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# LARGE $B_d$-FREE AND UNION-FREE SUBFAMILIES*

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**Abstract.** For a property $\Gamma$ and a family of sets $\mathcal{F}$, let $f(\mathcal{F}, \Gamma)$ be the size of the largest subfamily of $\mathcal{F}$ having property $\Gamma$. For a positive integer $m$, let $f(m, \Gamma)$ be the minimum of $f(\mathcal{F}, \Gamma)$ over all families of size $m$. A family $\mathcal{F}$ is said to be $B_d$-free if it has no subfamily $\mathcal{F}' = \{F_I : I \subseteq [d]\}$ of $2^d$ distinct sets such that for every $I, J \subseteq [d]$, both $F_I \cup F_J = F_{I\cup J}$ and $F_I \cap F_J = F_{I\cap J}$ hold. A family $\mathcal{F}$ is $a$-union-free if $F_1 \cup \cdots \cup F_a \neq F_{a+1}$ whenever $F_1, \ldots, F_{a+1}$ are distinct sets in $\mathcal{F}$. We verify a conjecture of Erdős and Shelah that $f(m, B_2\text{-free}) = \Theta(m^{2/3})$. We also obtain lower and upper bounds for $f(m, B_d\text{-free})$ and $f(m, a\text{-union free})$.

**Key words.** extremal set systems, $B_d$-free subfamilies, union-free subfamilies

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## 1. Introduction, results.

Let $A_1, A_2, \ldots, A_m$ be a collection of $m$ sets. A subfamily $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$ is *union-free* if $A_{i_{j_1}} \cup A_{i_{j_2}} \neq A_{i_{j_3}}$ for every triple of distinct sets $A_{j_1}, A_{j_2}, A_{j_3}$ with $1 \leq j_1 \leq r$, $1 \leq j_2 \leq r$, and $1 \leq j_3 \leq r$. Erdős and Komlós [4] considered the following problem of Moser: what is the size of the largest union-free subfamily $A_{i_1}, \ldots, A_{i_r}$?

Put $f(m) = \min r$, where the minimum is taken over all families of $m$ distinct sets. As mentioned in [4], Riddel pointed out that $f(m) > c\sqrt{m}$. Erdős and Komlós [4] showed $\sqrt{m} \leq f(m) \leq 2\sqrt{2}\sqrt{m}$. Kleitman [9] proved $\sqrt{2m} - 1 < f(m)$; Erdős and Shelah [5] obtained

$$f(m) < 2\sqrt{m} + 1. \tag{1}$$

The latter two conjectured $f(m) = (2 + o(1))\sqrt{m}$.

We define $f(\mathcal{F}, \Gamma)$ as the size of the largest subfamily of $\mathcal{F}$ having property $\Gamma$,

$$f(\mathcal{F}, \Gamma) := \max\{|\mathcal{F}'| : \mathcal{F}' \subseteq \mathcal{F},\ \ \mathcal{F}' \text{ has property } \Gamma\}.$$

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In this context, $f(E(K_r^n), \mathcal{H}\text{-free})$ is the Turán number $\mathrm{ex}_r(n, \mathcal{H})$. Let $f(m, \Gamma) = \min\{f(\mathcal{F}, \Gamma) : |\mathcal{F}| = m\}$. Generalizing the union-free property, a family $\mathcal{F}$ is *a-union-free* if there are no distinct sets $F_1, F_2 \ldots, F_{a+1}$ satisfying $F_1 \cup F_2 \cup \cdots \cup F_a = F_{a+1}$.

Erdős and Shelah [5] also considered $\Gamma$ to be the property that no four distinct sets satisfy $F_1 \cup F_2 = F_3$ and $F_1 \cap F_2 = F_4$. Such families are called *$B_2$-free*. Erdős and Shelah [5] gave an example showing $f(m, B_2\text{-free}) \leq (3/2)m^{2/3}$ and they also conjectured $f(m, B_2\text{-free}) > c_2 m^{2/3}$.

A family $\mathcal{B}$ of $2^d$ distinct sets forms a Boolean algebra of dimension $d$ if the sets can be indexed with the subsets of $[d] = \{1, 2, \ldots, d\}$ so that $B_I \cap B_J = B_{I\cap J}$ and $B_I \cup B_J = B_{I\cup J}$ hold for any $I, J \subseteq [d]$. If $\mathcal{F}$ does not contain any subfamily forming a Boolean algebra of dimension $d$, then it is called *$B_d$-free*, or we say that $\mathcal{F}$ *avoids* any Boolean algebra of dimension $d$. A result by Gunderson, Rödl, and Sidorenko [7] states that $f(2^{[n]}, B_d\text{-free}) = \Theta(2^n/n^{2^{-d}})$; here the case $d = 1$ is the classical Sperner's theorem [10], while the case $d = 2$ is due to Erdős and Kleitman [3]. In sections 2 and 3, we prove the aforementioned conjecture by Erdős and Shelah in the following more general form.

Theorem 1.1. *For any integer $d$, $d \geq 2$, there exist constants $c_d, c'_d > 0$, and exponents*

$$e_d := \frac{2^d - \lceil \log_2(d+2) \rceil}{2^d - 1}, \quad e'_d := \frac{2^d - 2}{2^d - 1}$$

*such that*

$$c_d m^{e_d} \leq f(m, B_d\text{-free}) \leq c'_d m^{e'_d}.$$

*In particular,*

$$(3 \cdot 2^{-7/3} + o(1))m^{2/3} \leq f(m, B_2\text{-free}) \leq \frac{3}{2} m^{2/3}. \tag{2}$$

In section 4, we consider $a$-union-free families. We generalize the construction giving (1) and prove the following theorem.

Theorem 1.2. *For any integer $a$, $a \geq 2$,*

$$\sqrt{2m} - \frac{1}{2} \leq f(m, a\text{-union free}) \leq 4a + 4a^{1/4}\sqrt{m}. \tag{3}$$

Since the first version of this manuscript, Fox, Lee, and Sudakov [6] verified the present authors' conjecture (see details in section 5) and proved a matching lower bound showing that $f(m, a\text{-union free}) \geq \max\{a, \frac{1}{3}\sqrt[4]{a}\sqrt{m})\}$. They also gave a sharp bound in (1), namely, $f(m) = \lfloor \sqrt{4m+1} \rfloor - 1$.

**2. Subfamilies avoiding Boolean algebras of dimension *d*.** In this section we prove the lower bounds in Theorem 1.1 by a probabilistic argument based on the first moment method.

Suppose that $\mathcal{B} = \{B_I : I \subseteq [d]\}$ forms a Boolean algebra of dimension $d$. Thus we have pairwise disjoint sets, $A_0, A_1, \ldots, A_d$, all except possibly $A_0$ nonempty, such that $B_I = A_0 \cup (\bigcup_{i\in I} A_i)$. Let us call these $A_i$'s *atoms*. A subfamily $\mathcal{C} \subseteq \mathcal{B}$ *determines* the $d$-dimensional Boolean algebra $\mathcal{B}$ if $\mathcal{B}$ is the only $d$-dimensional Boolean algebra that contains $\mathcal{C}$. Equivalently, every member of $\mathcal{B}$ can be obtained as a Boolean expression (using unions, intersections and differences but not complements) of some sets of $\mathcal{C}$. Obviously, the $d$ sets of the form $\{A_0 \cup A_i : i \in [d]\}$ determine $\mathcal{B}$. Much more is true.

LEMMA 2.1. *Suppose that the sets of $\mathcal{B}$ form a Boolean algebra of dimension $d$. Then there exists a subfamily $\mathcal{C} \subseteq \mathcal{B}$ determining $\mathcal{B}$ and of size $\lceil \log_2(d+2) \rceil$. Moreover, there is no subfamily of smaller size that determines $\mathcal{B}$.*

*Proof.* Let $k := \lceil \log_2(d+2) \rceil$. We define an appropriate $\mathcal{C}$ of size $k$ by considering a standard construction used for nonadaptive binary search. Namely, write each integer $i \in [d]$ in base 2, $i = \sum_{1 \le j \le k} \varepsilon_{i,j} 2^{j-1}$, and define $C_j = A_0 \cup (\bigcup_{\varepsilon_{i,j}=1} A_i)$, $j = 1, 2, \ldots, k$. Clearly

$$A_i = \bigcap_{j:\varepsilon_{i,j}=1} C_j \setminus \bigcup_{l:\varepsilon_{i,l}=0} C_l$$

holds for all $1 \le i \le d$, and as the atoms $A_0, A_1, \ldots, A_d$ determine $\mathcal{B}$, so does the family of the $C_j$'s.

On the other hand, any family of finite sets $\mathcal{C}$ has at most $2^{|\mathcal{C}|} - 1$ finite atoms. If they determine $\mathcal{B}$, these should be the distinct, disjoint sets $A_0, \ldots, A_d$. We obtain $2^{|\mathcal{C}|} - 1 \ge d + 1$. ∎

COROLLARY 2.2. *Given any family $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ of $m$ sets, $\mathcal{F}$ contains at most $\binom{m}{\lceil \log(d+2) \rceil}$ subfamilies forming a Boolean algebra of dimension $d$.*

If $d$ is fixed, then Corollary 2.2 gives the correct order of magnitude of the number of possible subfamilies forming a Boolean algebra of dimension $d$ contained in a family of $m$ sets, as shown by the family $\mathcal{F} = 2^{[n]}$, where $m = 2^n$ and the number of $B_d$'s is $\Theta((d+2)^n)$.

*Proof of the lower bound in Theorem* 1.1. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be any family of $m$ sets. Let us consider a random subfamily $\mathcal{F}'$, that is, we select every set in $\mathcal{F}$ independently with probability $p$. Let $X$ be the random variable denoting the number of sets in $\mathcal{F}'$, and let $Y$ be the random variable denoting the number of subfamilies in $\mathcal{F}'$ forming a Boolean algebra of dimension $d$. By Corollary 2.2,

$$\mathbb{E}(X - Y) \ge mp - p^{2^d} \binom{m}{\lceil \log_2(d+2) \rceil}.$$

If we remove a set from each subfamily in $\mathcal{F}'$ forming a Boolean algebra of dimension $d$, then we obtain a $B_d$-free subfamily $\mathcal{F}''$ of size at least $X - Y$. Substituting $p = m^{-h_d}$ where $h_d = \frac{\lceil \log(d+2) \rceil - 1}{2^d - 1}$ yields the lower bound. To get a better constant in the case $d = 2$, put $p = 2^{-1/3} m^{-1/3}$. ∎

In the case $d = 2$, one might try to improve the constant of the lower bound by improving Corollary 2.2 for families without large chains and antichains. However, the construction of Erdős and Shelah shows one cannot hope for anything better than $(\frac{1}{2} + o(1))\binom{m}{2}$, which would improve the constant of the lower bound in (2) only to 3/4.

**3. Upper bound using Turán theory.** In this section we prove the upper bounds in Theorem 1.1 by generalizing the ideas of Erdős and Shelah [5].

Let $\mathcal{K}(a_1, \ldots, a_d)$ denote the complete, $d$-partite hypergraph with parts of sizes $a_1, \ldots, a_d$, i.e., $V(\mathcal{K}) := X_1 \cup \cdots \cup X_d$, where $X_1, \ldots, X_d$ are pairwise disjoint sets with $|X_i| = a_i$, and $E(\mathcal{K}) := \{E : |E| = d,\ |X_i \cap E| = 1 \text{ for all } i \in [d]\}$. For simplicity we use $\mathcal{K}_d^{(k)}$ for $\mathcal{K}(k, k^2, \ldots, k^{2^{d-1}})$ and $K_{d*2}$ for $\mathcal{K}(2, \ldots, 2)$. The (generalized) *Turán number* of the $d$-uniform hypergraph $\mathcal{H}$ with respect to the other hypergraph $\mathcal{G}$, denoted by $\mathrm{ex}(\mathcal{G}, \mathcal{H})$, is the size of the largest $\mathcal{H}$-free subhypergraph of $\mathcal{G}$.

THEOREM 3.1. *For* $k, d \geq 2$, $\mathrm{ex}(\mathcal{K}_d^{(k)}, K_{d*2}) < \left(2 - \frac{1}{2^{d-1}}\right) k^{2^d-2}$.

*Proof.* We proceed by induction on $d$. When $d = 2$, let $H$ be a $K_{2,2}$-free subgraph of $K_{k,k^2}$. Let $v_1, v_2, \dots, v_{k^2}$ be the vertices of the larger part of $K_{k,k^2}$, and $d_i := \deg_H(v_i)$. Each pair of vertices in the smaller part of $K_{k,k^2}$ has at most one common neighbor in $H$. Therefore, $\sum \binom{d_i}{2} \leq \binom{k}{2}$. This yields

$$|E(H)| = \sum_{i=1}^{k^2} d_i \leq \sum_{i=1}^{k^2} \left( \binom{d_i}{2} + 1 \right) \leq \binom{k}{2} + k^2.$$

Fix $d$, $d > 2$, and a $K_{d*2}$-free subhypergraph $\mathcal{H}$ of $\mathcal{K}_d^{(k)}$. Let $v_i$ for $1 \leq i \leq k^{2^{d-1}}$ be the vertices of the largest part of $\mathcal{K}_d^{(k)}$, and $d_i := \deg_{\mathcal{H}}(v_i)$. Let $\mathcal{H}_i$ be the $(d-1)$-uniform $(d-1)$-partite hypergraph which we get by taking the set of edges of $\mathcal{H}$ containing $v_i$ and deleting $v_i$ from all of them. We have $|\mathcal{H}_i| = d_i$. The hypergraph $\mathcal{H}_i$ contains at least $d_i - \mathrm{ex}(\mathcal{K}_{d-1}^{(k)}, K_{(d-1)*2})$ copies of $K_{(d-1)*2}$. Since $\mathcal{H}$ is $K_{d*2}$-free, each copy of $K_{(d-1)*2}$ belongs to no more than one of the hypergraphs $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_{k^{2^{d-1}}}$. This implies

$$\sum_{i=1}^{k^{2^{d-1}}} \left[ d_i - \left(2 - \frac{1}{2^{d-2}}\right) k^{2^{d-1}-2} \right] \leq \binom{k}{2}\binom{k^2}{2} \cdots \binom{k^{2^{d-2}}}{2} < \frac{k^{2(2^{d-1}-1)}}{2^{d-1}},$$

and the claim follows by rearranging the inequality. ∎

*Proof of the upper bound in Theorem* 1.1. For $m = k^{2^d-1}$ we define a family $\mathcal{F}$ of size $m$ such that every subfamily $\mathcal{F}'$ avoiding $B_d$ has size at most $2k^{2^d-2}$. Then $f(m, B_d\text{-free}) \leq O(m^{e'_d})$ follows for all $m$ by the monotonicity of $f$.

Let $\mathcal{F}$ be a product of $d$ chains, the $i$th of which has size $k^{2^{i-1}}$, i.e., for $1 \leq i \leq d, 1 \leq j \leq k^{2^{i-1}}$, let $S_j^i$ be sets satisfying

- $|S_j^i| = j$, $S_{j_1}^i \subset S_{j_2}^i$ if $j_1 \leq j_2$,
- $S_{k^{2^{i-1}}}^i \cap S_{k^{2^{j-1}}}^j = \emptyset$ if $i \neq j$, and
- $\mathcal{F} := \{S_{j_1}^1 \cup S_{j_2}^2 \cup \cdots \cup S_{j_d}^d : 1 \leq i \leq d, 1 \leq j_i \leq k^{2^{i-1}}\}$.

Each set in $\mathcal{F}$ corresponds to a hyperedge in $\mathcal{K}_d^{(k)}$, and each copy of $B_d$ in $\mathcal{F}$ corresponds to a copy of $\mathcal{K}_{d*2}$ in $\mathcal{K}_d^{(k)}$. The $B_d$-free subfamilies of $\mathcal{F}$ correspond to $\mathcal{K}_{d*2}$-free subhypergraphs of $\mathcal{K}_d^{(k)}$. The bound in Theorem 3.1 on the size of a $\mathcal{K}_{d*2}$-free subfamily completes the proof. ∎

## 4. Union-free subfamilies.

*Proof of Theorem* 1.2. The lower bound proof of Erdős and Shelah [5] does not seem to work in the general $a$-union-free setting. Our approach is based on Kleitman's proof [9].

Let $\mathcal{F}$ be an arbitrary family of size $m$ and let $\ell$ be the size of a longest chain in it. Split $\mathcal{F}$ according the rank of the sets, $\mathcal{F} = \cup_{1 \leq k \leq \ell} \mathcal{F}_k$. Each $\mathcal{F}_k$ together with a chain of size $k$ with a top member from $\mathcal{F}_k$ form an $a$-union-free subfamily implying $f(\mathcal{F}, a\text{-union free}) \geq |\mathcal{F}_k| + k - 1$ for all $k$. Adding up we have $\ell \times f \geq m + \binom{\ell}{2}$ implying $f(\mathcal{F}, a\text{-union free}) \geq |\mathcal{F}|/\ell + (\ell - 1)/2$. Since the lower bound by Fox, Lee, and Sudakov [6] supersedes ours, we omit the details.

For the proof of the upper bound (3), first we consider the family $\mathcal{F}_{ES}(k)$ of size $k^2$ that Erdős and Shelah [5] used to obtain the upper bound (1) on $f(k^2, 2\text{-union free})$. The family $\mathcal{F}_{ES}$ is a product of two vertex disjoint chains of lengths $k$, that is, given the chains $\emptyset \neq A_1 \subset A_2 \subset \cdots \subset A_k$ and $\emptyset \neq B_1 \subset B_2 \subset \cdots \subset B_k$ with $A_k \cap B_k = \emptyset$, we define $\mathcal{F}_{ES}(k) := \{A_i \cup B_j : 1 \leq i, j \leq k\}$. We have $|\mathcal{F}_{ES}| = k^2$.

LEMMA 4.1. *If $\mathcal{G}$ is an $a$-union-free subfamily of $\mathcal{F}_{ES}(k)$, then*

$$|\mathcal{G}| \leq 2(\lceil \sqrt{a+1} \rceil - 1)k.$$

*Proof.* Associate a point set $P$ of the two-dimensional grid with the family $\mathcal{G}$ by $P := \{(i,j) : A_i \cup B_j \in \mathcal{G}\}$. The rectangle $R(i,j)$ is defined as $R(i,j) := \{(x,y) : 1 \leq x \leq i \text{ and } 1 \leq y \leq j\}$. The set $A_i \cup B_j$ is a union of $a$ distinct members of $\mathcal{G}$ if and only if the rectangle $R = R(i,j)$ contains at least $a$ distinct points apart from $(i,j)$, and at least one of these lies on the top boundary of $R$, i.e., on the segment $[(1,j),(i,j)]$, and at least one on the rightmost column $[(i,1),(i,j)]$.

Construct $P' \subseteq P$ by deleting the bottom $\lceil \sqrt{a+1} \rceil - 1$ elements of $P$ in each column of the grid. Suppose that $P'$ has a row with at least $\lceil \sqrt{a+1} \rceil$ elements, and let $(i,j)$ be the rightmost point. Then $P$ has at least $\lceil \sqrt{a+1} \rceil^2 \geq a+1$ points in the rectangle $R(i,j)$ and points on the top and the rightmost sides, a contradiction. Therefore, $P$ has at most $2(\lceil \sqrt{a+1} \rceil - 1)k$ elements. ∎

Now we are ready to define a family $\mathcal{F}$ of size $qk^2$ such that

$$f(\mathcal{F}, a\text{-union free}) < a - 2 + 2k(\lceil \sqrt{a+1} \rceil - 1) + (2k-1)(q-1). \tag{4}$$

The family $\mathcal{F}$ consists of $q$ *levels*, each of them isomorphic to $\mathcal{F}_{ES}(k)$. For all $1 \leq \ell \leq q$, let $\emptyset \neq A_1^\ell \subset A_2^\ell \subset \cdots \subset A_k^\ell$ and $\emptyset \neq B_1^\ell \subset B_2^\ell \subset \cdots \subset B_k^\ell$ be chains of length $k$ such that the $2q$ top sets $A_k^\ell$ and $B_k^{\ell'}$ are pairwise disjoint. Let us define

$$\mathcal{F}_\ell = \left\{ \bigcup_{s=1}^{\ell-1} (A_k^s \cup B_k^s) \cup A_i^\ell \cup B_j^\ell : 1 \leq i, j \leq k \right\} \text{ and } \mathcal{F} := \bigcup_{\ell=1}^{q} \mathcal{F}_\ell.$$

Observe that $|\mathcal{F}| = m = qk^2$ and indeed each $\mathcal{F}_\ell$ is isomorphic to $\mathcal{F}_{ES}$. Note that if $\ell < \ell'$ and $F \in \mathcal{F}_\ell, F' \in \mathcal{F}_{\ell'}$, then $F \subset F'$. Let $\mathcal{G}$ be an $a$-union-free subfamily of $\mathcal{F}$ and let us write $\mathcal{G}_\ell = \mathcal{G} \cap \mathcal{F}_\ell$. Let $t$ be the smallest integer with $\sum_{\ell=1}^{t} |\mathcal{G}_\ell| \geq a-2$. If there exists no such $t$, then $|\mathcal{G}| < a-2$, and we are done. The above reasoning proves the first two of the following three statements:

- $\sum_{\ell=1}^{t-1} |\mathcal{G}_\ell| < a-2$, by the definition of $t$.
- $|\mathcal{G}_t| \leq 2(\lceil \sqrt{a+1} \rceil - 1)k$ by Lemma 4.1 since $\mathcal{F}_t$ is isomorphic to $\mathcal{F}_{ES}$.
- The family $\mathcal{G}_\ell$ is 2-union free for each $\ell$ with $t < \ell \leq q$.

To prove the last statement, suppose on the contrary that $G' \cup G'' = G$ for some $G, G', G'' \in \mathcal{G}_\ell$. Pick any $a-2$ sets $G_1, G_2, \ldots, G_{a-2}$ from $\cup_{s=1}^{t} \mathcal{G}_s$, and we have $G = G' \cup G'' \cup G_1 \cup \cdots \cup G_{a-2}$, contradicting $\mathcal{G}$ being $a$-union-free. Therefore $|\mathcal{G}_\ell| \leq 2k-1$ by a slight strengthening of the result of Erdős and Shelah (see [6]). Putting these observations together, using $|\mathcal{G}| = \sum |\mathcal{G}_\ell|$ and $t \geq 1$, we obtain (4). Finally, substituting $q = \lceil \sqrt{a+1} \rceil$ into (4), we have $f(m, a\text{-union free}) \leq a + (4k-1)(q-1)$. With $k = \lceil \sqrt{m/q} \rceil$, a little calculation yields (3). ∎

## 5. Problems, concluding remarks.

CONJECTURE 5.1. *Let $m = 2^n$ and $d \geq 2$. Among all families with $m$ sets, $2^{[n]}$ has the maximum number of subfamilies that form Boolean algebras of dimension $d$.*

In Theorem 3.1 we considered $d$-partite hypergraphs with very uneven part sizes. There is a number of results of this type; see Győri [8]. Also, here the sizes grow exponentially, but one can easily generalize it to other sequences as well.

Concerning $a$-union-free families, we had the modest conjecture

$$\lim_{a\to\infty}\left(\liminf_{m\to\infty}\frac{f(m, a\text{-union free})}{\sqrt{m}}\right)\to\infty, \tag{5}$$

which has been resolved by Fox, Lee, and Sudakov [6]. Knowing their result it is natural to ask the following.

Problem 5.2. *Given $a$, does the limit*

$$\lim_{m\to\infty}\frac{f(m, a\text{-union free})}{a^{1/4}\sqrt{m}}$$

*exist? And if so, what is it?*

If it exists, it must be between 1/3 and 4.

One can improve the coefficient 4 of the factor $a^{1/4}$ in Theorem 1.2 if in section 4 we use different sizes. Namely, we construct $\mathcal{F}$ by using $\mathcal{F}_\ell = \mathcal{F}_{ES}(k_\ell)$, where $k_\ell = k(\frac{b-1}{b-2})^{2(\ell-1)}$ with $b = \lceil\sqrt{a+1}\rceil$. If $q/b$ tends to infinity, we obtain

$$f(m, a\text{-union free}) \le \sqrt{8}a^{1/4}\sqrt{m} + O(a).$$

A family $\mathcal{F}$ is $(a,b)$-*union free* if there are no distinct sets $F_1, F_2 \dots, F_{a+b}$ satisfying $F_1 \cup F_2 \cup \cdots \cup F_a = F_{a+1} \cup \cdots \cup F_{a+b}$. This is another frequently investigated property, especially the (2,2) case; see, e.g., [1, 2]. However, $f(m, (a,b)\text{-free}) = a+b-1$ if $a, b \ge 2$, as shown by the family consisting of all $(m-1)$-subsets of an $m$-set.

Many more problems remain open.

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