SIAM J. DISCRETE MATH.
Vol. 24, No. 3, pp. 964–978

ON REVERSE-FREE CODES AND PERMUTATIONS*

ZOLTAN FÜREDI†, IDA KANTOR‡, ANGELO MONTI§, AND BLERINA SINAIMERI§

Abstract. A set $\mathcal{F}$ of ordered k-tuples of distinct elements of an n-set is pairwise reverse free if it does not contain two ordered k-tuples with the same pair of elements in the same pair of coordinates in reverse order. Let $F(n,k)$ be the maximum size of a pairwise reverse-free set. In this paper we focus on the case of 3-tuples and prove $\lim F(n,3)/\binom{n}{3} = 5/4$, more exactly, $\frac{5}{24}n^3 - \frac{1}{2}n^2 - O(n \log n) < F(n,3) \leq \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{5}{8}n$, and here equality holds when n is a power of 3. Many problems remain open.

Key words. extremal combinatorics, ordered triples, permutations

AMS subject classification. 05D05

DOI. 10.1137/090774835

1. The problem.

Let k and n be natural numbers, and let X be an n-element underlying set. The set of k-element sequences is denoted by X^k, and its cardinality is n^k. The set of ordered k-tuples is denoted by $X_{(k)}$, and the set of k-subsets of X is denoted by $\binom{X}{k}$. We have $X_{(k)} \subset X^k$, $|X_{(k)}| = n(n-1)\ldots(n-k+1) = k!\binom{n}{k}$, and the cardinality of $\binom{X}{k}$ is $\binom{n}{k}$ (here $n \geq k$). Frequently the set X is identified with the set of first n integers $[n] = \{1, \ldots, n\}$.

A *code* $\mathcal{C}$ is simply a subset of X^k; k is called its length, and $|\mathcal{C}|$ is its size. A typical problem in coding theory is finding the maximum size of a code with some local side condition. In this paper we will deal with such a problem, with reverse-free codes.

Two sequences $\mathbf{x} = x_1, \ldots, x_k$ and $\mathbf{y} = y_1, \ldots, y_k$ are called *reverse free* if there are no two coordinates $i, j \in [k]$ such that $x_i \neq x_j$ but $(x_i, x_j) = (y_j, y_i)$. A code $\mathcal{F}$ is called *pairwise reverse free* if any two of its members are reverse free. Let $\overline{F}(n,k)$ be the maximum cardinality of a pairwise reverse-free code $\mathcal{F} \subset [n]^k$. Let $F(n,k)$ be the maximum cardinality of a pairwise reverse-free code $\mathcal{F} \subset [n]_{(k)}$; i.e., when the codewords have no repetition, $\mathcal{F}$ is a set of ordered k-tuples from $[n]$.

If one considers the $|\mathcal{F}| \times k$ *matrix* $M(\mathcal{F})$ of $\mathcal{F}$, where the rows are the members of $\mathcal{F}$, then it is reverse free if the rows are distinct and it does not contain a submatrix of type $\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$ with $a \neq b$. Many coding theory problems can be formulated in this way; for an extremal problem with forbidden submatrices, see, e.g., [18].

It seems difficult to determine the exact value of $F(n,k)$ and $\overline{F}(n,k)$, and thus we concentrate on estimating their asymptotic behavior as n tends to ∞ for k fixed. We also solve the first nontrivial cases and determine asymptotically $F(n,3)$, $\overline{F}(n,3)$. Moreover we establish the order of magnitude of $\overline{F}(3,k)$.

*Received by the editors October 23, 2009; accepted for publication (in revised form) June 1, 2010; published electronically August 17, 2010.
http://www.siam.org/journals/sidma/24-3/77483.html

†Rényi Institute of the Hungarian Academy, Budapest, P.O. Box 127, H-1364, Hungary (furedi@renyi.hu) and the Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801 (z-furedi@math.uiuc.edu). This author's research was supported in part by the Hungarian National Science Foundation under grants OTKA 062321 and 060427 and by the National Science Foundation under grant NFS DMS 06-00303.

‡Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801 (isvejda2@math.uiuc.edu).

§Department of Computer Science, Sapienza University of Rome, Via Salaria 113, Rome 00198, Italy (monti@di.uniroma1.it, sinaimeri@di.uniroma1.it).

2. Recurrences. Note that the sequence $F(n,k)/\big(k!\binom{n}{k}\big)$ is monotonically nonincreasing in n (with k fixed), so the asymptotic density

$$f(k) := \lim_{n\to\infty} \frac{F(n,k)}{k!\dbinom{n}{k}}$$

exists. Indeed, let $n_1 < n_2$ and let $\mathcal{F}$ be a pairwise reverse-free set of k-tuples from $[n_2]$ attaining the maximum cardinality $|\mathcal{F}| = F(n_2,k)$. The restriction of $\mathcal{F}$ to $A \subset [n_1]$, $\mathcal{F}[A] := \{\mathbf{x} \in \mathcal{F} : \text{all } x_i \in A\}$, is again a reverse-free family. Counting all pairs $(\mathbf{x}, A)$, where A is an n_1-subset of $[n_2]$ and $\mathbf{x}$ is a k-tuple of elements of A that belongs to $\mathcal{F}$, we obtain

$$\text{the number of } (\mathbf{x}, A) \text{ pairs } = \sum_{\mathbf{x}\in\mathcal{F}} |\{A : \mathbf{x} \in \mathcal{F}[A]\}| = F(n_2,k)\binom{n_2-k}{n_1-k}$$
$$= \sum_{A\in\binom{[n_2]}{n_1}} |\{\mathbf{x} : \mathbf{x} \in \mathcal{F}[A]\}| = \sum |\mathcal{F}[A]| \le \binom{n_2}{n_1} F(n_1,k).$$

Using the identity $\binom{n_2-k}{n_1-k}\binom{n_2}{k} = \binom{n_2}{n_1}\binom{n_1}{k}$, we obtain

$$\frac{F(n_2,k)}{k!\dbinom{n_2}{k}} \le \frac{F(n_1,k)}{k!\dbinom{n_1}{k}}. \tag{2.1}$$

Observe that all the increasing (nondecreasing) sequences of k elements from $[n]$ form a set of pairwise reverse-free k-tuples. Thus

$$\binom{n}{k} \le F(n,k) \quad \text{and} \quad \binom{n+k-1}{k} \le \overline{F}(n,k). \tag{2.2}$$

One can easily see that here equality holds for $k = 1$ and $k = 2$:

$$F(n,1) = \overline{F}(n,1) = n, \quad F(n,2) = \binom{n}{2}, \quad \text{and} \quad \overline{F}(n,2) = \binom{n+1}{2}. \tag{2.3}$$

One can also see that if the alphabet has only two symbols, then a reverse-free code forms a chain, and thus

$$\overline{F}(1,k) = 1 \quad \text{and} \quad \overline{F}(2,k) = k+1. \tag{2.4}$$

Consider the matrix M of a reverse-free code $\mathcal{F} \subset [n]_{(k)}$ and restrict its columns to a subset $I \subset [k]$. After removing the repeated rows, we obtain a smaller reverse-free code. So $M|I$ contains at most $F(n,i)$ distinct rows, $i := |I|$. Let $\mathbf{y}$ be a row of $M|I$. This subvector is contained in at most $F(n-i,k-i)$ members of $\mathcal{F}$ since $\{\mathbf{x}|([k]\setminus I) : \mathbf{x}|I = \mathbf{y}\}$ is again reverse free of length $k-i$ with alphabet $[n]\setminus \mathbf{y}$. We obtain

$$F(n,k) \le F(n,i)F(n-i,k-i). \tag{2.5}$$

Using (2.5) and (2.3) we obtain

$$F(n,k) \le F(n,2)F(n-2,k-2) \le F(n,2)F(n-2,2)F(n-4,2)\dots F(n-2i,k-2i)$$

$$\le \left(\prod_{0\le i\le \lfloor k/2\rfloor -1} \binom{n-2i}{2} \right) F(n-2\lfloor k/2\rfloor, k-2\lfloor k/2\rfloor) = \frac{k!\binom{n}{k}}{2^{\lfloor \frac{k}{2} \rfloor}}. \tag{2.6}$$

Take two reverse-free codes $\mathcal{F}_1 \subset X_{(k_1)}$ and $\mathcal{F}_2 \subset Y_{(k_2)}$. If X and Y are disjoint, then we can make a product code $\mathcal{F}$ with alphabet $X \cup Y$ by joining every $\mathbf{x} \in \mathcal{F}_1$ to each $\mathbf{y} \in \mathcal{F}_2$. Then $\mathcal{F}$ is reverse free, and we obtain

$$F(n_1,k_1)F(n_2,k_2) \le F(n_1+n_2,k_1+k_2). \tag{2.7}$$

3. Small constructions. In this section we collect a few small optimal constructions. The ideas of the constructions and proofs here are used again later. We have

$$F(3,3)=3,\ \ F(4,3)=6,\ \ F(5,3)=15, \tag{3.1}$$

$$F(4,4)=5,\ \ F(5,5)=13,\ \ \overline{F}(3,3)=11.$$

For $n=k=3$ it is easy to see that two ordered triples on three elements are reverse free if and only if one is a cyclic shift of the other. Thus, we have $F(3,3)=3$ and the optimal constructions are $\{abc, cab, bca\}$ and $\{cba, acb, bac\}$.

The lower bounds for $F(4,3)$ and $F(5,3)$ are given by considering the set of cyclic ordered triangles of some directed graphs on $n=4$ and $n=5$ vertices, respectively. Namely, one can consider the directed graph with directed edges 12, 23, 31, 14, and 43 (then the directed triangles are 123, 231, 312 and 143, 431, 314), and in the case of $n=5$ we take the directed triangles of the tournament with edge set $\{(i,i+1),(i,i+2)\}$ where all numbers are taken modulo 5.

The lower bounds for $F(4,4)$, $F(5,5)$, and $\overline{F}(3,3)$ are obtained from

$$\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \\ 2 & 3 & 4 & 1 \\ 3 & 1 & 2 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 2 & 5 & 4 & 1 & 3 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 4 & 2 & 1 \\ 3 & 5 & 2 & 1 & 4 \\ 3 & 1 & 5 & 2 & 4 \\ 4 & 3 & 5 & 2 & 1 \\ 4 & 1 & 5 & 3 & 2 \\ 4 & 3 & 1 & 5 & 2 \\ 5 & 4 & 1 & 3 & 2 \\ 5 & 4 & 2 & 1 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix} \quad L = \left\{ \begin{array}{l} AAA \\ BBB \\ CCC \\ AAB \\ AAC \\ ABB \\ CBB \\ CAC \\ CBC \\ ABC \\ CAB \end{array} \right. . \tag{3.2}$$

Upper bounds. The case $F(4,3)$ is easy, and $F(5,3) \le (5/2)F(4,3) = 15$ follows from (2.1).

Concerning a reverse-free code $\mathcal{F} \subset [4]_{(4)}$, suppose that there exists an element appearing at least twice in the same position. Say element 1 is in the first position twice, for example, $1234 \in \mathcal{F}$ and $1423 \in \mathcal{F}$. If another shift $1342 \in \mathcal{F}$, then there are no more members, $|\mathcal{F}|=3$, and we are done. Otherwise, the only other possible members of $\mathcal{F}$ are 2341 and 2431, then 3124 and 3142, and furthermore 4213 and 4312. However, these three pairs contain reversed pairs, so only one of each can belong to $\mathcal{F}$. This implies $|\mathcal{F}| \leq 5$, as stated.

The case $F(5, 5) = 13$ was proved by a computer search.

Concerning a reverse-free code L of length 3 of three symbols A, B, and C, we can have 3 sequences using one symbol (AAA, BBB, and CCC), or L can contain 2 sequences using both A and B but not C, so altogether L can have $3 + 3 \times 2 = 9$ members using at most two symbols and three sequences using all three symbols. These are $9 + 3 = 12$ sequences, but it is easy to see that if L contains ABC and two cyclic shifts, then it has no member with exactly two symbols, implying $\overline{F}(3, 3) < 9 + 3$.

From inequalities (2.2) and (2.6) we obtain that

$$\frac{1}{k!} \leq f(k) \leq \frac{1}{2^{\lfloor \frac{k}{2} \rfloor}}. \tag{3.3}$$

If $k = 2$, then these bounds are tight and we have $f(2) = 1/2$. When $k > 2$ it seems to be complicated to establish the exact value of $f(k)$. In this paper we are mainly concerned with the case of pairwise reverse-free sets of ordered 3-tuples (or simply triples). From (3.3) we derive that $0.1\bar{6} \leq f(3) \leq 0.5$, but both of these bounds are far from the truth. Indeed, we improve these by providing the exact value $f(3) = 5/24$.

4. An iterated construction of reverse-free triple systems. In this section we present a construction for $F(n, 3)$.

THEOREM 1. *For any $n \in \mathbb{N}$, we have*

$$F(n, 3) > \frac{5}{24}n^3 - \frac{1}{2}n^2 - O(n \log n).$$

If n is a power of 3, we have

$$F(n, 3) \geq \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{5}{8}n.$$

Proof. We build our construction $\mathcal{F}$ in a recursive manner. Suppose $n \geq 3$ and we partition $[n]$ into three nonempty disjoint sets A, B, and C of cardinalities a, b, and c, respectively. Given an ordered triple $\mathbf{x} = x_1x_2x_3 \in [n]_{(3)}$, we say that it is of *type* $X_1X_2X_3$, where $X_i \in \{A, B, C\}$ for $1 \leq i \leq 3$, if $x_1 \in X_1$, $x_2 \in X_2$, and $x_3 \in X_3$. All the ordered triples of our construction $\mathcal{F}$ will have a type from the list L defined in (3.2) in the previous section concerning $\overline{F}(3, 3) = 11$. Suppose that $\mathbf{x} = x_1x_2x_3$ is of the type $X_1X_2X_3$ and $\mathbf{y} = y_1y_2y_3$ is of the type $Y_1Y_2Y_3$ such that there exist two coordinates $i < j$ for which $(y_i, y_j) = (x_j, x_i)$. If $X_1X_2X_3$ and $Y_1Y_2Y_3$ both belong to L, then the definition of L implies that $X_i = X_j = Y_i = Y_j$. In other words, the eleven types defined by L generate very few conflicts. The ordered triples of types ABC and CAB especially have no conflict with any other ordered triples of the eleven types from L.

To avoid the possible conflicts, e.g., among the ordered triples of types AAA, AAB, and AAC, we proceed as follows. Consider the sets $\mathcal{F}_X$ for every $X \in \{A, B, C\}$ of reverse-free ordered triples of distinct elements from the set X attaining maximum cardinality, and let $\mathcal{F}'$ be the union of these three sets. Hence, $\mathcal{F}'$ is a set of reverse-free ordered triples of types AAA, BBB, and CCC of maximum size. Define a tournament, a complete oriented graph G_A with $\binom{a}{2}$ directed edges, such that it is compatible with $\mathcal{F}_A$, i.e., $x_1x_2x_3 \in \mathcal{F}_A$ implies $x_1x_2 \in \mathcal{E}(G_A)$. Then put into $\mathcal{F}''$

all ordered triples of the form x_1x_2y with $x_1x_2 \in \mathcal{E}(G_A)$ and $y \in B \cup C$. We added $\binom{a}{2}(b+c)$ ordered triples to $\mathcal{F}''$.

We proceed in the same way concerning $\mathcal{F}_B$, i.e., $x_1x_2x_3 \in \mathcal{F}_B$ implies $x_2x_3 \in \mathcal{E}(G_B)$, and put into $\mathcal{F}''$ all ordered triples of the form yx_2x_3 with $x_2x_3 \in \mathcal{E}(G_B)$ and $y \in A \cup C$; similarly we define G_C by having that $x_3x_1 \in \mathcal{E}(G_C)$ whenever $x_1x_2x_3 \in \mathcal{F}_C$ and put into $\mathcal{F}''$ all ordered triples of the form x_1yx_3 with $x_3x_1 \in \mathcal{E}(G_C)$ and $y \in A \cup B$.

Finally, we add to $\mathcal{F}''$ the ordered triples of types ABC and CAB and set $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}''$. We have obtained a family of pairwise reverse-free ordered triples of $[n]$.

It remains to compute the cardinality of $\mathcal{F}$. Clearly

$$|\mathcal{F}'| = F(a,3) + F(b,3) + F(c,3).$$

By definition we have

$$|\mathcal{F}''| = \binom{a}{2}(b+c) + \binom{b}{2}(a+c) + \binom{c}{2}(a+b) + 2abc.$$

We obtain

$$\begin{aligned} F(a+b+c,3) \geq |\mathcal{F}| = {} & F(a,3) + F(b,3) + F(c,3) \\ & + \binom{a}{2}(b+c) + \binom{b}{2}(a+c) + \binom{c}{2}(a+b) + 2abc \end{aligned}$$

or, equivalently, by rearranging the above formula,

$$\begin{aligned} &\left(F(a+b+c,3) - \binom{a+b+c}{3}\right) \\ &\geq \left(F(a,3) - \binom{a}{3}\right) + \left(F(b,3) - \binom{b}{3}\right) + \left(F(c,3) - \binom{c}{3}\right) + abc. \end{aligned} \tag{4.1}$$

In particular, if n is a power of 3, we can split the underlying set into three equal parts in each step, and the recurrence (4.1) together with the starting value $F(3,3) = 3$ yields

$$F(n,3) \geq \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{5}{8}n, \tag{4.2}$$

proving the second part of our theorem.

For general n to give a lower bound for $F(n,3)$ we seek an upper bound on the remainder term $r(n)$ defined by

$$F(n,3) = \binom{n}{3} + \frac{1}{24}n^3 + \frac{7}{24}n - r(n).$$

Note that the quantity in (4.2) equals $\binom{n}{3} + \frac{1}{24}n^3 + \frac{7}{24}n$. Theoretically, the remainder $r(n)$ might be negative, but that would only improve the lower bound. Substituting in (4.1), we obtain

$$\frac{(a+b+c)^3}{24} - r(a+b+c) \geq \frac{a^3+b^3+c^3}{24} + abc - r(a) - r(b) - r(c).$$

Letting a, b, and c be appropriate numbers with pairwise differences not exceeding 1, we have

$$\begin{aligned} r(3n) &\leq 3r(n), \\ r(3n-1) &\leq 2r(n) + r(n-1) + \frac{1}{4}n, \\ r(3n+1) &\leq 2r(n) + r(n+1) + \frac{1}{4}n. \end{aligned}$$

It easily follows by induction that, for some constant C,

$$r(n) \leq Cn \log n,$$

completing the proof of Theorem 1. □

5. An upper bound on reverse-free triple systems. The following theorem provides an upper bound for $F(n,3)$.

THEOREM 2.

$$F(n,3) \leq \begin{cases} \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{5}{8}n & \text{for } n \text{ odd}, \\ \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{1}{2}n & \text{for } n \text{ even}. \end{cases}$$

Proof. In this section we define the *support* of an ordered triple as the set of its three elements. Given a set $\mathcal{F}$ of pairwise reverse-free ordered triples from $[n]$, we say that a support $\{u,v,w\}$ has i occurrences in $\mathcal{F}$ if $\mathcal{F}$ contains i ordered triples with this support. Denote by T_i the set of supports having exactly i occurrences in $\mathcal{F}$. The following claim is the base of our proof.

CLAIM 1. *Let $\mathcal{F}$ be a set of pairwise reverse-free ordered triples from $[n]$. Then the following inequalities hold:*

(i) $|T_0| + |T_1| + |T_2| + |T_3| = \binom{n}{3}$.

(ii) $|T_2| + |T_3| \leq \begin{cases} \frac{n^3-n}{24} & \text{for } n \text{ odd}, \\ \frac{n^3-4n}{24} & \text{for } n \text{ even}. \end{cases}$

(iii) $|T_3| - |T_0| \leq \frac{n}{3}$.

Proof of (i). Recalling that two ordered triples of the same support are reverse free if and only if one is a cyclic shift of the other, we can deduce that any support has at most three occurrences in $\mathcal{F}$ (the set of all cyclic shifts). Hence, $|T_i| = 0$ for every $i > 3$. Thus the first equality of the claim follows by noting that the sets T_i (with $0 \leq i \leq 3$) form a partition of the whole $\binom{[n]}{3}$. □

Given the code $\mathcal{F}$, we create a directed graph G_1 on the vertex set $[n]$ by putting $uv \in E(G_1)$ whenever there exists a vertex x such that $uvx \in \mathcal{F}$. Note that $uv \in$

$E(G_1)$ implies $vu \notin E(G_1)$. If the resulting graph is not a tournament, we add edges arbitrarily to turn it into one. Similarly, let G_2 be a tournament that contains all edges uv such that, for some vertex x, we have $xuv \in \mathcal{F}$, and G_3 is a tournament containing all edges uv such that $vxu \in \mathcal{F}$ for some x.

Now define two additional directed graphs, $\mathcal{D}$ and $\mathcal{M}$, by putting

$$E(\mathcal{M}) = \{uv : uv \in E(G_i) \cap E(G_j) \text{ for some pair } i, j \in \{1,2,3\}\},$$
$$E(\mathcal{D}) = \{uv : uv \in E(G_1) \cap E(G_2) \cap E(G_3)\}.$$

For a directed graph G, let $c(G)$ be the number of oriented triangles in G. Let $d^+(v)$ and $d^-(v)$ be the out-degree and in-degree of v, respectively. It is well known (see, for example, [17]) that, for a tournament G on n vertices,

$$c(G) = \binom{n}{3} - \sum_{v \in V(G)} \binom{d^+(v)}{2} \leq \begin{cases} \dfrac{n^3 - n}{24} & \text{for } n \text{ odd,} \\[2ex] \dfrac{n^3 - 4n}{24} & \text{for } n \text{ even.} \end{cases}$$

Proof of (ii). If uvw and vwu both belong to $\mathcal{F}$, then $uv \in E(G_1)$ and $uv \in E(G_3)$, so $uv \in E(\mathcal{M})$. Similarly, $vw, wu \in E(\mathcal{M})$. We therefore have $|T_2| + |T_3| = |T_2 \cup T_3| \leq c(\mathcal{M})$ and the result follows. ☐

Proof of (iii). For any vertex u of the graph $\mathcal{D}$ define the sets of supports

$$S_u = \{\{u, v, w\} : uv \in E(\mathcal{D}) \text{ and } uw \in E(\mathcal{D})\}.$$

It is easy to see that $\bigcup_{u \in [n]} S_u \subseteq T_0$. The set S_u consists of all three-element sets containing u and two of its out-neighbors in $\mathcal{D}$; hence $|S_u| = \binom{d^+(u)}{2}$ (the out- and in-degree of the vertex u refer to the directed graph $\mathcal{D}$). For any two different vertices u, v the sets S_u and S_v are disjoint, so $|T_0| \geq \sum \binom{d^+(u)}{2}$. Similarly, $|T_0| \geq \sum \binom{d^-(u)}{2}$ holds too. It follows that

$$|T_0| \geq \frac{1}{2} \sum_{u \in [n]} \left[\binom{d^+(u)}{2} + \binom{d^-(u)}{2} \right]. \tag{5.1}$$

Each three-element set from T_3 induces a cyclic triangle in $\mathcal{D}$. A vertex u in $D(\mathcal{F})$ is in at most $d^+(u) \cdot d^-(u)$ cyclic triangles and hence

$$|T_3| \leq c(\mathcal{D}) \leq \frac{1}{3} \sum_{u \in [n]} d^+(u) \cdot d^-(u). \tag{5.2}$$

Subtract (5.1) from (5.2). Using that for nonnegative integers x, y, we have $\frac{1}{3}xy - \frac{1}{4}(x^2 - x) - \frac{1}{4}(y^2 - y) \leq \frac{1}{3}$. We obtain

$$|T_3| - |T_0| \leq \sum_{u \in [n]} \left[\frac{1}{3} d^+(u) \cdot d^-(u) - \frac{1}{2}\binom{d^+(u)}{2} - \frac{1}{2}\binom{d^-(u)}{2} \right] \leq \frac{n}{3}.$$

This concludes the proof of the claim. ☐

Finally, we finish the proof of the theorem by adding the three inequalities of Claim 1:

$$
\begin{aligned}
|\mathcal{F}| &= |T_1| + 2|T_2| + 3|T_3| \\
&= \binom{n}{3} + (|T_2| + |T_3|) + (|T_3| - |T_0|) \\
&\le \begin{cases} \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{5}{8}n & \text{for } n \text{ odd,} \\ \frac{5}{24}n^3 - \frac{1}{2}n^2 + \frac{1}{2}n & \text{for } n \text{ even.} \end{cases} \qquad \square
\end{aligned}
$$

6. Reverse-free triple systems: The case of repetitions. In this section we determine the asymptotics of $\overline{F}(n,3)$ exactly. When k is fixed and $n \to \infty$, using (2.2) we have

$$\overline{F}(n,k) = F(n,k) + O(n^{k-1}),$$

since $n^k - k!\binom{n}{k} = O(n^{k-1})$. Then $\overline{F}(n,k)$ has the same order of magnitude of $F(n,k)$, namely, $\Theta(n^k)$. In the case of $k = 3$ a more exact result is proved.

THEOREM 3. *For any $n \in \mathbb{N}$, we have*

$$\frac{5}{24}n^3 + \frac{1}{2}n^2 - O(n \log n) < \overline{F}(n,3) \le \frac{5}{24}n^3 + \frac{1}{2}n^2 + \frac{7}{24}n$$

and equality holds when n is a power of 3.

Proof of the lower bound. One can use the same arguments as in the proof of Theorem 1. Build a reverse-free set $\mathcal{F}$ in a recursive manner. Assume $n \ge 3$ and partition $[n]$ into three nonempty sets A, B, and C of cardinality a, b, and c, respectively. All the ordered triples of our construction will have a type from the list L defined in (3.2). Let $\mathcal{F}'$ be a reverse-free set of all ordered triples (with repetitions) of types AAA, BBB, and CCC, attaining the maximum size $\overline{F}(a,3) + \overline{F}(b,3) + \overline{F}(c,3)$. Consider the set $\mathcal{F}'' \subseteq [n]_{(k)}$ as described in the proof of Theorem 1 (note that $\mathcal{F}''$ contains ordered triples without repetitions). Consider the types AAB and AAC, and observe that since we allow repetition of symbols inside an ordered triple, for every $x \in A$, we can add to $\mathcal{F}''$ the ordered triples xxy and xxz for every $y \in B$ and $z \in C$. It is not difficult to see that these ordered triples cannot generate conflicts with the ordered triples in $\mathcal{F}'$ and in $\mathcal{F}''$. Using the same argument for the types CBB, ABB and CAC, CBC we have that

$$
\begin{aligned}
\overline{F}(a+b+c,3) \ge |\mathcal{F}| &= \overline{F}(a,3) + \overline{F}(b,3) + \overline{F}(c,3) \\
&\quad + \binom{a+1}{2}(b+c) + \binom{b+1}{2}(a+c) + \binom{c+1}{2}(a+b) + 2abc. \qquad (6.1)
\end{aligned}
$$

In case n is a power of 3, inequality (6.1) together with the base case $\overline{F}(3,3) = 11$ yields the following lower bound:

$$\overline{F}(n,3) \ge \frac{5}{24}n^3 + \frac{1}{2}n^2 + \frac{7}{24}n. \qquad (6.2)$$

For a general n it is possible to prove that (6.2) is at most $O(n \log n)$ far from $\overline{F}(n,3)$.

Proof of the upper bound. Let $\mathcal{F} \subseteq [n]^k$ be a reverse-free code attaining the maximum cardinality $\overline{F}(n,3)$. As in the proof of Theorem 2 we associate to an ordered triple in $\mathcal{F}$ its support, i.e., the set of its symbols. Let $\mathcal{F}_i$ be the set of all ordered triples of $\mathcal{F}$ whose support has cardinality i, and note that obviously $i \in [3]$. Clearly, $|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3|$. Trivially $|\mathcal{F}_1| \leq n$. Moreover, $|\mathcal{F}_3| \leq F(n,3)$ since it is a set of reverse-free ordered triples without repetitions. It remains to bound $|\mathcal{F}_2|$. Given any two distinct elements x, y from $[n]$, at most one ordered triple from each of the sets $\{xxy, xyx, yxx\}$ and $\{yyx, yxy, xyy\}$ belongs to $\mathcal{F}_2$, and hence at most two ordered triples of support $\{x, y\}$ appear in $\mathcal{F}_2$. Consequently, $|\mathcal{F}_2| \leq 2\binom{n}{2}$, but we can still improve this bound as follows. According to the set $\mathcal{F}_3$ consider the graph $\mathcal{D}$ as it was defined in the proof of Claim 1. It is not difficult to see that if $xy \in E(\mathcal{D})$, then none of the six ordered triples of support $\{x, y\}$ can be in $\mathcal{F}_3$. Hence $|\mathcal{F}_2| \leq 2\binom{n}{2} - 2E(\mathcal{D})$. Now, using the same arguments as in Theorem 2 we deduce

$$\begin{aligned}|\mathcal{F}| &= |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| \\ &\leq n + 2\binom{n}{2} - 2E(\mathcal{D}) + \binom{n}{3} + (|T_2| + |T_3|) + (|T_3| - |T_0|).\end{aligned} \tag{6.3}$$

Using (5.1) and (5.2) we have that

$$\begin{aligned}&|T_3| - |T_0| - 2E(\mathcal{D}) \\ &\leq \sum_{u \in [n]} \left[\frac{1}{3} d^+(u) \cdot d^-(u) - \frac{1}{2}\binom{d^+(u)}{2} - \frac{1}{2}\binom{d^-(u)}{2} - d^+(u) - d^-(u)\right] \leq 0,\end{aligned} \tag{6.4}$$

where in the last inequality we use the fact that for nonnegative p, q we have $\frac{1}{3}pq - \frac{1}{4}(p^2 - p) - \frac{1}{4}(q^2 - q) - p - q \leq 0$. From (6.3), using Claim 1 and (6.4) we have

$$|\mathcal{F}| \leq \begin{cases} \dfrac{5}{24}n^3 + \dfrac{1}{2}n^2 + \dfrac{7}{24}n & \text{for } n \text{ odd}, \\[2ex] \dfrac{5}{24}n^3 + \dfrac{1}{2}n^2 + \dfrac{1}{6}n & \text{for } n \text{ even}. \end{cases}$$

This concludes the proof. ∎

The argument for the upper bound sheds some additional light on the lower bound as well. Notice that the value in (6.2) equals $F(n,3) + n + 2\binom{n}{2} - \frac{n}{3}$. Indeed, the lower bound construction may be obtained from the construction for $F(n,3)$ by adding ordered triples xxx for all x, two ordered triples with support $\{x, y\}$ for each pair x, y (one from $\{xxy, xyx, yxx\}$ and one from $\{yyx, yxy, xyy\}$), and by removing an ordered triple xyz of three distinct elements if $\{x, y, z\}$ supported three ordered triples (there were $n/3$ of those). Notice that the induction step doesn't introduce any edges of $\mathcal{D}$, so the lastly mentioned operation removed all of them.

7. A related problem: Codes with many flips. A code $\mathcal{G} \subset [n]^k$ is *full of flips* if there is a reversed pair of distinct symbols for every pair of its members. Let $G(n,k)$, ($\overline{G}(n,k)$, etc.) be the maximum size of a code $\mathcal{G} \subset [n]_{(k)}$, ($\mathcal{G} \subset [n]^k$, etc.) full of flips. Obviously $G(2,2) = G(3,3) = 2$. We have

$$G(4,4) = 4 \quad \text{and} \quad G(5,5) = 8. \tag{7.1}$$

The lower bounds are given by the following constructions:

$$\text{(7.2)} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \qquad \begin{pmatrix} 5 & 1 & 2 & 3 & 4 \\ 5 & 2 & 1 & 4 & 3 \\ 5 & 3 & 4 & 1 & 2 \\ 5 & 4 & 3 & 2 & 1 \\ 4 & 1 & 2 & 3 & 5 \\ 4 & 2 & 1 & 5 & 3 \\ 4 & 3 & 5 & 1 & 2 \\ 4 & 5 & 3 & 2 & 1 \end{pmatrix},$$

and computer searches show that these lower bounds are tight.

Similarly as in (2.7), one can prove

$$G(n_1, k_1)G(n_2, k_2) \leq G(n_1 + n_2, k_1 + k_2). \tag{7.3}$$

These lead to

$$\frac{1}{8}(1.515\ldots)^k < 8^{\lfloor k/5 \rfloor} \leq G(k, k). \tag{7.4}$$

8. Long permutations. In this section we summarize some of the consequences of the above recurrences and bounds, especially $F(k, k)$. First, one more recurrence:

$$F(n, a)^b F(b, b) \leq F(nb, ba). \tag{8.1}$$

Indeed, take the reverse-free codes $\mathcal{F}^i \subset X^i_{(a)}$, where $X^1, \ldots, X^b$ are disjoint n-sets and $|\mathcal{F}^i| = F(n, a)$, and consider a reverse-free code $\mathcal{G} \subset [b]_{(b)}$ of size $F(b, b)$. One can create a reverse-free code $\mathcal{F}$ of size $(\Pi|\mathcal{F}^i|)|\mathcal{G}|$ with underlying set $\cup X^i$ and of length ab by taking a codeword $\mathbf{x}_i$ from each $\mathcal{F}^i$ and a member $\sigma \in \mathcal{G}$ and creating all the ab-tuples of the form $(\mathbf{x}_{\sigma(1)}, \ldots, \mathbf{x}_{\sigma(b)})$.

Starting with $F(5, 5) = 13$ induction gives $F(5^t, 5^t) \geq 13^{\frac{5^t}{4} - \frac{1}{4}}$. Now for any fixed integer k write $k = \sum_{i=0}^{\lfloor \log_5 k \rfloor} c_i 5^i$ with $0 \leq c_i \leq 4$. Then using (2.7) we obtain

$$F(k, k) \geq \prod_{i=0}^{\lfloor \log_5 k \rfloor} F(5^i, 5^i)^{c_i} \geq 13^{\frac{k}{4} - \frac{1}{4}\sum_{i=0}^{\lfloor \log_5 k \rfloor} c_i}.$$

As $\sum_{i=0}^{\lfloor \log_5 k \rfloor} c_i \leq 4 \log_5 k$, we get

$$\frac{1}{k^2}(1.898\ldots)^k = \frac{1}{k^2} 13^{k/4} \leq F(k, k). \tag{8.2}$$

Concerning an upper bound for $F(k, k)$, one can see that $G(k, k)F(k, k) \leq k!$. More generally

$$G(k, k)F(n, k) \leq k!\binom{n}{k}. \tag{8.3}$$

Indeed, consider a reverse-free code $\mathcal{F} \subset [n]_{(k)}$ and a code $\mathcal{G} \subset [k]_{(k)}$ full of flips. Consider all k-tuples of the form $(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$, where $\mathbf{x} \in \mathcal{F}$ and $\sigma \in \mathcal{G}$. We claim these are all distinct. Indeed, suppose that $(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = (y_{\tau(1)}, \ldots, y_{\tau(k)})$, $\mathbf{x}, \mathbf{y} \in \mathcal{F}$, $\mathbf{x} \neq \mathbf{y}$, and $\sigma, \tau \in \mathcal{G}$, $\sigma \neq \tau$. Then by the definition of $\mathcal{G}$ there are $i \neq j$

with $\alpha := \sigma(i) = \tau(j)$ and $\beta := \sigma(j) = \tau(i) \neq \alpha$. Then $x_{\sigma(i)} = y_{\tau(i)}$ and $x_{\sigma(j)} = y_{\tau(j)}$ imply that (x_α, x_β) and (y_α, y_β) are reversed pairs, a contradiction.

Inequalities (7.4) and (8.3) improve (2.6) and (3.3):

$$F(n,k) \leq \frac{k!\binom{n}{k}}{G(k,k)}, \quad F(k,k) \leq \frac{k!}{G(k,k)}, \quad \text{and} \tag{8.4}$$
$$f(k) \leq \frac{1}{G(k,k)} \leq \frac{8}{(1.515\dots)^k}.$$

One can achieve further improvements using Theorem 2 and (2.5). We have

$$\begin{aligned} F(n,k) &\leq F(n,3)F(n-3,3)F(n-6,3)\dots \\ &\leq \left(\frac{5}{4}+o(1)\right)^{\lfloor k/3\rfloor} \frac{n(n-1)\dots(n-k+1)}{6^{\lfloor k/3\rfloor}} \\ &\leq \left(\frac{5}{24}+o(1)\right)^{k/3} k!\binom{n}{k} = \frac{k!\binom{n}{k}}{(1.686\dots+o(1))^k}. \end{aligned} \tag{8.5}$$

The proof of the following inequality is the same as the proof of (2.5):

$$\overline{F}(n,k) \leq \overline{F}(n,i)\overline{F}(n,k-i). \tag{8.6}$$

Then Theorem 3 and (8.6) give

$$\overline{F}(n,k) \leq \overline{F}(n,3)^{\lfloor k/3\rfloor} n^{k-3\lfloor k/3\rfloor} \leq \left(\frac{5}{24}+o(1)\right)^{k/3} n^k = \frac{n^k}{(1.686\dots+o(1))^k}. \tag{8.7}$$

9. Small alphabets. In this section we deal with the case of n fixed and the length of the codewords $k \to \infty$. The true order of magnitude of the maximum size is polynomial in k.

THEOREM 4. *Let $n \geq 2$, $k \geq 2$. Then*

$$\left(\frac{k}{\binom{n}{2}}\right)^{\binom{n}{2}} \leq \overline{F}(n,k) \leq \binom{k}{\leq n-1}\binom{k}{\leq n-2}\cdots\binom{k}{\leq 1} = O\left(k^{\binom{n}{2}}\right),$$

where $\binom{k}{\leq \ell}$ stands for $\sum_{0\leq i\leq \ell}\binom{k}{i}$.

Proof of the upper bound. We use induction on n and k. In the case in which n or k is small, we have $\overline{F}(1,k)=1$, $\overline{F}(2,k)=k+1$, and $\overline{F}(n,1)=n$, $\overline{F}(n,2)=\binom{n+1}{2}$.

Let $\mathcal{F} \subset [n]^k$ be a reverse-free code, $n,k \geq 3$. Let $\mathbf{x} \in [n]^k$ be an arbitrary codeword. Define its i-support, $\text{supp}_i(\mathbf{x})$, as the subset of the coordinates when $\mathbf{x}$ takes value i,

$$\text{supp}_i(\mathbf{x}) := \{\ell : x_\ell = i\}.$$

Let $\mathcal{F}_1$ be the family of 1-supports, and for each $A \in \mathcal{F}_1$ let

$$\mathcal{F}_A := \{\mathbf{y} \in \mathcal{F} : \text{supp}_1(\mathbf{y}) = A\}.$$

Our first observation is that A cannot appear as a 1-support too many times,

$$|\mathcal{F}_A| \leq \overline{F}(n-1, k-|A|). \tag{9.1}$$

Indeed, consider the projection $\mathcal{F}_A|([k]\setminus A)$. It is a reverse-free code of length $k-|A|$ using $n-1$ symbols (namely, $\{2,3,\ldots,n\}$), so (9.1) follows.

We claim that

$$|\mathcal{F}_1| \le \binom{k}{n-1} + \binom{k}{n-2} + \cdots + \binom{k}{0}. \tag{9.2}$$

Then the upper bound follows from (9.1) by induction.

Our main tool for the proof of (9.2) is the following theorem, which was discovered independently and about the same time by Sauer [23], Shelah [24], and Vapnik and Chervonenkis [26] and became an important result in different contexts. Let M be an $m \times k$ matrix of 0's and 1's with distinct rows. We say that the *VC-dimension* of M is at least s if one can find s columns S such that $M|S$, the matrix restricted to these columns, contains all the 2^s possible 0-1 rows. It is known (see [23, 24, 26]) that if the VC-dimension is at most s, then

$$m \le \sum_{0\le i\le s} \binom{k}{i}. \tag{9.3}$$

We claim that the VC-dimension of $\mathcal{F}_1$ is at most $n-1$. Thus (9.3) implies (9.2). Suppose, on the contrary, that there exists an n-set $S \subset [k]$, $S := \{s^{(1)},\ldots,s^{(n)}\}$, such that $\mathcal{F}$ induces all possible traces, $\mathcal{F}|S = 2^S$. Then there are members $F^{(i)} \in \mathcal{F}_1$ such that $F^{(i)} \cap S = S \setminus \{s^{(i)}\}$, $1 \le i \le n$. This means that there exist $\mathbf{x}^{(i)} \in \mathcal{F}$ with coordinates

$$\mathbf{x}_s^{(i)} = \begin{cases} 1 & \text{for } s \in S,\ s \ne s^{(i)}, \\ \alpha_i & \text{for } s = s^{(i)}, \end{cases}$$

where $\alpha_i \in \{2,3,\ldots,n\}$, $i \in [n]$. By the pigeonhole principle we obtain a $i,j \in [n]$, $i \ne j$, but $\alpha_i = \alpha_j$. Then $\mathbf{x}^{(i)}$ and $\mathbf{x}^{(j)}$ contains a reversed pair (at coordinates $s^{(i)}$ and $s^{(j)}$). This final contradiction completes the proof of the upper bound in the theorem. ∎

Proof of the lower bound. We explicitly construct a reverse-free code $\mathcal{F} \subset [n]^k$ of the desired size.

Split $[k]$ into $n(n-1)/2$ almost equal parts $[k] = \cup_{1\le i<j\le n} V_{i,j}$. Here $|V_{i,j}| \ge \lfloor k/\binom{n}{2}\rfloor$. Take a reverse-free family $\mathcal{F}_{i,j}$ of $|V_{i,j}|+1$ vectors with coordinates $V_{i,j}$ such that each $\mathbf{x} \in \mathcal{F}_{i,j}$ takes values i and j only. Define $\mathcal{F}$ as the product of all of these families:

$$\mathcal{F} := \{\mathbf{y} \in [n]^k : \mathbf{y}|V_{i,j} \in \mathcal{F}_{i,j} \text{ for all } 1 \le i < j \le n\}.$$

This $\mathcal{F}$ is reverse free. Suppose, on the contrary, that there exist $\mathbf{x}, \mathbf{y} \in \mathcal{F}$ such that (x_α, x_β) and (y_α, y_β) are reversed pairs, i.e., $x_\alpha = y_\beta = i$ and $x_\beta = y_\alpha = j > i$. Then $\alpha, \beta \in V_{i,j}$, but $\mathcal{F}_{i,j}$ has no reversed pairs, a contradiction. ∎

10. Hypergraph problems.

Throughout this paper we dealt with a problem of ordered k-tuples from a set of n elements. However, it is worthwhile to look at this problem also from some different perspectives.

We can consider it in the spirit of Turán-type problems. Turán problems [25] deal with questions of the following type: Given a family of graphs $\mathcal{H}$, what is the maximum number of edges that an n-vertex graph may have if it does not contain any of the graphs in $\mathcal{H}$ as a subgraph? This type of problem has been studied extensively and has been generalized to different types of combinatorial structures. As we move from ordinary graphs to directed graphs these problems become much more complicated, and for hypergraphs they are notoriously difficult. In this case exact results are rare and even the asymptotic behavior is poorly understood (see, e.g., [13]). In particular the case of uniform hypergraphs has drawn considerable attention [11, 14, 16, 15]. Now, the problem of pairwise reverse-free k-tuples is a Turán-type problem for a particular generalization of this structure, directed uniform hypergraphs. For the sake of completeness we state the definition of a directed k-uniform hypergraph according to [8]

DEFINITION 1. *A k-uniform directed hypergraph H is a pair (V, E), where V is a finite set of vertices and E is a family of ordered k-tuples of vertices (all vertices in each k-tuple must be distinct, i.e., we do not allow loops).*

Therefore in the light of this reformulation it would be nice to determine at least the asymptotic behavior of this nontrivial problem.

11. Traces. We can consider our problem from another point of view. Note that the pairwise reverse-free property of a set of k-tuples trivially implies that any projection on two coordinates of such a set contains at most $\binom{n}{2}$ different pairs. Thus, loosely speaking we ask how large a set of k-tuples can be if its projections (in this case on two coordinates) are somehow "small." This question is very much in the spirit of Vapnik–Chervonenkis dimension-type problems. The result (9.3) was generalized by Frankl [12] and Alon [1] as follows.

Let $\mathcal{F} \subset [n]^k$ be a set of sequences. We say that $\mathcal{F} \to (s, r)$, or that $\mathcal{F}$ is (s, r)-dense, if there exists an s-set $S \subset [k]$ such that it *induces* at least r traces, i.e., $|\mathcal{F}|S| \geq r$, where $\mathcal{F}|S := \{\mathbf{x}|S : \mathbf{x} \in \mathcal{F}\}$, the projection of $\mathcal{F}$ to the coordinates from S. The family $\mathcal{F}$ is *monotone* if $\mathbf{x} \in \mathcal{F}$, $\mathbf{x} = (x_1, \ldots, x_k)$, $\mathbf{y} \in [n]^k$, $1 \leq y_i \leq x_i$ $(1 \leq i \leq k)$ imply $\mathbf{y} \in \mathcal{F}$. The Frankl–Alon theorem says that in case $\mathcal{F} \not\to (s, r)$ one can find a monotone family $\mathcal{F}' \subset [n]^k$ of the same size $|\mathcal{F}'| = |\mathcal{F}|$ such that $\mathcal{F}' \not\to (s, r)$.

One can find further information on matrices with forbidden configurations in the papers of Anstee and Sali; see [2, 3].

If we restrict ourselves to the quantitative version of the pairwise reverse-free property, it is not difficult to see that the exact asymptotic can be determined. Indeed, a pairwise reverse-free family induces at most $\binom{n}{2}$ pairs on a projection on any two coordinates; in other words, $\mathcal{F} \not\to (2, \binom{n}{2} + 1)$. It is not difficult to see that the Alon–Frankl result implies that the maximum size of such a family of ordered k-tuples is

$$\max |\mathcal{F}| = \left(\frac{n}{\sqrt{2}}\right)^k + O(n^{k-1}).$$

Another proof for this upper bound can be given by applying Shearer's lemma [10]. It is trivial to find a construction achieving it, namely, the family $[m]_{(k)}$ with $m := \lfloor n/\sqrt{2} \rfloor - 1$.

This result suggests that the difficulty of the problem of the pairwise reverse-free sets stems from the fact that this property settles some kind of qualitative requirement regarding the structure of the projections. Now, several papers [4, 5, 21, 22] dealt with this kind of requirement regarding set families (or equivalently binary strings representing their characteristic vectors), and we think it would be interesting to consider this type of problem on families of ordered sets, too.

12. Conclusion, more problems. In this paper we tackle the problem of determining the maximum cardinality $F(n,k)$ of a pairwise reverse-free code of length k over a set of n symbols. In particular we established the first nontrivial case regarding $F(n,3)$. However, in spite of our efforts, determining the exact formula for the general case $F(n,k)$ remains an open problem. In this context it seems reasonable to concentrate on estimating their asymptotic behavior with k fixed and n tending to infinity. Thus the aim is to exactly determine the value of $f(k)$. This is also an open question, as the best bounds provided in (3.3) seem to be far from the truth. Hence, it would be interesting even to decide if the true order of magnitude of $f(k)$ is $\exp(-ck)$ and if it is much bigger than $1/k!$, e.g., whether $k!f(k) \to \infty$.

Furthermore we consider the problem of determining the maximum cardinality $G(n,k)$ of a code $\mathcal{G} \subset [n]^k$ full of flips where we require the reverse property to hold for every pair of its codewords. As might be expected, very little is known even in this case, and this remains also an open problem.

Almost the same situation prevails if we consider the reverse-free property on permutations. Again it seems very hard to establish the true order of magnitude of $F(k,k)$ and $G(k,k)$, and the best bounds we provide are the following:

$$\frac{1}{k^2}(1.898\ldots)^k \le F(k,k) \le \frac{k!}{(1.686\ldots + o(1))^k},$$
$$\frac{1}{8}(1.515\ldots)^k \le G(k,k) \le \frac{k^2 k!}{(1.898\ldots)^k}.$$

However, we believe that these bounds are far from the truth.

Besides the above, we approached the reverse-free problem in the case in which the size n of the alphabet is fixed and the length k of the codewords goes to infinity. In this case we established the true order of magnitude of the maximum size of such a code, but the exact formula is still unknown.

Finally, we would like to mention two further seemingly related open problems.

Two permutations σ and τ of $[n]$ are *colliding* if there is an i with $|\sigma(i) - \tau(i)| = 1$. Let $\rho(n)$ be the maximum cardinality of a set of pairwise colliding permutations. Körner and Malvenuto [20] showed that $\rho(n) > c^n$ for some $c = 1.661\ldots$ and $\rho(n) \le \binom{n}{\lfloor n/2 \rfloor}$, and here equality holds for $n = 1, 2, \ldots, 7$. They conjecture that equality holds for all n. Actually the lower bound has been improved in [7], where it is proved that $\rho(n) > c^n$ with $c = 1.8155\ldots$.

Sperner's theorem states that $\overline{G}(2,n) = \binom{n}{\lfloor n/2 \rfloor}$. Körner conjectures [19] that $\overline{G}(3,n) = \overline{G}(2,n)$; i.e., the maximum number of ternary sequences such that any two have a reversed pair is the same as the maximum number of binary sequences with the same property. Using Sperner capacities, an upper bound 2^n is known (by Blokhuis [6] and by Calderbank et al. [9]).

Acknowledgment. We would like to thank J. Körner for very valuable discussions and useful suggestions.

REFERENCES

[1] N. Alon, *On the density of sets of vectors*, Discrete Math., 46 (1983), pp. 199–202.
[2] R. Anstee, *A Survey of Forbidden Configuration Results*, manuscript, 2010; available online from http://www.math.ubc.ca/~anstee/FCsurvey10.pdf.
[3] R. Anstee and A. Sali, *Small forbidden configurations.* IV. *The* 3 *rowed case*, Combinatorica, 25 (2005), pp. 503–518.

[4] J. Balogh and B. Bollobás, *Unavoidable traces of set systems*, Combinatorica, 25 (2005), pp. 633–643.
[5] J. Balogh, P. Keevash, and B. Sudakov, *Disjoint representability of sets and their complements*, J. Combin. Theory Ser. B, 95 (2005), pp 12–28.
[6] A. Blokhuis, *On the Sperner capacity of the cyclic triangle*, J. Algebraic Combin., 2 (1993), pp. 123–124.
[7] G. Brightwell, G. Cohen, E. Fachini, M. Fairthorne, J. Körner, G. Simonyi, and Á. Tóth, *Permutation capacities of families of oriented infinite paths*, SIAM J. Discrete Math., 24 (2010), pp. 441–456.
[8] W. G. Brown and M. Simonovits, *Digraph extremal problems, hypergraph extremal problems, and the densities of graph structures*, Discrete Math., 48 (1984), pp. 147–162.
[9] A. R. Calderbank, P. Frankl, R. L. Graham, W. C. W. Li, and L. A. Shepp, *The Sperner capacity of the cyclic triangle for linear and non linear codes* J. Algebraic Combin., 2 (1993), pp. 31–48.
[10] F. Chung, P. Frank, R. Graham, and J. Shearer, *Some intersection theorems for ordered sets and graphs*, J. Combin. Theory Ser. A, 43 (1986), pp. 23–37.
[11] D. De Caen and Z. Füredi, *The maximum size of* 3*-uniform hypergraphs not containing a Fano plane*, J. Combin. Theory Ser. B, 78 (2000), pp. 274–276.
[12] P. Frankl, *On the trace of finite sets*, J. Combin. Theory Ser. A, 34 (1983), pp. 41–45.
[13] Z. Füredi, *Turán type problems*, in Surveys in Combinatorics, London Math. Soc. Lecture Note Ser. 166, Cambridge University Press, Cambridge, UK, 1991, pp. 253–300.
[14] Z. Füredi and M. Simonovits, *Triple systems not containing a Fano configuration*, Combin. Probab. Comput., 14 (2005), pp. 467–484.
[15] P. Keevash and B. Sudakov, *On a hypergraph Turán problem of Frankl*, Combinatorica, 25 (2005), pp. 673–706.
[16] P. Keevash and B. Sudakov, *The Turán number of the Fano plane*, Combinatorica, 25 (2005), pp. 561–574.
[17] M. G. Kendall and B. Babington Smith, *On the method of paired comparisons*, Biometrika, 33 (1940), pp. 324–345.
[18] J. Körner, *On the extremal combinatorics of the Hamming space*, J. Combin. Theory Ser. A, 71 (1995), pp. 112–126.
[19] J. Körner, *personal communication*, 2009.
[20] J. Körner and C. Malvenuto, *Pairwise colliding permutations and the capacity of infinite graphs*, SIAM J. Discrete Math., 20 (2006), pp. 203–212.
[21] B. Patkós, *l-trace k-Sperner families of sets*, J. Combin. Theory Ser. A, 116 (2009), pp. 1047–1055.
[22] B. Patkós, *Traces of uniform families of sets*, Electron. J. Combin., 16 (2009), article 8.
[23] N. Sauer, *On the density of families of sets*, J. Combin. Theory Ser. A, 13 (1972), pp. 145–147.
[24] S. Shelah, *A combinatorial problem: Stability and order for models and theories in infinitary languages*, Pacific J. Math., 41 (1972), pp. 247–261.
[25] P. Turán, *On the theory of graphs*, Colloq. Math., 3 (1954), pp. 19–30.
[26] V. N. Vapnik and A. Y. Chervonenkis, *On the uniform convergence of relative frequencies of events to their probabilities*, Theory Probab. Appl., 16 (1971), pp. 264–280.