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Large B_d-free and union-free subfamilies

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Abstract

For a property Γ and a family of sets $\mathcal{F}$, let $f(\mathcal{F},\Gamma)$ be the size of the largest subfamily of $\mathcal{F}$ having property Γ. For a positive integer m, let $f(m,\Gamma)$ be the minimum of $f(\mathcal{F},\Gamma)$ over all families of size m. A family $\mathcal{F}$ is said to be B_d-free if it has no subfamily $\mathcal{F}' = \{F_I : I \subseteq [d]\}$ of 2^d distinct sets such that for every $I, J \subseteq [d]$, both $F_I \cup F_J = F_{I\cup J}$ and $F_I \cap F_J = F_{I\cap J}$ hold. A family $\mathcal{F}$ is a-union free if $F_1 \cup \ldots \cup F_a \neq F_{a+1}$ whenever $F_1, \ldots, F_{a+1}$ are distinct sets in $\mathcal{F}$. We verify a conjecture of Erdős and Shelah that $f(m, B_2\text{-free}) = \Theta(m^{2/3})$. We also obtain lower and upper bounds for $f(m, B_d\text{-free})$ and $f(m, a\text{-union free})$.

Keywords: extremal set theory, union-free subfamilies, B_d-free subfamilies

1 Introduction, results

Moser proposed the following problem: Let $A_1, A_2, \ldots, A_m$ be a collection of m sets. A subfamily $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$ is *union-free* if $A_{i_{j_1}} \cup A_{i_{j_2}} \neq A_{i_{j_3}}$ for every triple of distinct sets $A_{j_1}, A_{j_2}, A_{j_3}$ with $1 \le j_1 \le r$, $1 \le j_2 \le r$, and $1 \le j_3 \le r$. Erdős and Komlós [2] considered the following problem of Moser: what is the size of the largest union-free subfamily $A_{i_1}, \ldots, A_{i_r}$?

Put $f(m) = \min r$, where the minimum is taken over all families of m distinct sets. As mentioned in [2], Riddel pointed out that $f(m) > c\sqrt{m}$. Erdős and Komlós [2] showed $\sqrt{m} \le f(m) \le 2\sqrt{2}\sqrt{m}$. Kleitman proved $\sqrt{2m} - 1 < f(m)$; Erdős and Shelah [3] obtained

$$f(m) < 2\sqrt{m} + 1. \tag{1}$$

The latter two conjectured $f(m) = (2 + o(1))\sqrt{m}$.

We define $f(\mathcal{F},\Gamma)$ as the size of the largest subfamily of $\mathcal{F}$ having property Γ,

$$f(\mathcal{F},\Gamma) := \max\{|\mathcal{F}'| : \mathcal{F}' \subseteq \mathcal{F},\ \ \mathcal{F}' \text{ has property } \Gamma\}.$$

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In this context, $f(E(K_r^n), \mathcal{H}\text{-free})$ is the Turán number $\mathrm{ex}_r(n, \mathcal{H})$. Let $f(m, \Gamma) = \min\{f(\mathcal{F}, \Gamma) : |\mathcal{F}| = m\}$. Generalizing the union-free property, a family $\mathcal{F}$ is *a-union free* if there are no distinct sets $F_1, F_2 \ldots, F_{a+1}$ satisfying $F_1 \cup F_2 \cup \ldots \cup F_a = F_{a+1}$.

Erdős and Shelah [3] also considered Γ to be the property that no four distinct sets satisfy $F_1 \cup F_2 = F_3$ and $F_1 \cap F_2 = F_4$. Such families are called B_2-*free*. Erdős and Shelah [3] gave an example showing $f(m, B_2\text{-free}) \le (3/2)m^{2/3}$ and they also conjectured $f(m, B_2\text{-free}) > c_2 m^{2/3}$.

A family $\mathcal{B}$ of 2^d distinct sets is forming a Boolean algebra of dimension d if the sets can be indexed with the subsets of $[d] = \{1, 2, \ldots, d\}$ so that $B_I \cap B_J = B_{I \cap J}$ and $B_I \cup B_J = B_{I \cup J}$ hold for any $I, J \subseteq [d]$. If $\mathcal{F}$ does not contain any subfamily forming a Boolean algebra of dimension d, then it is called B_d-*free*, or we say that $\mathcal{F}$ *avoids* any Boolean algebra of dimension d. A result by Gunderson, Rödl, and Sidorenko [5] states that $f(2^{[n]}, B_d\text{-free}) = \Theta(2^n / n^{2^{-d}})$; here the case $d = 1$ is the classical Sperner's theorem [6], the case $d = 2$ is due to Erdős and Kleitman [1]. We were able to prove the aforementioned conjecture by Erdős and Shelah in the following more general form.

Theorem 1.1 *For any integer d, $d \ge 2$, there exist constants $c_d, c'_d > 0$, and exponents*

$$e_d := \frac{2^d - \lceil \log_2(d+2) \rceil}{2^d - 1}, \quad e'_d := \frac{2^d - 2}{2^d - 1}$$

such that

$$c_d m^{e_d} \le f(m, B_d\text{-free}) \le c'_d m^{e'_d}.$$

In particular,

$$(2) \qquad (3 \cdot 2^{-7/3} + o(1)) m^{2/3} \le f(m, B_2\text{-free}) \le \frac{3}{2} m^{2/3}.$$

The lower bound in Theorem 1.1 follows from a first moment method argument and a lemma bounding the number of B_d's that a family of m sets can contain. The construction for the upper bound is a generalization of the construction by Erdős and Shelah. To calculate the bound that this construction gives we consider the following Turán-type problem.

Let $\mathcal{K}(a_1, \ldots, a_d)$ denote the complete, d-partite hypergraph with parts of sizes $a_1, \ldots, a_d$, i.e., $V(\mathcal{K}) := X_1 \cup \ldots \cup X_d$ where $X_1, \ldots, X_d$ are pairwise disjoint sets with $|X_i| = a_i$, and $E(\mathcal{K}) := \{E : |E| = d, |X_i \cap E| = 1$ for all $i \in [d]\}$. For short we use $\mathcal{K}_d^{(k)}$ for $\mathcal{K}(k, k^2, \ldots, k^{2^{d-1}})$ and K_{d*2} for $\mathcal{K}(2, \ldots, 2)$. The (generalized) *Turán number* of the d-uniform hypergraph $\mathcal{H}$ with respect to the other hypergraph $\mathcal{G}$, denoted by $\mathrm{ex}(\mathcal{G}, \mathcal{H})$, is the size of the largest

$\mathcal{H}$-free subhypergraph of $\mathcal{G}$.

Theorem 1.2 *For* $k, d \geq 2$, $\mathrm{ex}(\mathcal{K}_d^{(k)}, K_{d*2}) < \left(2 - \frac{1}{2^{d-1}}\right) k^{2^d-2}$.

We also considered a-union free families. We generalize the construction giving (1) and prove the following

Theorem 1.3 *For any integer* a, $a \geq 2$,

$$\sqrt{2m} - \frac{1}{2} \leq f(m, a\text{-union free}) \leq 4a + 4a^{1/4}\sqrt{m}. \tag{3}$$

Since we obtained our results, Fox, Lee, and Sudakov [4] verified the present authors' conjecture and proved a matching lower bound showing that $f(m, a\text{-union free}) \geq \max\{a, \frac{1}{3}\sqrt[4]{a}\sqrt{m})\}$. They also gave a sharp bound in (1), namely $f(m) = \lfloor\sqrt{4m+1}\rfloor - 1$.

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