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COVERS FOR CLOSED CURVES OF LENGTH TWO

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Abstract

The least area α_2 of a convex set in the plane large enough to contain a congruent copy of every closed curve of length two lies between 0.385 and 0.491, as has been known for more than 38 years. We improve these bounds by showing that $0.386 < \alpha_2 < 0.449$.

1. Covers in the plane

Let $\mathcal{G}$ be a transitive group of motions of the plane and $\mathcal{F}$ a family of figures in the plane (by motion we mean congruence). A set X is a $\mathcal{G}$*-cover* for $\mathcal{F}$ if for each F in $\mathcal{F}$ there is a motion μ in $\mathcal{G}$ so that $\mu(F) \subset X$. For given families of figures, interest lies in covers, usually but not always convex, that are small in some specific sense (measure, perimeter, width, etc.). Problems of determining such covers, sometimes of prescribed shapes, are called *worm problems* for the family. Many such problems can be found in the literature, but few have been solved.

For example, Besicovitch and Rado [3] and Kinney [11] have constructed closed plane sets of measure zero containing a circle of every radius. Besicovitch

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[1], [2], constructed a closed plane set of measure zero that is a translation cover for the family of all unit line segments, and Ward [22], Davies [6], and Marstrand [14] showed more generally that there is such a cover for the family of unions of all finite sets of lines. In contrast, Marstrand [15] showed that every set large enough to contain a congruent copy of every unit arc must have positive measure. Concerning triangular covers of triangles, see Post [19], Kovalev [12], and the present authors [9]. One of the nicest results is due to K. Bezdek and Connelly [4], who showed that each plane convex body of constant width one is a translation cover for the family of all plane closed curves of length two. The smallest area of these is attained by the Reuleaux triangle, whose area is about 0.705.

Our interest is in bounds for the least area α_2 of convex covers for the family $\mathcal{C}$ of all closed curves in the plane of length at most two, i.e., in bounds for $\alpha_2 = \inf\{\text{area}(X)\colon X$ is convex and contains a congruent copy of every closed curve of length at most two$\}$. The value of α_2 is not known, but the bounds

$$0.385\,31 < \alpha_2 < 0.490\,95 \tag{1}$$

have been known for more than 35 years ([5], [20]). In [9] we mentioned in passing that we can improve these bounds a little to $0.386\,67 < \alpha_2 < 0.470\,16$. Here we improve both of these bounds.

Theorem 1. *The value of* α_2 *lies in the interval*

$$0.386\,778 < \alpha_2 < 0.448\,504. \tag{2}$$

We give three increasing lower bounds for α_2 (see (4), (5) and (6)) and five decreasing upper bounds (see (7), (8), (14), (15), and (18)). We conclude with an overview for the analogous question in $\mathbb{R}^d$.

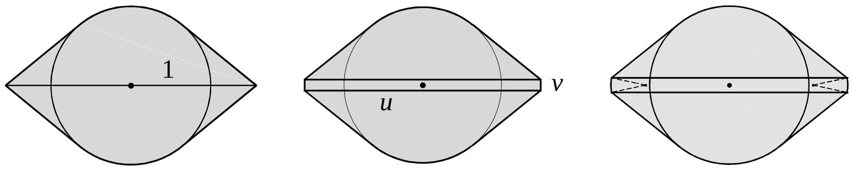
2. Lower bounds

We need the following theorem of Fáry and Rédei [7]: Let K and L be centrally symmetric convex bodies having the same center, and let $\mathbf{v}$ be a fixed vector. Then

$$\text{vol}(\text{conv}(K \cup (L + \mathbf{v}))) \geq \text{vol}(\text{conv}(K \cup L)). \tag{3}$$

Any convex cover for the family $\mathcal{C}$ of all closed curves in the plane of length two must contain a circle C_0 of radius $1/\pi$ and a line segment I of unit length. It follows from (3) that the arrangement of the circle and the line segment whose convex hull has least area is the one with the midpoint of I at the center of C_0 (Figure 1a). The previous best lower bound is the area of this hull:

$$\alpha_2 \geq \frac{1}{\pi^2}\Big(\pi + \sqrt{\pi^2 - 4} - 2\arccos\frac{2}{\pi}\Big) > 0.385\,318. \tag{4}$$



a. Good b. Better c. Best

FIGURE 1. Lower bounds

One can do better by replacing the unit line segment by a $u \times v$ rectangle with $u + v = 1$, $0 \leq v \leq \frac{1}{2}$ (Figure 1b). Again the hull of least area occurs when the rectangle and the disk have the same center, and the area of their convex hull is

$$f(v) = \frac{1}{2}v(1-v) + \frac{1}{\pi}\sqrt{2v^2 - 2v - \frac{4}{\pi^2} + 1} + \\ + \frac{1}{\pi} - \frac{2}{\pi^2}\left(\arctan\frac{v}{1-v} + \arccos\frac{2}{\pi\sqrt{2v^2-2v+1}}\right).$$

The function $f(v)$ on the interval $[0, \frac{1}{2}]$ has a unique maximum greater than 0.386 675 at the point $v \approx 0.052\,337\,2$. Hence

$$\alpha_2 \geq f(0.052\,337\,2) > 0.386\,675. \tag{5}$$

A further slight improvement can be achieved as follows. Let γ_1 be the curvilinear rectangle of perimeter two formed by two parallel line segments of equal length and two symmetrically located circular arcs (Figure 1c). The convex hull of least area spanned by the disk C_0 and this curvilinear rectangle γ_1 occurs when they are concentric.

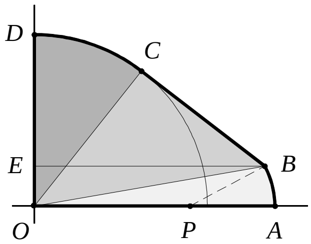


FIGURE 2. Slight improvement

Figure 2 shows one quarter of this convex hull, with P and r the center and radius of the circular end, respectively. Let $x = |OP|$ and $\theta = \angle APB$. Then $|EB| = x + r\cos\theta$ and $r\theta + |EB| = \frac{1}{2}$, and it follows that

$$r = \frac{\frac{1}{2} - x}{\theta + \cos\theta}.$$

Then

$$\begin{aligned} s &= |OB| = \sqrt{x^2 + 2xr\cos\theta + r^2}, \\ \alpha &= \angle AOB = \arctan\frac{r\sin\theta}{x + r\cos\theta}, \\ \beta &= \angle BOC = \arccos\frac{1}{\pi s}, \\ \varphi &= \angle COD = 1/2\pi - \alpha - \beta. \end{aligned}$$

Consequently,

$$\begin{aligned} \text{area}(ABO) &= 1/2 rx\sin\theta + 1/2 r^2\theta, \\ \text{area}(BCO) &= \frac{|BC|}{2\pi} = \frac{1}{2\pi}\sqrt{s^2 - \frac{1}{\pi^2}}, \\ \text{area}(COD) &= \frac{\varphi}{2\pi^2}. \end{aligned}$$

Finally, define

$$f(x,\theta) = 4\left(\text{area}(ABO) + \text{area}(BCO) + \text{area}(COD)\right).$$

Then

$$\alpha_2 \ge f(x,\theta) > f(0.335, 0.212) > 0.386\,778 \tag{6}$$

(we omit the computational details). Figure 1c shows the region when $x = 0.335$ and $\theta = 0.212$, about 12.1°.

One can easily see that if γ in $\mathcal{C}$ is centrally symmetric about the center of C_0, then the area of $\text{conv}(\gamma \cup C_0)$ is maximized when γ is the curvilinear rectangle γ_1.

3. Upper bounds

We turn next to a series of five decreasing upper bounds.

3.1. The smallest rectangular cover

It follows from Cauchy's formula that a closed *convex* curve γ of length ℓ has parallel support lines ℓ/π apart (see [5], [20]); and for $\ell = 2$ this implies that an $l \times w$ rectangular region $\mathbf{R} = \text{conv}\{A, B, C, D\}$ with

$$\begin{aligned} |AB| = |CD| = l &= \frac{1}{\pi}\sqrt{\pi^2 - 4} \approx 0.771\,178, \\ |BC| = |DA| = w &= \frac{2}{\pi} \approx 0.636\,620 \end{aligned}$$

(Figure 3) is a cover for the family of all closed convex curves of length two, by [20, Lemma 1]. Consequently it is a cover for $\mathcal{C}$, because the boundary curve of the convex hull of a closed curve is no longer than the curve.

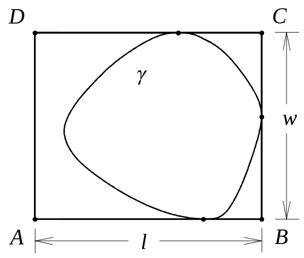


FIGURE 3. The smallest rectangular cover.

No smaller rectangle can be a cover for $\mathcal{C}$ because both a circle of diameter $2/\pi$ and a line segment of unit length must be accommodated. The area of this smallest rectangular cover $\mathbf{R}$,

$$lw = \frac{2}{\pi^2}\sqrt{\pi^2 - 4} < 0.490\,948, \tag{7}$$

is the known upper bound (1).

3.2. A covering pentagon

A curve γ that lies in $\mathbf{R}$ and gets close to each of the four corners of $\mathbf{R}$ must have length approaching $2l + 2w \approx 2.815$, which suggests that some region near at least one corner of $\mathbf{R}$ is not really needed. Indeed, suppose an isosceles right triangular region of leg 0.2 is marked in each corner of $\mathbf{R}$.

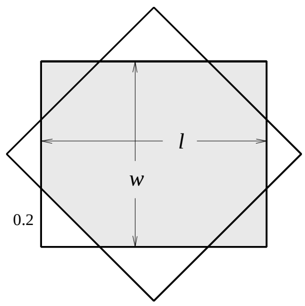


FIGURE 4. A pentagonal cover

The four 45° lines of these isosceles triangles determine a square whose diagonal is $l + w - 0.4$, about $1.007\,798$, and it follows that every path that meets each of these four corner triangles has length greater than 2.015. Thus every closed curve of length two must miss the interior of at least one of these four corner triangles, and by a suitable motion we may arrange for the curve to miss the lower left corner triangle.

Consequently the pentagonal region formed by removing this corner triangle (shaded in Figure 4) is a cover for $\mathcal{C}$ with area $lw - 0.02 < 0.470\,948$. So

$$\alpha_2 < 0.470\,948. \tag{8}$$

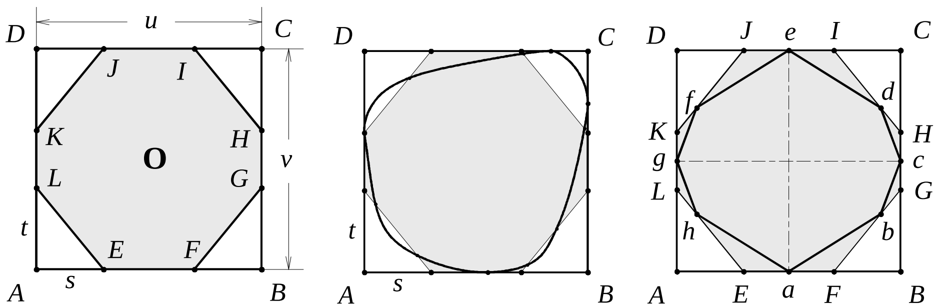
This is already better than the upper bound in (1).

3.3. Eight-point curves in a u × v rectangle R

We examine more closely how closed curves can fit in a clipped rectangle, and we begin by setting some notation.

Notation. We write $|XY|$ for the distance between the points X and Y, $[X, Y]$ for the closed segment with endpoints X and Y, (X, Y) for the open segment with endpoints X and Y excluded, $[X, Y)$ for the segment with endpoint X included and endpoint Y excluded, $[X, Y\rangle$ for the ray with endpoint X (included) through the point Y, and $\langle XY\rangle$ for the line determined by X and Y. And we write $\partial(S)$ for the boundary of a plane set S.

Fix positive reals u and v. Suppose that $\mathbf{R} = \operatorname{conv}\{A, B, C, D\}$ is a $u \times v$ rectangle, with $|AB| = |CD| = u$ and $|BC| = |DA| = v$. Take reals s and t with $0 < s < \frac{1}{2}u$, and $0 < t < \frac{1}{2}v$, and mark congruent right triangles with legs s, t symmetrically placed at each corner of $\mathbf{R}$. Let E, F, G, H, I, J, K, L be the points on the sides of $\mathbf{R}$ so that $|AE| = |BF| = |CI| = |DJ| = s$ and $|DK| = |AL| = |BG| = |CH| = t$ (Figure 5a). Then right triangles AEL, BFG, CIH, and DJK are congruent and disjoint. Let $\mathbf{O} = \operatorname{conv}\{E, F, G, H, I, J, K, L\}$ be the closed central octagonal region remaining when the corner triangles are removed from $\mathbf{R}$.



a. The octagonal region **O** b. An eight-point curve c. A minimal curve

Figure 5. Eight-point curves

DEFINITION. A closed curve γ in the rectangle R with marked corner right triangles is an eight-point curve if it meets each side, $[A,B]$, $[B,C]$, $[C,D]$, $[D,A]$, of R and each hypotenuse, $[L,E]$, $[F,G]$, $[H,I]$, and $[J,K]$, of the corner triangles. Let $\mathcal{F}$ be the family of all eight-point curves, and let $\mathcal{F}_{\min}$ be the subfamily of $\mathcal{F}$ formed by the eight-point curves whose length is as small as possible.

Note that an eight-point curve is *not* assumed to be convex. A typical (albeit convex) eight-point curve is shown in Figure 5b.

LEMMA 2. *The family* $\mathcal{F}_{\min}$ *is not empty, and every minimal eight-point curve is a convex polygon having at most eight vertices.*

PROOF. If an eight-point curve is not convex, then the boundary curve of its convex hull is a shorter eight-point curve. If γ is a minimal eight-point curve, then there are points a, b, c, d, e, f, g, and h of γ so that $a \in [A,B]$, $b \in [F,G]$, $c \in [B,C]$, $d \in [H,I]$, $e \in [C,D]$, $f \in [J,K]$, $g \in [D,A]$, and $h \in [L,E]$. The polygon γ' having these points in their order on γ as vertices is no longer than γ, and $\gamma'' = \partial\,(\mathrm{conv}(\gamma'))$ is convex and still shorter than γ, unless $\gamma'' = \gamma'$. Finally, compactness implies that among all curves of the form γ'' there is at least one whose length is as small as possible, so $\mathcal{F}_{\min} \neq \emptyset$.

□

Note that at this point we know only that the n-gon γ has vertices a, c, e, g in $[A,B]$, $[B,C]$, $[C,D]$, and $[D,A]$, respectively, but we cannot assume that they lie in the medial subintervals $[E,F]$, $[G,H]$, $[I,J]$, and $[K,L]$ or that they fall in order along γ. Compare, for example, the eight-point curves in Figures 5b and 5c.

Before investigating the properties of *minimal* eight-point curves we recall two useful elementary geometric facts.

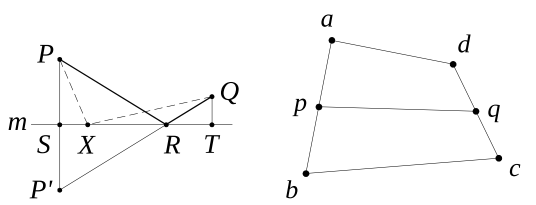
LEMMA 3. a. (The shortest path property) *In the notation of Figure* 6a, *let* S *and* T *be the orthogonal projections on a line* m *of points* P *and* Q (*not both on* m). *The case* $S = T$ *being trivial, we assume that* $S \neq T$. *Let* R *be the point on* $[S,T]$ *so that*

$$|SR| = \frac{|SP|}{|SP| + |TQ|}|ST|.$$

Then

$$|PX| + |XQ| \geq |PR| + |RQ| \tag{9}$$

for every point X *on* m, *with equality precisely when* $X = R$. *Moreover, as* X *moves on* $\langle S,T\rangle$ *in the sense of the vector* $\overrightarrow{ST}$, *the sum* $f(X) = |PX| + |XQ|$ *is strictly decreasing on the ray* $\langle S,R]$ *and strictly increasing on the ray* $[R,T\rangle$. *Further, apart* *from degenerate cases, angles* $\angle SRP$ *and* $\angle QRT$ *are equal.*



a. Shortest path b. Bimedian inequality

FIGURE 6. Two elementary geometric facts

b. (The bimedian inequality) *If* a, b, c, d *are arbitrary points and* p *and* q *are the midpoints of the segments* $[a, b]$ *and* $[c, d]$ *(Figure* 6b*), then*

$$|pq| \leq \frac{|ad| + |bc|}{2}, \tag{10}$$

with equality precisely when $[b, c]$ *and* $[d, a]$ *are parallel and have opposite orientation.*

LEMMA 4. *Every minimal eight-point curve* γ *lies entirely in* $\mathbf{O}$*, and it has the form*

$$\partial(\mathrm{conv}\{a, b, c, d, e, f, g, h\}),$$

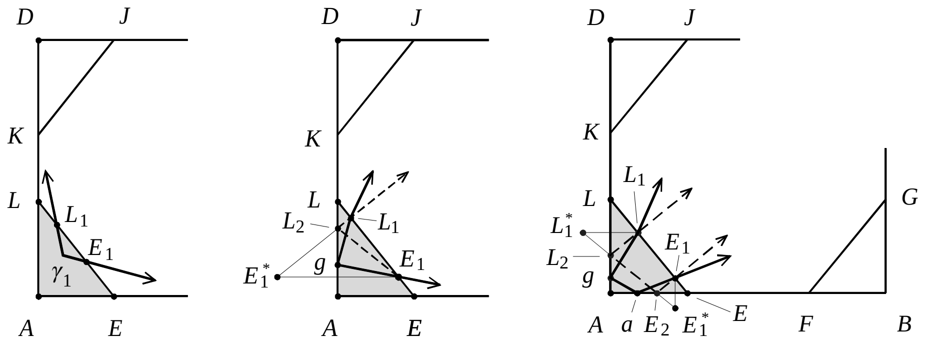
with $a \in [E, F]$*,* $b \in [F, G]$*,* $c \in [G, H]$*,* $d \in [H, I]$*,* $e \in [I, J]$*,* $f \in [J, K]$*,* $g \in [K, L]$*, and* $h \in [L, E]$*. (Some of these eight vertices might coincide.)*

PROOF. Suppose $\gamma \in \mathcal{F}_{\min}$. Then, according to Lemma 2, there are points a, b, c, d, e, f, g, h with $a \in [A, B]$, $b \in [F, G]$, $c \in [B, C]$, $d \in [H, I]$, $e \in [C, D]$, $f \in [J, K]$, $g \in [D, A]$, and $h \in [L, E]$ so that $\gamma = \partial(\mathrm{conv}\{a, b, c, d, e, f, g, h\})$. We show that γ does not meet the interior of any corner right triangle.

Suppose to the contrary that γ enters the triangular region $\Delta = \mathrm{conv}(AEL) \setminus [EL]$. Then there are two distinct points E_1 and L_1 in the order L-L_1-E_1-E on $[L, E]$ so that the closed convex region $\mathrm{conv}(\gamma)$ meets $[L, E]$ in the segment $[L_1, E_1]$. We suppose $L \neq L_1$ and $E_1 \neq E$; the cases with $L = L_1$ or $E = E_1$ are similar and simpler.

Incorporating L_1 and E_1 as vertices into γ, we see that γ is divided into two disjoint parts, a subarc γ_1 (missing its endpoints L_1 and E_1) lying in Δ and an open subarc γ_2 with endpoints E_1 and L_1 that lies across $[E, L]$ in $\mathbf{R} \setminus \Delta$. There are three possibilities.

Case 1. If γ_1 meets neither $[A, E)$ nor $[A, L)$, then replacing γ_1 by the line segment $[L_1, E_1]$ produces a curve in $\mathcal{F}$ that is strictly shorter than γ (Figure 7a), a contradiction.



a. γ_1 interior b. γ_1 meets one leg c. γ_1 meets both legs

FIGURE 7. The three possibilities

Case 2. If γ_1 meets $[A, L)$ but not (A, E), then g (in $[A, L)$) is the only vertex of γ that lies in Δ, and $\gamma_1 = (L_1, g] \cup [g, E_1)$ (Figure 7b). According to Lemma 3a, the shortest curve connecting L_1 and E_1 and touching (A, L) is $\gamma_1' = (L_1 L_2] \cup [L_2 E_1)$. Then the eight-point curve $\gamma_1' \cup \gamma_2$ cannot be shorter than γ, and it follows that $\gamma_1 = \gamma_1'$ and $g = L_2$. The point L_2 is the intersection of $[A, L)$ with the line through L_1 and the reflected image E_1^* of E_1 across $\langle AB \rangle$. But then

$$\angle DL_2L_1 = \angle AL_2E_1 > \angle ALE = \angle DKJ,$$

so that the rays $[gL_1\rangle$ $(= [L_2L_1\rangle)$ and $[KJ\rangle$ are disjoint. Hence γ does not meet $[KJ]$, a contradiction.

A similar argument shows that γ cannot meet $[A, E)$ and miss (A, L).

Case 3. The remaining possibility is that γ_1 meets both (A, L), and (A, E). Then g and a are the only vertices of V in Δ, $g \in (A, L)$, $a \in (A, E)$ and $\gamma_1 = (L_1, g] \cup [g, a] \cup [a, E_1)$ (Figure 7c). The line that joins the point E_1^* symmetric to E_1 in the side $[A, E]$ and the point L_1^* symmetric to L_1 in the side $[A, L]$ meets those sides at points E_2 and L_2, respectively, and the path

$$\gamma_1' = (E_1 E_2] \cup [E_2 L_2] \cup [L_2 L_1)$$

is the shortest path that connects E_1 to L_1 and touches both legs (A, E) and (A, L). Then the eight-point curve $\gamma_1' \cup \gamma_2$ cannot be shorter then γ, and it follows that $\gamma_1 = \gamma_1'$ and $a = E_2$, $g = L_2$. Hence the rays $[a, E_1\rangle$ $(= [E_2, E_1\rangle)$ and $[g, L_1\rangle$ $(= [L_2, L_1\rangle)$ are parallel, and either $[a, E\rangle$ does not meet $[F, G]$ or $[g, L_1\rangle$ does not meet $[K, J]$. Thus γ cannot be an eight-point curve, a contradiction.

This completes the proof that the curve γ lies in the octagonal region **O**.

□

LEMMA 5. *There is a minimal eight-point curve γ_2 that is an octagon with vertices a, b, c, g, e, f, g, h that passes through the midpoint of each of the four sides of the rectangle* $\mathbf{R}$, *a, c, e, g (Figure* 5c*), it is inscribed in* $\mathbf{O}$, *and it is doubly symmetric, with axis $\langle ae\rangle$ and $\langle cg\rangle$.*

PROOF. Let $\gamma = [abcdefgha]$ be a curve in $\mathcal{F}_{\min}$ (Figure 8). Its reflection $\gamma' = [a'b'c'd'e'f'g'h'a']$ in the vertical axis of symmetry of $\mathbf{R}$ also lies in $\mathcal{F}_{\min}$. Note the labeling of the vertices of γ'. According to Lemma 3b, the length of the bimedian that connects the sides $[a', a]$ and $[b', b]$ of the quadrilateral $aa'b'b$ is $\frac{1}{2}(|ab| + |a'b'|)$ at most, and similarly all around the figure.

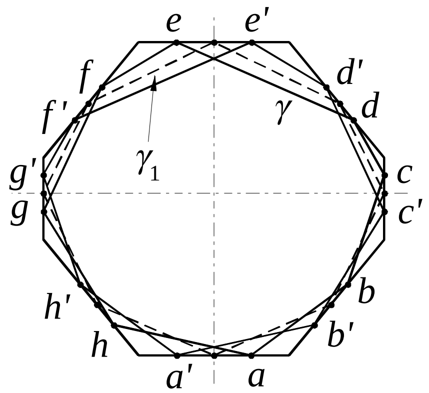


FIGURE 8. The symmetrization of γ

Summing, we see that the closed curve γ_1 formed by the successive bimedians (dashed in Figure 8) is a minimal eight-point curve, and it passes through the midpoints of the sides $[A, B]$ and $[C, D]$ of the $u \times v$ rectangle $\mathbf{R}$. Now reflect γ_1 through the horizontal axis of symmetry of $\mathbf{R}$ and apply the same argument. The resulting eight-point curve (dashed in Figure 8) is minimal and passes through the midpoints of the top and bottom sides and the midpoints of the left and right sides of $\mathbf{R}$, and it is symmetric about the perpendicular bisectors of the sides of $\mathbf{R}$. □

3.4. Pentagonal cover for eight-point curves in R

We have established that each eight-point curve of minimal length in the $u \times v$ rectangle $\mathbf{R}$ lies in the central octahedron $\mathbf{O}$. We have shown further that every eight-point curve γ in $\mathbf{R}$ has length

$$\ell(\gamma) \geq 4f\Big(\frac{u}{2}, \frac{v}{2}, s, t\Big), \tag{11}$$

where $f(\frac{1}{2}u, \frac{1}{2}v, s, t)$ is the length of each of the four congruent shortest polygonal paths that join the midpoints of two adjacent sides of $\mathbf{R}$ and touch the hypotenuse of the corresponding corner right triangle. Consider, for example, the shortest path $[ahg]$ pictured in Figure 5c, which joins the midpoint a of $[AB]$ to the midpoint g of $[DA]$, and meets the hypotenuse $[EL]$ at the point h for which $|ah|+|hg|$ is as small as possible. The determination of $f(\frac{u}{2},\frac{v}{2},s,t)$ is elementary; there are just three possibilities: the minimal path meets the segment $[EL]$ at the endpoint E, at the endpoint L, or at an inner point of $[EL]$, as described in Lemma 3a.

Hence we have the following corollary:

COROLLARY 6. *Let* $\mathbf{R}$ *be a* $u\times v$ *rectangle, and let* s *and* t *be given, with* $0\le s<u/2$ *and* $0\le t<v/2$*. Let* $\mathbf{P}$ *be the pentagon obtained by removing an* $s\times t$ *corner triangle from the* $u\times v$ *rectangle* $\mathbf{R}$ *(for example, in Figure* 5a *let* $\mathbf{P}=\mathrm{conv}\{B,C,D,L,E\}$*). If* γ *is an eight-point curve in* $\mathbf{R}$ *whose length is at most* $4f(\frac{u}{2},\frac{v}{2},s,t)$*, then* $\mathbf{P}$ *contains a congruent copy of* γ*.*

PROOF. Every closed curve in $\mathbf{R}$ of length at most $4f(u/2,v/2,s,t)$ must miss the interior of at least one of the four corner triangles, and by a suitable motion we may arrange for it to miss the lower left corner triangle.

□

We pause to establish a formula for $f(u/2,v/2,s,s)$ we shall need in the next section. In Figure 9, a and g are the midpoints of the sides of lengths u and v, respectively, g^* is symmetric to g in the hypotenuse $\langle EL\rangle$, and $\langle ag^*\rangle$ meets the line $\langle EL\rangle$ at h.

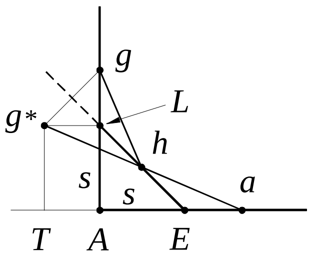


FIGURE 9 The case $s=t$

Then $h\in[EL]$, $|Tg^*|=s$, and because $|TA|=|g^*L|=|Lg|=1/2v-s$ and $|Ta|=|TA|+|Aa|=1/2v-s+1/2u$, it follows that

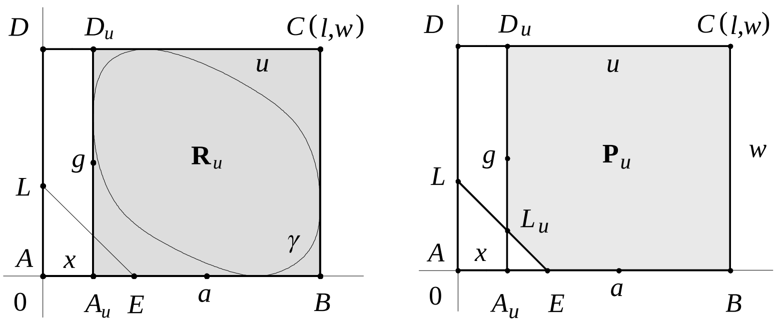
$$f\Big(\frac{1}{2}u,\frac{1}{2}v,s,s\Big)=|ah|+|hg|=|ag^*|=\sqrt{\Big(\frac{1}{2}(u+v)-s\Big)^2+s^2}. \tag{12}$$

An analogous but more complicated formula for the case $s\neq t$ can be proved in a similar manner.

3.5. A smaller pentagonal cover

In Section 3.2 we described a pentagon with area about 0.471 that is a cover for the family $\mathcal{C}$ of all closed curves of length two. Here we use the results of the previous section to produce a smaller pentagonal cover for this family. Throughout we are content to take $t = s$.

Place the best rectangle $\mathbf{R}$ (Section 3.1) in the coordinate plane with A at the origin and C at (l, w). Suppose $0 \le s < \frac{1}{2}w$, and take E and L on $[AB]$ and $[AD]$, respectively, with $|AE| = |AL| = s$. For x with $0 \le x \le s$, let $u = l - x$, and take A_u on $[AB]$ and D_u on $[DC]$ with $|AA_u| = |DD_u| = x$. Suppose $[A_uD_u]$ meets $[EL]$ at L_u, let $\mathbf{R}_u = \operatorname{conv}\{A_u, B, C, D_u\}$ and $\mathbf{P}_u = \operatorname{conv}\{B, C, D_u, L_u, E\}$ (See Figure 10.)



a. $\mathbf{R}_u \subset \mathbf{R}$ b. $\mathbf{P}_u \subset \mathbf{P}$

FIGURE 10. $\mathbf{R}$ and $\mathbf{P}$

Then $\mathbf{R}_u$ is $(l - x) \times w$, $|A_uE| = |A_uL_u| = s - x$, and $\mathbf{P}_u = \mathbf{P}_u(l - x, w, s - x, s - x)$; observe that $s - x < \min\{\frac{1}{2}w, \frac{1}{2}(l - x)\}$. Formula (12) asserts that

$$f\Big(\frac{1}{2}u, \frac{1}{2}w, s, s\Big) = \sqrt{\Big(\frac{1}{2}(l + w - x) - (s - x)\Big)^2 + (s - x)^2}, \tag{13}$$

and it follows from Corollary 6 that every eight-point curve γ of at most this length fits in $\mathbf{P}_u$.

Now take

$$s_0 = l + w - 1/2\sqrt{5} \approx 0.289\ 764,$$

and note that $s_0 < \frac{1}{2}w$ (the reason for this choice of s will appear shortly), We claim that $\mathbf{P} = \operatorname{conv}\{B, C, D, L, E\}$ is a cover for the family $\mathcal{C}$ of all closed curves of length two.

THEOREM 7. *The pentagon* $\mathbf{P}$ *with* $|AE| = |AL| = s_0$ *is a cover for the family* $\mathcal{C}$ *of all closed curves of length two.*

PROOF. Let γ be a convex curve in $\mathcal{C}$. As in Section 3.1, γ has two parallel support lines at distance w apart. Place γ in $\mathbf{R}$ touching the top, bottom, and right edge of $\mathbf{R}$, and suppose the support line of γ parallel to and not containing $[BC]$ meets $[AB]$ at A_u and $[CD]$ at D_u (Figure 10a). Let $x = |AA_u| = |DD_u|$, $u = l - x$, and $\mathbf{R}_u$ and $\mathbf{P}_u$ as above. Since $\mathbf{R}_u$ surely fits in $\mathbf{P}$ when $u < l - s_0$, we suppose from now on that $l - s_0 \le u \le l$, i.e., $0 \le x \le s_0$. The intersection of $\mathbf{P}$ and $\mathbf{R}_u$ is the pentagonal region $\mathbf{P}_u = \mathbf{P}(u, w, s, s)$, with $s = s_0 - x$. According to Corollary 6 and equation (13), $\mathbf{P}_u$ is a cover for every closed curve of length two in $\mathbf{R}_u$ when $4f(1/2u, 1/2w, s_0, s_0) \ge 2$, i.e., when

$$\Big(\frac{1}{2}(l + w - x) - (s_0 - x)\Big)^2 + (s_0 - x)^2 \ge \frac{1}{4}.$$

It is a calculus exercise to show that the minimum of this parabolic arc on the interval $[0, s_0]$ is $1/4$, and it occurs at

$$x = \frac{1}{\pi}(2 + \sqrt{\pi^2 - 4}) - \frac{3}{5}\sqrt{5} \approx 0.066\,157.$$

It follows as claimed that $\mathbf{P}$ is a cover for the family $\mathcal{C}$ of all closed curves of length two.

□

Computing the area of $\mathbf{P}$, we find the upper bound

$$\alpha_2 < \text{area}(\mathbf{P}) = lw - 1/2(l + w - 1/2\sqrt{5})^2 < 0.448\,966. \tag{14}$$

3.6. A slight improvement

With a similar argument but a more involved calculation one can see that the pentagon $\mathbf{P}(l, w, s, t)$ is a cover for $\mathcal{C}$ if $s = 0.284\,044$ and $t = 0.296\,300$. Hence

$$\alpha_2 \le lw - \frac{1}{2}\,(0.284\,044)\,(0.296\,300) < 0.448\,866, \tag{15}$$

which is a slight improvement over (14).

3.7. A covering hexagon having one elliptic arc boundary

Position the $l \times w$ rectangle $\mathbf{R}$ as in Section 3.5, and for $1 - w \le u \le l$ let $x = l - u$. (Observe that the $u \times v$ rectangle contains the convex curve γ, hence has perimeter at least 2.) Take $\mathbf{R}_u$ and $\mathbf{P}_u$ as pictured in Figure 10. We seek the locus of the corner L_u of $\mathbf{P}_u$ when $s = s(x)$ is chosen (less than $\min\{\frac{1}{2}u, \frac{1}{2}v\}$) so as to make $4f(\frac{1}{2}u, \frac{1}{2}w, s, s) = 2$.

The point L_u has coordinates (x, y) with $y = s(x) - x$, and we see from equation (13) that the locus of L_u lies on the ellipse ε whose equation is

$$\sqrt{\Big(\frac{1}{2}(l + w - x) - y\Big)^2 + y^2} = \frac{1}{2}, \tag{16}$$

i.e.,

$$x^2 + 4xy + 8y^2 - 2(l+w)x - 4(l+w)y + 2lw = 0, \tag{17}$$

as pictured in part in Figure 11.

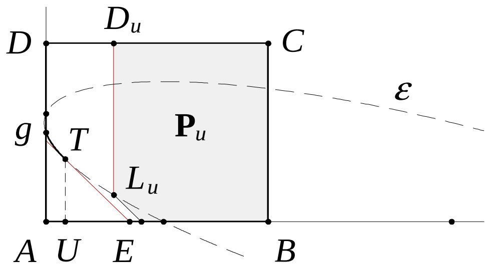


FIGURE 11. The ellipse ε

The center of ε is the point $(l+w,0)$, marked in the figure. The ellipse ε meets the x-axis at the points $(l+w\pm 1,0)$ and the y-axis at the midpoint g of $[AD]$ and at the point $(0,\frac{1}{2}l)$, which lies between g and D. Observe that for $0 \le x \le l+w-1$ (17) has two solutions $y_1(x) < y_2(x)$, but $\{(x, y_2(x)) \mid 0 \le x \le l+w-1\}$ lies above the chord $[(0,\frac{1}{2}l),(l+w-1,\frac{1}{2})]$ of ε, so $L_u = (x, y_1(x))$.

It is a calculus exercise to show that the line m with slope -1 that is tangent to the lower half of ε touches it at the point T with coordinates $(l+w-\frac{3}{5}\sqrt{5}, \frac{1}{10}\sqrt{5})$ and has the equation

$$x + y = l + w - 1/2\sqrt{5}.$$

This line cuts from $\mathbf{R}$ an isosceles right triangle with legs

$$l + w - 1/2\sqrt{5} = s_0,$$

leaving the region $\mathbf{P}$ described in Section 3.5. Note that the foot U of the perpendicular from T to the x-axis has coordinates $(l+w-\frac{3}{5}\sqrt{5},0)$, and

$$|UE| = l + w - \frac{1}{2}\sqrt{5} - \left(l + w - \frac{3}{5}\sqrt{5}\right) = \frac{1}{10}\sqrt{5}.$$

Finally, let

$$\mathbf{Q} = \bigcup_{1-w \le u \le l} \mathbf{P}_u.$$

The region $\mathbf{Q}$ is a curvilinear hexagonal region bounded by five line segments and the arc $\widehat{gT}$ of the ellipse ε. Clearly $\mathbf{Q}$ is a cover for $\mathcal{C}$, because each curve γ in $\mathcal{C}$ lies in $\mathbf{R}_u$ for some u and consequently fits in the corresponding $\mathbf{P}_u$.

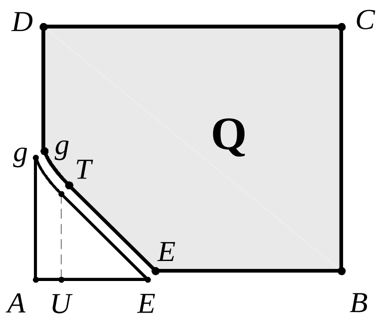


FIGURE 12. The regions $\mathbf{Q}$ and $\mathbf{R} \setminus \mathbf{Q}$

The area of the region $\mathbf{R} \setminus \mathbf{Q}$ is given by

$$\begin{aligned}\operatorname{area}(\mathbf{R} \setminus \mathbf{Q}) &= \int_0^{l+w-\frac{3}{5}\sqrt{5}} y(x)dx + \operatorname{area}(UET) \\ &> 0.017\,443\,078 + 0.025 = 0.042\,443\,078.\end{aligned}$$

Consequently

$$\alpha_2 \leq lw - \operatorname{area}(\mathbf{R} \setminus \mathbf{Q}) < 0.448\,504. \tag{18}$$

Some numerical experimentation suggests that improvements when $s \neq t$ are tiny.

4. Higher dimensions

Little is known about the analogous problem in $\mathbb{E}^d$ for $d \geq 3$. Write α_d for the least content (volume) of a convex cover for all closed curves of length two in $\mathbb{E}^d$. Then we have the following quite weak bounds:

$$\frac{2^d}{d!\sqrt{d^d(d+1)^{d+1}}} \leq \alpha_d \leq \frac{1}{d^{d/2}}. \tag{19}$$

The lower bound is the largest volume of a d-simplex in $\mathbb{R}^d$ that has a Hamiltonian cycle of length 2, a consequence of a recent inequality of Fiedler [8, Th. 2.5, p. 67]. The upper bound follows from the fact that every closed curve of length two in $\mathbb{R}^d$ lies in some hypercube of edge $1/\sqrt{d}$ (see [20, Theorem 3]).

For $d = 3$, 4, and 5, for example, (19) gives the feeble bounds

$$\begin{aligned}0.016\,037 &< \alpha_3 < 0.193\,450,\\ 0.000\,745 &< \alpha_4 < 0.062\,500,\\ 0.000\,022 &< \alpha_5 < 0.017\,886.\end{aligned}$$

4.1. Leo Moser's worm problem

Let $\mathcal{C}_1^d$ the family of continuous rectifiable arcs of unit length in $\mathbb{E}^d$. A region C is a *cover* if it contains a directly congruent copy of each member of $\mathcal{C}_1^d$ (only translations and rotations are allowed, no reflection). Let β_d denote the minimum (infimum) of the volumes of the convex covers. The determination of β_2 is a version of the "worm" problem of Leo Moser [16]. An overview of more general problems can be found in [21] (See especially page 316).

There is a series of papers constructing increasingly better upper bounds, e.g., $\beta_2 < 0.275\,237$ (Norwood, Poole and Laidacker [18], 1992), $\beta_2 < 0.260\,437$ (Norwood and Poole [17], 2003), $\beta_3 < 0.159\,530$ (Lindström [13], 1997), $\beta_3 < 0.075\,803$ (Håstad, Linusson, and Wästlund [10], 2001).

For general d Håstad et al. [10] showed $\beta_d < (c\sqrt{\log d})^d/d^{3d/2}$. Since $\alpha_d \leq 2^d\beta_d$, this significantly improves the upper bound in (19).

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