



Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta



Completing partial packings of bipartite graphs

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ARTICLE INFO

Article history:

Received 22 October 2010

Available online 19 July 2011

Keywords:

Graph packings

Graph embeddings

Designs

ABSTRACT

Given a bipartite graph H and an integer n , let $f(n; H)$ be the smallest integer such that any set of edge disjoint copies of H on n vertices can be extended to an H -design on at most $n + f(n; H)$ vertices. We establish tight bounds for the growth of $f(n; H)$ as $n \rightarrow \infty$. In particular, we prove the conjecture of Füredi and Lehel (2010) [4] that $f(n; H) = o(n)$. This settles a long-standing open problem.

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1. Introduction

Let H be a simple graph. A *partial H -packing of order n* , or simply *H -packing*, is a set $\mathcal{P} := \{H_1, H_2, \dots, H_m\}$ of edge-disjoint copies of H whose union forms a simple graph on n vertices. We say that an H -packing of order n is *complete* or an *H -design* if the edge sets of H_i , $i = 1, \dots, m$, partition the edge set of the complete graph on n vertices. More generally, we say that a graph G can be *edge-decomposed* into copies of H if G is the union of some H -packing.

A long-standing problem in design theory is to find a way of completing an H -packing into an H -design of a larger size, using as few new vertices as possible. We define $f(n; H)$ to be the smallest integer such that any H -packing on n vertices, can be extended to an H -design on at most $n + f(n; H)$ vertices.

The existence of $f(n; H)$ for any n and H follows from Wilson's theorem [16], see Section 3 for details. Many bounds of the type of $f(n; H) \leq c(H)n$ have been proved for various graphs H by explicit constructions. A (by no means complete) list of references includes Hoffman, Küçükçifçi, Lindner,

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¹ Research supported in part by the Hungarian National Science Foundation OTKA, and by the National Science Foundation under grant NFS DMS 09-01276 ARRA.

Roger, Stinson [8,10–14], Jenkins [9], Bryant, Khodkar and El-Zanati [3]. See also Füredi and Lehel [4] for a survey of these results.

Hilton and Lindner [7] achieved a breakthrough, having proved a sub-linear bound on $f(n; H)$ for a particular H . More precisely, they showed that a C_4 -packing can be completed by adding $O(n^{3/4})$ new vertices.

Füredi and Lehel [4] applied methods from extremal graph theory and managed to find the right order of magnitude for $f(n; C_4)$. They proved that

$$f(n; C_4) = \Theta(\sqrt{n}).$$

This settled the case $H = C_4$ (up to a constant factor) and solved a problem proposed decades ago (see [14]). Based on their theorem, Füredi and Lehel [4] conjectured that for *any* fixed bipartite graph H the packing can be completed by adding $o(n)$ new vertices. Our aim in this article is to give a proof of their conjecture.

Theorem 1. *For every bipartite graph H there is a function $f(n; H) = o(n)$ such that every H -packing of order n can be completed to an H -design on at most $n + f(n; H)$ vertices.*

In fact we determine the asymptotic growth of the function $f(n; H)$ exactly.

To present our main result, we need to define a new property of graphs. We say that a (not necessarily bipartite) graph H is *matching-friendly* if its vertex set $V(H)$ can be partitioned into V_1 and V_2 such that V_2 is an independent set of vertices and the induced graph $H[V_1]$ consists of a non-empty matching and a set of isolated vertices. For example, C_4 is not matching-friendly, but every other cycle is. At the moment we do not know much about how ‘matching-friendly’ is related to other graph properties, see Section 12 for a discussion. The choice of the name ‘matching-friendly’ should become clear in the course of the proof.

Theorem 2. *If H is matching-friendly, then*

$$f(n; H) = \Theta(\text{ex}(n, H)/n).$$

If H is not matching-friendly, then

$$f(n; H) = \Theta(\max\{\text{ex}(n, H)/n, \sqrt{n}\}).$$

Here, as usual, $\text{ex}(n, H)$ stands for the extremal number of H , see next section for its definition.

Theorem 2 applies to all graphs H , not just bipartite ones. However if H is not bipartite, it just states that $f(n; H) = \Theta(n)$. This is rather easy to deduce: take a packing \mathcal{P}_n , whose union consists of two complete graphs on $n/2$ vertices each. Such a packing exists for infinitely many values of n by Wilson’s theorem, to be stated in Section 3. It is not hard to check that \mathcal{P}_n needs $\Omega(n)$ vertices in order to be extended to an H -design. On the other hand, every H -packing can be extended to an H -design by adding $O(n)$ new vertices; this is a consequence of Gustavsson’s theorem, to be stated in Section 3.

Thus from now on we shall assume that H is bipartite. Note that Theorem 2 implies Theorem 1.

2. Notation and basic tools

As usual, we write $|G|$, $e(G)$, $\delta(G)$ and $\Delta(G)$ for the number of vertices, number of edges, minimum degree and maximum degree of a graph G . These quantities will also be used for multigraphs and (multi)hypergraphs. Denote by $N(v)$ the neighbourhood of v , excluding v .

Let K_n and $K_{m,n}$ denote the complete graph on n vertices and the complete bipartite graph with bipartition classes of size m and n . The graph $K_{1,k}$ is also called a k -star. It has a *central vertex* of degree k and k *endvertices* or *leaves* of degree 1.

The *degeneracy* of G is $\text{dg}(G) := \max(\delta(G'))$, where the maximum is taken over all induced non-empty subgraphs G' of G . Suppose that the vertices of G are numbered v_1, v_2, \dots, v_n so that v_i

is a minimum degree vertex of $G_{(i)} := G[v_1, \dots, v_i]$, the subgraph of G induced by the vertices v_1 through v_i , for every $i = 1, 2, \dots, n$. It is easy to see that $\text{dg}(G) = \max \delta(G_{(i)})$. In other words, given a graph G , we can choose an ordering of its vertices such that the (maximum) *downdegree* $\bar{\Delta}(G)$, defined as the maximum of the number of edges from a vertex v_i to vertices v_j , $j < i$, over all $i = 1, 2, \dots, n$, equals $\text{dg}(G)$.

A *transversal* of a graph G (also known as a *vertex-cover*) is a subset U of its vertices such that every edge of G has at least one endpoint in U . In other words, transversals are complements of independent sets. The *transversal number* $\tau(G)$ is the size of the smallest transversal of the graph G .

A graph G not containing H as a (not necessarily induced) subgraph is called *H-free*. Let us denote by $\text{ex}(n, H)$ the *extremal number* for H , i.e. the maximum number of edges of an H -free graph on n vertices. More generally, let $\text{ex}(G, H)$ be the maximum number of edges in an H -free subgraph of G . Then $\text{ex}(n, H) = \text{ex}(K_n, H)$. Also, if $F \subset H$ then $\text{ex}(n, F) \leq \text{ex}(n, H)$.

In our proof of Theorem 2 we shall use the following crude bound on *symmetric Zarankiewicz numbers* $z = z(m, n, s, s) = \text{ex}(K_{m,n}, K_{s,s})$, see for instance [1].

Theorem 3. For all $m, n \geq s$, and $s \geq 1$ we have

$$z(m, n, s, s) \leq 2nm^{1-1/s} + sm.$$

It is a well-known fact that $z(n, n, s, s) \geq 2\text{ex}(n, K_{s,s})$, see [1]. Since every bipartite graph H is a subgraph of $K_{s,s}$ for some s , it follows that an H -free graph G on n vertices has at most $c(H)n^{2-\epsilon(H)}$ edges, where $\epsilon = \epsilon(H)$ is a small positive number. Therefore $\delta(G) \leq cn^{1-\epsilon}$. Furthermore, since a subgraph of an H -free graph is also H -free, we may conclude that $\text{dg}(G) \leq cn^{1-\epsilon}$. A more careful estimate on the degeneracy of an H -free graph is given by the following lemma.

Lemma 4. For every H -free graph G ,

$$\text{dg}(G) \leq \frac{4\text{ex}(n, H)}{n} + 2|H| \leq C_H \frac{\text{ex}(n, H)}{n}.$$

In other words, every H -free graph G of order $m \leq n$ has a vertex of degree at most $C_H \text{ex}(n, H)/n$, where C_H is a constant that depends only on H .

Proof. The second inequality follows from the fact that $\text{ex}(n, H) \geq n/2$ for all graphs H containing more than one edge (the case $e(H) \leq 1$ is trivial), thus we can take $C_H = 4 + 4|H|$.

To prove the first inequality, notice that every H -free graph G on m vertices contains a vertex of degree at most $2e(G)/m \leq 2\text{ex}(m, H)/m$. Hence, it suffices to show that

$$\frac{\text{ex}(m, H)}{m} \leq \frac{2\text{ex}(n, H)}{n} + |H| \quad (2.1)$$

for all $1 \leq m \leq n$. We claim that

$$\lfloor n/m \rfloor \cdot \text{ex}(m, H) - |H| \cdot m \cdot \lfloor n/m \rfloor \leq \text{ex}(n, H) \quad (2.2)$$

for all $1 \leq m \leq n$. It is easy to check that (2.2) implies (2.1), no matter if the left-hand side is positive or not.

To see that (2.2) holds, consider an m -vertex H -free graph G with the maximum number of edges, and take $\lfloor n/m \rfloor$ of its vertex disjoint copies G_1, G_2, \dots . If their union is H -free (e.g., in the case when H is connected) then

$$\lfloor n/m \rfloor \text{ex}(m, H) \leq \text{ex}(m \lfloor n/m \rfloor, H) \leq \text{ex}(n, H).$$

If H is disconnected with components C_1, \dots, C_t then let s be the maximum integer that the graph F_s with components C_1, \dots, C_s appears in G ; by assumption that G is H -free we have $s < t$. Let $G' = G \setminus V(F_s)$, that is remove F_s and all edges adjacent to it from G ; we have deleted at most $m|H|$

edges. Then the graph comprising $\lfloor n/m \rfloor$ vertex disjoint copies of G' is F_{s+1} -free, and therefore H -free as well, which implies (2.2). \square

We shall need two basic facts about graph colouring. Their proofs can be found in any standard textbook on graph theory e.g. [2]. One is the fact that a graph of maximal degree Δ can be $\Delta + 1$ -coloured by a greedy algorithm. We shall apply it in Section 6 in order to construct an edge-colouring of a certain hypergraph. The other theorem we need is Vizing's theorem: a simple graph of maximal degree Δ can be edge-coloured using $\Delta + 1$ -colours or, equivalently, can be decomposed into $\Delta + 1$ matchings.

3. A primer on graph decompositions

In this section we shall state various theorems on graph decompositions that we shall use in the proof of Theorem 2.

Let H be a bipartite simple graph of order d with vertices v_1, v_2, \dots, v_d and let $\deg(v_i)$ denote the degree of v_i . Denote $\gcd(H) = \gcd(\deg(v_1), \dots, \deg(v_d))$. For an H -design of order n to exist we need the following obvious conditions:

$$e(H) \mid \binom{n}{2} \quad \text{and} \quad \gcd(H) \mid (n-1).$$

If these conditions hold we say that n is H -divisible. If n admits an H -design, we call it H -admissible. Wilson [16] proved the following fundamental theorem.

Theorem 5. *There exists an integer n_0 , depending on H , such that every $n > n_0$ that is H -divisible is also H -admissible.*

Wilson's theorem implies that $f(n; H)$ exists for every H and n . Indeed, the union of an H -packing \mathcal{P} on n vertices can be considered as our new 'building block' H' . By Theorem 5 there exists an H' -design \mathcal{P}' for a sufficiently large H' -divisible number n' . By decomposing each copy of H' in \mathcal{P}' into copies of H , we obtain an H -design on n' vertices. Since for a given n there are only finitely many H -packings on n vertices, and each of them can be completed to an H -design as above, $f(n; H)$ is well-defined.

More generally, let us say a graph G is H -divisible if all degrees of G are multiples of $\gcd(H)$ and $e(H) \mid e(G)$.

A very deep and powerful extension of Wilson's theorem was proved by Gustavsson [5].

Theorem 6. *For any digraph D there exist $\epsilon_D > 0$ and $N_D > 0$ such that if G is a digraph satisfying:*

1. $e(G)$ is divisible by $e(D)$;
2. there exist non-negative integers a_{ij} such that

$$\sum_{v_i \in V(D)} a_{ij} d_D^+(v_i) = d_G^+(u_j), \quad \sum_{v_i \in V(D)} a_{ij} d_D^-(v_i) = d_G^-(u_j)$$

for every $u_j \in V(G)$;

3. if there exists $u_1 \vec{u}_2 \in E(G)$ such that $u_2 \vec{u}_1 \notin E(G)$ then there exists $v_1 \vec{v}_2 \in E(D)$ such that $v_2 \vec{v}_1 \notin E(D)$;
4. $|V(G)| \geq N_D$;
5. $\delta^+, \delta^- > (1 - \epsilon_D)|V(G)|$

then G can be written as an edge-disjoint union of copies of D .

Viewing simple graphs G and H as digraphs, by orienting each edge in both directions, the above theorem translates to

Theorem 7. For every H there exist m_0 and ϵ_0 such that every H -divisible graph G on $m > m_0$ vertices with minimum degree at least $(1 - \epsilon_0)m$ can be edge-decomposed into copies of H .

In the proof of Theorem 2 we shall need the analogue of Wilson's theorem for H -packings into complete bipartite graphs $K_{m,n}$, in which case the obvious divisibility conditions are

$$e(H) | mn, \quad \gcd(H) | m \quad \text{and} \quad \gcd(H) | n.$$

Theorem 8. Let H be a bipartite graph. There exists an integer n_0 , depending on H , such that every H -divisible $K_{m,n}$ with $m, n > n_0$ can be edge-decomposed into copies of H .

This was proved by Häggkvist [6] for the case when H is regular, $m = n$, and under stronger divisibility assumptions. However, Häggkvist's proof was given before Gustavsson's theorem. With Theorem 6 at our disposal, we can give a proof of Theorem 8. While it is almost certain that its statement has been well known, we could not find any explicit reference. Thus, we shall give a proof sketch, skipping some technical details.

Proof. First suppose that $m = n$. The graph $K_{n,n}$ on vertices $\{1, \dots, n\}$ and $\{1', \dots, n'\}$ can be thought of as a directed graph with loops on $\{1, \dots, n\}$ by replacing each edge ab' with a directed edge a to b . By embedding H into $K_{n,n}$ avoiding 'vertical' edges, that is edges of type kk' , we can regard H as a directed graph H' without loops. By removing n copies of H from $K_{n,n}$ first, where each copy has exactly one vertical edge (a tedious but very straightforward check shows that this is always possible), we reduce to the case of decomposing a dense digraph G (without loops) into copies of the digraph H' . Here 'dense' means that we must ensure that $\delta^\pm(G) > (1 - \epsilon)n$. The packing of G can be done provided (a) n is large enough; (b) the number of edges is divisible by $e(H)$; and (c) the in- and out-degrees of any vertex of G are representable as a non-negative linear combination of the in- and out-degrees of vertices of H' . This last condition should translate to the assumption that n is divisible by both the gcd of the degrees of the vertices in A and the gcd of the degrees of the vertices in B , where (A, B) is the bipartition of H . (This assumes one wants to pack all the copies of H the same way round. If not, pack $H \cup H^r$ where H^r is H with the bipartition reversed, and possibly remove one extra copy of H initially to ensure that $2e(H)$ divides $e(G)$. Then n needs only be divisible by the $\gcd(H)$.)

So there is an integer n'_0 such that the theorem holds for all $K_{n,n}$ with $n > n'_0$. In fact, the same construction works for $K_{m,n}$ if $n \leq m \leq (1 + \epsilon'(H))n$. To see this, remove some copies of H in order to isolate $m - n$ vertices in the larger partition class, making sure that we do not reduce the degrees of the remaining vertices too much. Having done that, apply the above digraph reduction to the remaining graph, which can be viewed as a subgraph of $K_{n,n}$. Then apply Theorem 6 as above.

Given H -divisible $m, n \geq n_0 = (n'_0)^2$, we can partition both sets $\{1, \dots, m\}$ and $\{1, \dots, n\}$ into subsets of size between n_0 and $(1 + \epsilon')n_0$ each, such that each complete bipartite graph (X, Y) induced on two partition classes $X \subset \{1, \dots, m\}$ and $Y \subset \{1, \dots, n\}$ is H -divisible. Pack every such graph with copies of H as described above. \square

4. Upper bound: outline of the proof

In this section we would like to describe our strategy for proving the upper bound in Theorem 2.

Consider an H -packing $\mathcal{P} = \{H_1, H_2, \dots, H_m\}$ on n vertices. We want to complete it to an H -design by adding few vertices. We consider the *uncovered graph* $G_0 = (K_n) \setminus \bigcup_{i=1, \dots, m} E(H_i)$ i.e., the graph consisting of edges that are not covered by copies of H .

We proceed in three steps:

Step 1: Reducing the transversal. We add some new vertices and all possible edges from those to other vertices. Now we delete an edge-disjoint collection of copies of H from the resulting graph, so that the resulting graph has a smaller transversal than the graph we started with. This step constitutes a major part of the proof of Theorem 2 and will be carried out in Sections 5 through 7.

More precisely, in Section 5 we shall construct a ‘nice’ collection of disjoint k -stars on the edges of any given graph G . This construction will be applied in Section 6 to G_0 in order to construct a hypergraph M with a small edge-chromatic number, related to G_0 . Then in Section 7 we shall use M and its edge-colouring in order to extend \mathcal{P} to a packing on a larger vertex set, such that the uncovered graph has a small transversal.

In Section 8 we shall describe how we iterate Step 1 in order to obtain further packings with yet smaller transversals of the uncovered graphs.

Step 2: Decreasing the number of uncovered edges. Starting with an uncovered graph G_1 that has a small transversal we extend the new packing to obtain a new uncovered graph G_2 with very few edges. This will be established in Section 9.

Step 3: Completing the packing. This will be done by applying Theorem 7 and Theorem 8 in Section 10.

5. Degeneracy

The aim of this section is to prove Proposition 9: this will be our main tool for reducing the transversal of the uncovered graph. We also believe that the statement of Proposition 9 is interesting in its own right; see Section 12 for related questions.

Recall that a k -star is a copy of $K_{1,k}$.

Proposition 9. *For every integer k and a graph G of degeneracy d there is a maximal collection \mathcal{C} of edge disjoint k -stars on G such that each vertex of G is an endvertex to at most $d + k - 1$ stars in \mathcal{C} .*

Case $k = 2$ was proved by Füredi and Lehel [4]. We are following their approach, using downdegree instead of updegree since this feels more natural to us. Let us choose an ordering v_1, v_2, \dots, v_n of vertices of G such that the downdegree $\bar{\Delta}(G)$, defined in Section 2, equals $d = \text{dg}(G)$.

Let us construct \mathcal{C} as follows: take a maximal collection of edge-disjoint k -stars whose central vertex is smaller in the given ordering than any of its endvertices, and then extend it to a maximal collection of edge-disjoint k -stars. Then $u \in G$ appears as an endvertex of a star of the first kind, or as such endvertex of a star of the second kind which is greater than its centre at most $\bar{\Delta}(G)$ times. It appears as an endvertex smaller than the centre of a star of the second kind at most $k - 1$ times since otherwise we could form a star of the first kind with u at its centre – this is a contradiction as we started taking stars of the second kind in a graph containing no stars of the first kind.

It follows that u can appear at most $\bar{\Delta}(G) + k - 1 = d + k - 1$ times as an endvertex of a star in \mathcal{C} , which proves Proposition 9. Note that the maximality of \mathcal{C} implies $\Delta(G \setminus \bigcup \mathcal{C}) \leq k - 1$.

6. A hypergraph and its colouring

We continue carrying out the plan outlined in Section 4. Recall that we are given an H -packing $\mathcal{P} = \{H_1, H_2, \dots, H_m\}$ on n vertices and $G_0 = (K_n) \setminus \bigcup_{i=1, \dots, m} E(H_i)$ is our uncovered graph.

In this section we shall give a construction of a certain hypergraph M on a vertex set of G_0 along with its edge-colouring; we shall need it in order to extend \mathcal{P} to a packing on a larger set of vertices, in which the uncovered graph will have a small transversal.

First of all, we can assume without loss of generality that G_0 is H -free (by removing a maximal set of edge-disjoint copies of H from G_0). By Lemma 4 we know that $\text{dg}(G_0) = O(\text{ex}(n, H)/n)$.

For a fixed vertex v of H , let $k = \text{deg}(v)$ and $W_1 = N(v)$. Let (U, W) be a bipartition of H such that $v \in U$ and $W_1 \subset W$. Let $s = |U|$ and $t = |W|$ be the sizes of the bipartition classes. For convenience we can assume that $s \geq t$, perhaps choosing another v .

By Proposition 9 there is a collection \mathcal{C} of disjoint k -stars on G_0 with the property that each vertex of G_0 is an endvertex to at most $\text{dg}(G_0) + k - 1$ stars in \mathcal{C} . Define a multi- k -graph (k -uniform hypergraph with several edges on the same set of vertices allowed) called M as follows: for every star of \mathcal{C} there is a k -edge containing precisely the leaves of the star. The maximum degree $\Delta(M)$ (i.e., the maximum number of edges containing any given vertex) is bounded by $\text{dg}(G_0) + k - 1 \leq c_3 \cdot \text{ex}(n, H)/n$, where c_3 is a positive constant depending only on H . We shall denote edges of M by

(c, e) where $c \in G_0$ is the centre of the respective star and e is the hyperedge consisting precisely of the leaves of the star.

Let us introduce an edge-colouring on M so that each colour class forms a vertex-disjoint collection of hyperedges. Since every hyperedge intersects at most $k(\Delta(M) - 1)$ other hyperedges, it can be done, using at most $k(\Delta(M) - 1) + 1 = c_4 \cdot \text{ex}(n, H)/n$ colours: let us colour greedily as many hyperedges with colour 1 as we can, then with colour 2 and so on (again c_4 is a positive constant depending only on H).

Split every colour class i into $R = \lceil |W|/|W_1| \rceil$ (almost) equal parts $i.1$ through $i.R$. For every colour class i , fix a map σ_i which, for every j , takes hyperedges coloured $i.j$ to disjoint $|W_1|(R - 1)$ -subsets of vertices inside the union of hyperedges coloured with one of the colours $i.l$, $l \neq j$. Note that this mapping takes hyperedges into sets which are disjoint from the hyperedge itself.

Now we are ready to extend \mathcal{P} in order to reduce G_0 to a new uncovered graph G_1 that has a new transversal.

7. Construction of a transversal

Write $V = V(G_0)$. We shall prove that, by adding a small set of new vertices Q , we can use up all the edges inside G_0 in edge-disjoint copies of H and end up with a graph G_1 on the vertex set $V \cup Q$ with no edges inside V (i.e., with transversal Q).

The following construction decreases the degrees of the vertices in V below k .

Construction 1. Covering all k -stars. Consider $v \in H$, $k = \deg(v)$, the bipartition $H = (U, W)$ and the colouring of the multihypergraph M as before. For every colour $i.j$ add to G_0 a set $Q^{i.j} = \{q_1^{i.j}, \dots, q_{|U|-1}^{i.j}\}$ of $|U| - 1$ new vertices and place a copy of $H = (U, W)$ in the obvious way on every star (c, e) of colour $i.j$ such that $U = \{c, q_1^{i.j}, \dots, q_{|U|-1}^{i.j}\}$ and $W \subset e \cup \sigma_i(e)$ (if $|W|$ is divisible by $|W_1|$ then we have $W = e \cup \sigma_i(e)$). Note that the sets $e \cup \sigma_i(e)$ for different hyperedges e of colour $i.j$ are pairwise disjoint and so the copies of H are placed edge-disjointly. We needed $O(\text{ex}(n, H)/n)$ new vertices.

The following construction takes care of all the edges within V .

Construction 2. Covering the remaining edges. By Vizing's theorem, the set of remaining edges inside V can be partitioned into (at most) k matchings L_1, \dots, L_k . Consider the smallest r such that $\binom{r}{2} \geq e(H) \frac{n}{2}$ and K_r can be packed completely with copies of H . By Theorem 5 we can pick $r = O(n^{1/2})$. For each matching L_i , add to G_0 a set Q^{L_i} of r new vertices, and pack the copies of H into $K_r \cup L_i$ so that the packing is almost like the complete packing of K_r , except with all edges in L_i covered by an edge from different copies of H . This way we clearly pack copies of H edge-disjointly. Note that $|Q^{L_i}| = O(n^{1/2})$ for every i , so we need $O(n^{1/2})$ new vertices for this construction. Notice that the factor $n/2$ we used for the choice of r reflects the fact that a matching in a graph of order n contains at most $n/2$ edges. If we know that the graph has a transversal of size q , we may replace $n/2$ with q and r becomes $O(q^{1/2})$.

However, if H is matching-friendly, we can do much better. Recall, H is matching-friendly if $V(H)$ can be partitioned into V_1 and V_2 , where V_2 is independent and V_1 is 'almost' independent, i.e., the V_1 -induced subgraph of H is a non-empty matching and some isolated vertices. This implies that we can cover at least one edge of an uncovered matching L_i by adding $|V_2|$ new vertices such that no edge between the new vertices will be used. It follows easily that the whole L_i can be covered using at most a constant number of $c(H)$ new vertices.

Let $Q = \bigcup_{i,j} Q^{i.j} \cup \bigcup_i Q^{L_i}$. We have constructed a graph G_1 on vertex set $V \cup Q$ with transversal Q . By removing copies of H , we can assume that G_1 is H -free. For the number of added vertices we have the bound $|Q| \leq c_5 \cdot \max\{\text{ex}(n, H)/n, \sqrt{n}\}$. If H is matching-friendly, we obtain $|Q| \leq c_6 \cdot \text{ex}(n, H)/n$.

8. Further transversals

We can add some more vertices to G_1 to reduce the transversal number of the resulting graph even further. This procedure can be repeated many times.

It suffices to prove the following lemma.

Lemma 10. *Let G be an H -free graph on n vertices, containing a transversal Q of size $q = o(n)$. Then there is an ordering v_1, \dots, v_n of the vertices of G such that $\tilde{\Delta}(G) \leq Cq^{1-\epsilon}$, where C and $0 < \epsilon = \epsilon(H) < 1$ are constants depending only on H . In particular, $\text{dg}(G) \leq Cq^{1-\epsilon}$.*

Proof. Let us write $Y = V(G) \setminus Q$ and consider the bipartite graph G' with bipartition (Y, Q) , whose edges are the edges of G having precisely one vertex in each of Q and Y . Let $G'' = G[Q]$ be the subgraph of G induced by Q . Then the edge sets of G' and G'' partition the edge set of G .

Since G'' is an H -free graph on q vertices, its degeneracy is at most $c''q^{1-\epsilon}$ for a positive constant c'' depending only on H . Let us fix an ordering u_1, u_2, \dots, u_q of the vertices in Q such that $\tilde{\Delta}(G'') = \text{dg}(G'')$.

Select s and t with $s \geq t$ such that $H \subset K_{s,t} \subset K_{s,s}$ and s is chosen as small as possible. By Theorem 3 we have that

$$z(|Q|, |Y|, s, s) \leq 2|Y||Q|^{1-1/s} + s|Q|.$$

Let $\epsilon \leq 1/s$. We find that

$$\text{ex}(K_{|Q|,|Y|}, H) \leq \text{ex}(K_{|Q|,|Y|}, K_{s,s}) = z(|Q|, |Y|, s, s) \leq 2|Y||Q|^{1-1/s} + s|Q|.$$

Therefore, as long as $|Y| \geq q^{1/s}$, the minimal degree in Y satisfies $\delta(Y) = O(q^{1-1/s})$.

Let v_1 be a vertex of Y of smallest possible degree in the graph G' , let v_2 be a vertex of Y of minimal degree in $G'[V(G) \setminus \{v_1\}]$, take v_3 to be a vertex of Y of minimal degree in $G'[V(G) \setminus \{v_1, v_2\}]$ and so on, until v_r , where $r = |Y| - q^{1/s}$. Each of those degrees is $O(q^{1-\epsilon})$, by the previous paragraph. Let $v_{r+1}, v_{r+2}, \dots, v_{n-q}$ be the remaining vertices in Y .

Define the ordering $v_{r+1}, v_{r+2}, \dots, v_{n-q}, u_1, u_2, \dots, u_q, v_1, v_2, \dots, v_r$. It follows from the construction that $\tilde{\Delta}(G') \leq c'q^{1-\epsilon(H)}$. \square

The lemma allows us to iterate the construction of Sections 6 and 7. An H -free uncovered graph with a transversal of size q has by Lemma 10 degeneracy $Cq^{1-\epsilon}$. Hence we can define a hypergraph as in Section 6 and use it in order to construct a new packing as in Section 7. The number of new vertices needed in Construction 1 will be $O(q^{1-\epsilon})$ and in Construction 2 of Section 7 each matching has cardinality at most q , so we need to add a set Q^{L_i} of $O(q^{1/2})$ additional vertices for every matching L_i . Hence, the total number of new vertices will be at most $C(H)q^{1-\epsilon(H)}$. By construction, this set of vertices will be a transversal of the new packing, so we can just repeat the procedure, using the new transversal. We iterate as long as $Cq^{1-\epsilon} \leq q/2$, that is $q \geq C'(H) = (2C)^{1/\epsilon}$. The number of new vertices halves after each step, thus by adding $O(\max\{\text{ex}(n, H)/n, \sqrt{n}\})$ new vertices, or $O(\text{ex}(n, H)/n)$ if H is matching-friendly, we can make the transversal smaller than the constant $C'(H)$.

9. Decreasing the number of uncovered edges

Our next objective is to reduce the number of uncovered edges.

Lemma 11. *Given a partial H -packing \mathcal{P}_2 whose uncovered graph G_2 has a transversal Q of size $|Q| < C'(H)$, we can add a constant number of vertices and remove some copies of H to obtain a new packing \mathcal{P}_3 that leaves constantly many edges uncovered. In addition, the number of vertices in \mathcal{P}_3 will be congruent 1 modulo $e(H)$.*

Proof. Let $Y = V(G_2) \setminus Q$ and $g = \gcd(H)$. By adding a few new vertices to Q we may also assume that $|G_2| \equiv 1 \pmod{e(H)}$. Since G_2 is the complement of a partial packing and $g|e(H)$ (because H is

bipartite), all degrees in G_2 must be multiples of g . This implies that every vertex in Y is either isolated or has at least g neighbours in Q . We shall add a set Z of new vertices of size $m|Q|^g$ and remove some copies of H from the resulting graph, in order to reduce G_2 to a new uncovered graph G_3 in which every subset of vertices of Q of size g has at most m common neighbours in Y and every vertex in Y has either none or at least g neighbours in Q . That would bound the number of edges between Y and Q by $m|Q|^g$. In addition every vertex from Z will have at most m uncovered edges in Y incident with it. Then G_3 would have at most $m|Q|^g + m|Z| + 1/2(|Z| + |Q|)^2 = C''(H)$ edges.

Let $m = 2n_0$, where n_0 is a multiple of $e(H)$ that satisfies Theorem 8 for H , that is any H -divisible complete bipartite graph with at least n_0 vertices in each partition class can be edge-decomposed into copies of H .

Let us pick a set $K = \{q_1, q_2, \dots, q_g\}$ of some g vertices in Q and write N for their common neighbourhood in Y : $N = N(q_1) \cap N(q_2) \cap \dots \cap N(q_g) \cap Y$. If $|N| > m$, we are going to add to G_2 an additional set $Q_{q_1, \dots, q_g}^* = Q^* = \{q_1^*, q_2^*, \dots, q_m^*\}$ of m vertices. If $|N| \leq m$, we just pick the next K .

We are going to cover almost all the edges in the complete bipartite graphs $(K \cup Q^*, N)$ and $(Q^*, Y \setminus N)$. Since $|Q^*|$ and $|K \cup Q^*|$ are both divisible by g , to make those graphs H -divisible, it suffices to omit less than $e(H)$ vertices from each of the sets N and $Y \setminus N$ – so that we obtain respectively sets N' and Y' . By Theorem 8 it follows that both complete bipartite graphs $(K \cup Q^*, N')$ and (Q^*, Y') can be packed completely with edge-disjoint copies of H .

The uncovered graph has obtained m new vertices, each of which has at most m (in fact at most $2e(H)$) uncovered edges into Y and the vertices in Q^* have now at most m common neighbours inside Y . Also, for each vertex in Y , the number of its remaining neighbours in Q is a multiple of g .

If we repeat the procedure for all possible sets $K \subset Q$ of size g , we obtain the desired graph G_3 , taking Z to be the union over all K . Notice also that by adding m vertices at a time, we make sure that $|G_3| \equiv 1 \pmod{e(H)}$. \square

10. Completing the packing

We shall now apply Theorems 7 and 8 to complete the packing. Since the uncovered graph G_3 has a constant number of edges, the number of non-isolated vertices in it is also constant. Let Q be a set of vertices of constant size such that all vertices in $Y = G_3 \setminus Q$ are isolated and $|Y| \equiv 0 \pmod{e(H)}$; hence also $|Y| \equiv 0 \pmod{g}$, where g is the greatest common divisor of all degrees in H , as before. By the construction in the previous section we may assume that $|Q| + |Y| = |G_3| \equiv 1 \pmod{e(H)}$, thus $|Q| \equiv 1 \pmod{e(H)}$.

We now apply Theorem 7 to $G_3[Q]$ to extend the packing by adding a set X of few new vertices. More precisely, we pick X to be a set of new vertices of size $\max\{m_0, (1/\epsilon_0)|Q|\}$, where m_0 and ϵ_0 are as in Theorem 7, this is a constant of H . Also let $|X| \equiv |Y| + |Q| - 1 \pmod{2e(H)}$. To complete the packing it suffices to make sure that the uncovered graph on $Q \cup X$ and the complete bipartite graph $K_{X,Y}$ are H -divisible.

One divisibility condition requires $|X| + |Q| \equiv 1 \pmod{g}$ for the former graph and $|X|, |Y| \equiv 0 \pmod{g}$ for the latter. Both conditions are satisfied since $|X| \equiv 0 \pmod{g}$.

The other divisibility condition requires the number of edges in each graph to be divisible by $e(H)$. This is certainly true for $K_{X,Y}$, by the choice of Y . So we only need to make sure that $e(H)$ divides the number of edges of the uncovered graph on $Q \cup X$, in other words

$$e(H) \mid \left(\binom{|X| + |Q|}{2} - \binom{|Q|}{2} + e(G_3) \right).$$

Since G_3 is the complement of an H -packing, we know that

$$e(G_3) \equiv \binom{|Q| + |Y|}{2} \pmod{e(H)}.$$

Therefore we need $e(H)$ to divide

$$\binom{|X| + |Q|}{2} - \binom{|Q|}{2} + \binom{|Q| + |Y|}{2} = \binom{|X| + |Y| + |Q|}{2} - |X||Y|.$$

This is true whenever $|X| \equiv |Y| + |Q| - 1 \pmod{2e(H)}$.

Hence we can satisfy all divisibility conditions in order to apply Theorems 7 and 8 to complete the packing. This finishes the proof of the upper bound in Theorem 2.

11. Lower bound

In this section we want to show the existence of H -packings that need $\Omega(\text{ex}(n, H)/n)$ vertices in order to be completed. If H is not matching-friendly, there exist also packings that need $\Omega(\sqrt{n})$ new vertices.

Let us start with the second claim. If H is not matching-friendly, we need $\Omega(\sqrt{n})$ new vertices in order to cover the edges of a complete matching L on n vertices. Indeed, any time we place a copy of H that covers at least one edge of L , we must use an edge between two new vertices (otherwise H would be matching-friendly). Hence, in order to cover $n/2$ edges of L we need about \sqrt{n} new vertices.

Now we have to make sure that the complement of a perfect matching is the union of an H -packing for infinitely many n . Take two disjoint copies of H and view their union H' as a bipartite graph with equal partition classes, i.e. one copy of H is 'upside down'. Let s be the size of the partition classes. By Theorem 5, if n is sufficiently large, there is a complete packing \mathcal{P} of K_n with copies of H' . Now take two identical copies of \mathcal{P} , one on $\{a_1, a_2, \dots, a_n\}$ and another on $\{b_1, b_2, \dots, b_n\}$ and add a copy of H' between a_{i_1}, \dots, a_{i_s} and b_{j_1}, \dots, b_{j_s} and another one between a_{j_1}, \dots, a_{j_s} and b_{i_1}, \dots, b_{i_s} for each copy of H' in \mathcal{P} between a_{i_1}, \dots, a_{i_s} and a_{j_1}, \dots, a_{j_s} , in the obvious way. We obtain a packing on $2n$ vertices, whose union is the complement of a matching between vertices a_i and b_i .

Now let us prove the first claim. Suppose we have found an H -packing \mathcal{P} , whose complement is an H -free graph with about $\text{ex}(n, H)$ edges. In order to cover each edge of it, every copy of H would use at least one out of $kn + \binom{k}{2} = (1 + o(1))kn$ new edges, where k is the number of new vertices. Since we need at least $\text{ex}(n, H)/e(H)$ copies of H to cover all edges of the uncovered graph, we must have $k = \Omega(\text{ex}(n, H)/n)$.

Hence, it remains to prove that such a packing \mathcal{P} exists for arbitrarily large values of n . Take an (extremal) H -free graph \bar{G} on n vertices with $\text{ex}(n, H)$ edges. We would like to remove a small proportion of edges from \bar{G} in order to make the complement of the remaining graph satisfy the conditions of Theorem 7. This would ensure the existence of the desired packing.

Let us first eliminate vertices of high degree. Suppose \bar{G} has $\log n$ vertices of degree at least $\epsilon_0 n$, where ϵ_0 is as in Theorem 7. Then by Theorem 3, for a sufficiently large n the bipartite graph between $m = \log n$ such vertices and the rest of \bar{G} contains $K_{s,s} \supset H$, contradicting the assumption that \bar{G} is H -free. It follows that \bar{G} has less than $\log n$ vertices of degree at least $\epsilon_0 n$. Removing them, we lose at most $n \log n$ edges obtaining (unless H is a forest, in which case $\text{ex}(n, H)/n$ is a constant, thus the first claim is trivial) a new H -free graph \bar{G}' with $(1 - o(1))\text{ex}(n, H)$ edges and no vertices of high degree.

Next we would like to remove a few more edges from \bar{G}' in order to fulfil the divisibility conditions. A theorem of Pyber [15] states that a graph F that has at least $n \log n \cdot 32r^2$ edges contains a (not necessarily spanning) r -regular subgraph. Let us set $r = 2e(H)$. Remove edge sets of r -regular subgraphs $G_1 \subset \bar{G}'$, $G_2 \subset \bar{G}' \setminus G_1$ etc. until the remaining graph $\bar{G}'' \setminus (G_1 \cup G_2 \cup \dots \cup G_k)$ has less than $n \log n \cdot 32r^2$ edges. Then the graph $\bar{G}'' = G_1 \cup G_2 \cup \dots \cup G_k$ satisfies all conditions of Theorem 7 and still has about $\text{ex}(n, H)$ edges, whence we obtain the desired packing \mathcal{P} .

12. Outlook

There is a simple sufficient condition for a graph H to be *not* matching friendly: H cannot be matching-friendly if every edge of it is contained in a 4-cycle. However, in this case, since $C_4 \subset H$ and $\text{ex}(n, C_4) = \Theta(n^{3/2})$, we obtain $\text{ex}(n, H)/n = \Omega(n^{1/2})$, thus being not matching-friendly does

not matter, as far as Theorem 2 is concerned. There are examples of bipartite graphs that are not matching-friendly and C_4 -free; take for instance C_8 and connect the opposite pairs of vertices by paths of length 2. Or, alternatively, take the incidence graph of the Fano plane. However, we do not know much about the extremal numbers of such graphs, so the question is: *does ‘matching-friendly’ ever make a difference?* In other words, is it always true that $\text{ex}(n, H) = \Omega(n^{3/2})$ for a non-matching-friendly graph H ? If this is indeed the case, then the statement of Theorem 2 would simplify to $f(n; H) = \Theta(\text{ex}(n, H)/n)$ for all graphs H .

The constant C_H in the proof of Lemma 4 depends on H only when H is a disconnected forest. Is it possible to prove Lemma 4 with an absolute constant, perhaps even $C_H = 2 + o(1)$?

We believe that Proposition 9 is an interesting statement in its own right and would like to know how tight the bound of $d + k - 1$ is. Apart from the obvious double counting argument that gives a lower bound of $d/2$ for regular graphs, there is not much we know about it.

The following question was also inspired by Proposition 9.

Conjecture 12. *For every integer k and a graph G of degeneracy d there is a maximal collection \mathcal{C} of edge disjoint paths of length $2k$ on G such that each vertex of G is an endvertex to at most $c_k d$ paths in \mathcal{C} .*

This cannot hold for odd-length paths, as can be seen by taking, for instance paths of length 3 and $G = K_{2,m}$, where m is large. Case $k = 1$ of Conjecture 12 is the special case of Proposition 9; it was first proved by Füredi and Lehel [4]. It seems likely that using an elaboration of the method of proof of Proposition 9, one can also prove Conjecture 12 for $k = 2$ and $k = 3$. However, for $k \geq 4$ one would probably need a genuinely different approach.

More generally, can the $2k$ -path in the statement of Conjecture 12 (or Proposition 9) be replaced by a tree, in which all distances between the leaves are even?

Acknowledgments

The authors would like to thank Jenő Lehel for introducing the problem and Paul Balister and Béla Bollobás for many useful discussions.

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