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A new short proof of the EKR theorem ☆

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ABSTRACT

A family $\mathcal{F}$ is *intersecting* if $F \cap F' \neq \emptyset$ whenever $F, F' \in \mathcal{F}$. Erdős, Ko, and Rado (1961) [6] showed that

$$|\mathcal{F}| \leqslant \binom{n-1}{k-1} \tag{1}$$

holds for an intersecting family of k-subsets of $[n] := \{1, 2, 3, \ldots, n\}$, $n \geqslant 2k$. For $n > 2k$ the only extremal family consists of all k-subsets containing a fixed element. Here a new proof is presented by using the Katona's shadow theorem for t-intersecting families.

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1. Definitions: shadows, *b*-intersecting families

$\binom{X}{k}$ denotes the family of k-element subsets of X. For a family of sets $\mathcal{A}$ its *s-shadow* $\partial_s \mathcal{A}$ denotes the family of s-subsets of its members $\partial_s \mathcal{A} := \{S\colon\ |S| = s,\ \exists A \in \mathcal{A},\ S \subseteq A\}$. E.g., $\partial_1 \mathcal{A} = \bigcup \mathcal{A}$. Suppose that $\mathcal{A}$ is a family of a-sets such that $|A \cap A'| \geqslant b \geqslant 0$ for all $A, A' \in \mathcal{A}$. Katona [10] showed that then

$$|\mathcal{A}| \leqslant |\partial_{a-b} \mathcal{A}|. \tag{2}$$

We show that this inequality quickly implies the EKR theorem. This way it is even shorter than the classical proof of Katona [11] using cyclic permutations, or the one found by Daykin [2] applying the Kruskal–Katona theorem.

2. The proof

Let $\mathcal{F} \subset \binom{[n]}{k}$ be intersecting. Define a partition $\mathcal{F}_0 := \{F \in \mathcal{F}\colon\ 1 \notin F\}$, $\mathcal{F}_1 := \{F \in \mathcal{F}\colon\ 1 \in F\}$ and define $\mathcal{G}_1 := \{F \setminus \{1\}\colon\ 1 \in F \in \mathcal{F}\}$. Consider $\mathcal{F}_0$ as a family on $[2, n]$. Its complementary family $\mathcal{G}_0 :=$

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$\{[2,n]\setminus F\colon\ F\in\mathcal{F}_0\}$ is $(n-1-k)$-uniform. The intersection property of $\mathcal{F}$ implies that any member of $\mathcal{G}_1$ is not contained in any member of $\mathcal{G}_0$. We obtain

$$\mathcal{G}_1\cap\partial_{k-1}\mathcal{G}_0=\emptyset.$$

Since both $\mathcal{G}_1$ and $\partial_{k-1}\mathcal{G}_0$ are subfamilies of $\binom{[2,n]}{k-1}$ we obtain that $|\mathcal{G}_1|+|\partial_{k-1}\mathcal{G}_0|\leqslant\binom{n-1}{k-1}$. The intersection size $|G\cap G'|$ of $G,G'\in\mathcal{G}_0$ is at least $n-2k$, since

$$\left|G\cap G'\right|=\left|\bigl([2,n]\setminus F\bigr)\cap\bigl([2,n]\setminus F'\bigr)\right|=(n-1)-2k+\left|F\cap F'\right|.$$

Then (2) gives (with $a=n-k-1$, $b=n-2k\geqslant 0$) that $|\mathcal{G}_0|\leqslant|\partial_{k-1}\mathcal{G}_0|$. Summarizing

$$|\mathcal{F}|=|\mathcal{F}_1|+|\mathcal{F}_0|=|\mathcal{G}_1|+|\mathcal{G}_0|\leqslant|\mathcal{G}_1|+|\partial_{k-1}\mathcal{G}_0|\leqslant\binom{n-1}{k-1}. \qquad\square \tag{3}$$

Extremal families. Equality holds in (2) if and only if $a=b$, or $\mathcal{A}=\emptyset$, or $\mathcal{A}\equiv\binom{[2a-b]}{a}$. Thus, for $n>2k$, equality in (3) implies either $\mathcal{G}_0=\emptyset$ and $1\in\bigcap\mathcal{F}$, or $\mathcal{G}_0\equiv\binom{[2,n-1]}{n-1-k}$ and $n\in\bigcap\mathcal{F}$.

3. Two algebraic reformulations

Given two families of sets $\mathcal{A}$ and $\mathcal{B}$, the *inclusion matrix* $I(\mathcal{A},\mathcal{B})$ is a 0-1 matrix of dimension $|\mathcal{A}|\times|\mathcal{B}|$, its rows and columns are labeled by the members of $\mathcal{A}$ and $\mathcal{B}$, respectively, the element $I_{A,B}=1$ if and only if $A\supseteq B$. In the case $\mathcal{F}\subseteq 2^{[n]}$ the matrix $I(\mathcal{F},\binom{[n]}{1})$ is the usual *incidence matrix* of $\mathcal{F}$, and $I(\mathcal{F},\binom{[n]}{s})$ is the *generalized* incidence matrix of order s.

Suppose that L is a set of non-negative integers, $|L|=s$, and for any two distinct members A,A' of the family $\mathcal{A}$ one has $|A\cap A'|\in L$. The Frankl, Ray-Chaudhuri, and Wilson [8,13] theorem states that in the case of $\mathcal{A}\subseteq\binom{[n]}{k}$, $s\leqslant k$ the row vectors of the generalized incidence matrix $I(\mathcal{A},\binom{[n]}{s})$ are linearly independent. Here the rows are taken as real vectors (in [13]) or as vectors over certain finite fields (in [8]). Note that this statement generalizes (2) with $L=\{b,b+1,\ldots,a-1\}$, $s=a-b$.

Matrices and the EKR theorem. Instead of using (2) one can prove directly that the row vectors of the inclusion matrix $I(G_0\cup G_1,\binom{[2,n]}{k-1})$ are linearly independent. For more details see [8,13].

Linearly independent polynomials. One can define homogeneous, multilinear polynomials $p(F,\mathbf{x})$ of rank $k-1$ with variables $x_2,\ldots,x_n$

$$p(F,\mathbf{x})=\begin{cases}\sum\{x_S\colon\ S\subset[2,n]\setminus F,\ |S|=k-1\} & \text{for } 1\notin F\in\mathcal{F},\\ x_{F\setminus\{1\}} & \text{for } 1\in F\in\mathcal{F},\end{cases}$$

where $x_S:=\prod_{i\in S}x_i$. To prove (1) one can show that these polynomials are linearly independent. For more details see [9].

4. Remarks

The idea of considering the shadows of the complements (one of the main steps of Daykin's proof [2]) first appeared in Katona [10] (p. 334) in 1964. He applied a more advanced version of his intersecting shadow theorem (2), namely an estimate using $\partial_{a-b+1}\mathcal{A}$.

Linear algebraic proofs are common in combinatorics, see the book [1]. For recent successes of the method concerning intersecting families see Dinur and Friedgut [4,5]. There is a relatively short proof of the EKR theorem in [9] using linearly independent polynomials. In fact, our proof here can be considered as a greatly simplified version of that one.

Since the algebraic methods are frequently insensitive to the structure of the hypergraphs in question it is much easier to give an upper bound $\binom{n}{k-1}$ which holds for *all* n and k (see [3]). To decrease this formula to $\binom{n-1}{k-1}$ requires further insight. Our methods resemble to those of Parekh [12] and Snevily [14] who succeeded to handle this for various related intersection problems.

Generalized incidence matrices proved to be extremely useful, see, e.g., the ingenious proof of Wilson [15] for another Frankl–Wilson theorem, namely the exact form of the classical Erdős–Ko–Rado theorem concerning the maximum size of a k-uniform, t-intersecting family on n vertices. They proved [7,15] that the maximum size is exactly $\binom{n-t}{k-t}$ if and only if $n \geqslant (t+1)(k-t+1)$.

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