 2$.
- Case 3: $t = n - d - 1$, $n_d = 1$ but there is an i with $n_i = n_{i+1} = 2$,
- Case 4: the graphs in $\mathcal{H}_2(n, d)$.

These exhaust all possibilities. We will show that the sum (6) in each of the above cases is $o(h_2(n, d))$, except in Case 4. Recall that $2h_2(n, d) = n_{(d+1)} d^s 3^s$.

Case 1. $t < n - d - 1 := s$.

Every graph in this class can be obtained by the following five-step procedure.

(1) Take a path $P := v_0, v_1, \dots, v_d$, there are $n_{(d+1)}$ ways to do it. We will have $v_i \in N_i$.

(2) Choose $d - t$ indices from $[d]$, the corresponding classes and v_0 are the singletons, there are $\binom{d}{t} \le d^t/t!$ ways to do this.

(3) Put a second element to the non-singleton classes from the s vertices outside the path, there are

$$s_{(t)} = \binom{s}{t} t! = \binom{s}{s-t} t! \le s^{s-t} t!$$

ways to proceed.

(4) Distribute the remaining $s - t$ vertices arbitrarily among the non-singleton classes, there are t^{s-t} ways of this. We now have a partition $(N_0, N_1, \dots, N_d)$ together with a path P.

(5) Finally, call a pair xy *open* if either it is contained in some N_i or $x \in N_i$, $y \in N_{i+1}$ with $|N_i| > 1$ and it is not an edge of P. There are

$$E := \sum \binom{n_i}{2} + \sum_{n_i > 1} n_i n_{i+1} - 1 \tag{23}$$

open pairs. With given P and a partition $(N_0, N_1, \dots, N_d)$ we can select at most 2^E subsets of open pairs to create a graph from $\mathcal{G}(x_0, N_1, \dots, N_d)$.

Define $x_i := n_i - 1$ and use (10) and then (9) from Lemma 1. Note that $m \le s - (t - 3)$, since there are t positive x_i's. We obtain that the right hand side of (23) is at most

$$f(\mathbf{x}) + \frac{5s}{2} \le \frac{3}{4}(s - t + 3)s + \frac{5s}{2} < s(s - t) + 5s.$$

So the number of graphs counted in Case 1 is at most

$$\sum_{1 \le t < s} n_{(d+1)} \times \frac{d^t}{t!} \times s^{s-t} t! \times t^{s-t} \times 2^{s(s-t)+5s} = 2h_2(n, d) \left(\frac{32}{3}\right)^s \sum_{s-t \ge 1} \left(\frac{st2^s}{d}\right)^{s-t}.$$

This is $o(h_2(n, d))$ since the base of the geometric series is $o((32/3)^{-s})$ if $s = n - d - 1 < (\log_2 n)/6$.

Case 2. $n_j \le 2$ for all $1 \le j \le d$, and $\max\{n_1, n_2, n_{d-2}, n_{d-1}, n_d\} = 2$.

We consider the case $n_d = 2$ only, the other cases can be handled in the same way. In this case (5) gives at most $2^s 9^s$ graphs. Furthermore there are $\binom{d-1}{s-1} \le sd^{s-1}/s!$ ways to select the s indices of the 2-element blocks. So the number of partitions with $n_d = 2$ is

$$\frac{sd^{s-1}}{s!} \times \binom{n}{2} \binom{n-2}{2} \cdots \binom{n - 2(s-1)}{2} (n - 2s)!.$$

So the number of graphs in this case is at most

$$2^s 3^{2s} \times \frac{sd^{s-1}}{s!} \frac{n!}{2^s} = 2h_2(n, d) \frac{s3^s}{d}.$$

Case 3. $n_j \le 2$, for all $1 \le j \le d$, $n_d = 1$ and there is an i with $n_i = n_{i+1} = 2$.

Inequality (5) gives at most $2^s 9^s$ graphs. Furthermore, there are

$$\binom{d-1}{s} - \binom{d-s}{s} \le (s-1)\binom{d-2}{s-1} \le s\binom{d-1}{s-1} \le \frac{s^2}{d} \frac{d^s}{s!}$$

ways to select the s indices of the 2-element blocks from $\{1, 2, \dots, d-1\}$ such a way that two are next to each other. So the number of graphs in this case is at most

$$2^s 3^{2s} \times \frac{s^2}{d} \frac{d^s}{s!} \binom{n}{2} \binom{n-2}{2} \cdots \binom{n-2(s-1)}{2} (n - 2s)! = 2h_2(n, d) \frac{s^2 3^s}{d}.$$

Adding up the above three cases, we get that the number of graphs of $\mathcal{G}(n, \text{diam} = d) \setminus \mathcal{H}_2(n, d)$ is at most $o(h_2(n, d))$, completing the proof of the upper bound in Theorem 2.

7. Eccentricity

The *eccentricity* of a vertex x in the graph G is the maximum over all vertices of the length of a shortest path from x to that vertex. Actually, in both theorems above, we proved asymptotic formulas for the number of n-vertex graphs of eccentricity d.

The error terms in the asymptotics are exponentially small. For $3 \le d \le n - c_1 \log n$ we have

$$\frac{|\mathcal{G}(n, \mathrm{diam} = d)|}{h_1(n, d)} = 1 + O\left(d^2 s^4 \left(\frac{11}{12}\right)^s\right), \tag{24}$$

and for $d > n - c_2 \log n$ we have

$$\frac{|\mathcal{G}(n, \mathrm{diam} = d)|}{h_2(n, d)} = 1 + O\left(\frac{s^2 (64/3)^s}{d}\right). \tag{25}$$

8. Phase transition

It would be interesting to investigate the *phase transition*, i.e., the case of $n - d = \Theta(\log n)$.

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