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Note

Tight embeddings of partial quadrilateral packings

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ABSTRACT

Let $f(n; C_4)$ be the smallest integer such that, given any set of edge disjoint quadrilaterals on n vertices, one can extend it into a complete quadrilateral decomposition by including at most $f(n; C_4)$ additional vertices. It is known, and it is easy to show, that $\sqrt{n} - 1 \leqslant f(n; C_4)$. Here we settle the longstanding problem that $f(n; C_4) = \sqrt{n} + o(\sqrt{n})$.

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1. *H*-packings and *H*-designs

Let H be a simple graph. An *H-packing of order n* is a set $\mathcal{P} := \{H_1, H_2, \ldots, H_m\}$ of edge disjoint copies of H whose union forms a graph with n vertices. If this graph is the *complete graph* K_n, then $\mathcal{P}$ is called an H-decomposition on n vertices, or following the terminology of design theory, it is called an *H-design* of order n. In this case $\binom{n}{2}/e(H)$ is an integer, where $e(H)$ denotes the number of edges of H, and we have the obvious congruence properties

$$e(H) \,\Big|\, \binom{n}{2} \quad \text{and} \quad \gcd\big(\deg(v_1), \ldots, \deg(v_k)\big) \,\Big|\, (n-1) \tag{1}$$

where H has vertices $v_1, v_2, \ldots, v_k$, and $\deg(v_i)$ denotes the degree of v_i, $i = 1, \ldots, k$. An integer n for which there exists an H-design of order n should satisfy the divisibility constraints (1), and is called *H-admissible*. Wilson [21] proved that given any graph H, there exists an integer $n_0(H)$ such

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that every $n > n_0(H)$ satisfying (1) is H-admissible. The case of the complete graph $H = K_t$ is equivalent to the existence of Steiner systems $S(n,t,2)$. Further H-designs and graph decompositions can be found in [1,2].

Let $\Lambda(n;H)$ be the least H-admissible integer N_0 such that, for every H-admissible integer $N \geqslant N_0$, any H-packing of order n can be extended into an H-design of order N. The existence of $\Lambda(n;H)$, for every H, follows from a far more general result of Wilson [20]. The determination of Λ is a difficult research area with full of problems and results. Let us list a few explicit bounds. Concerning C_ℓ, the cycle of length ℓ, Hoffman, Lindner, Rodger, and Stinson (see [9,16]) showed

$$\Lambda(n;C_{2k}) \leqslant kn + O\big(k^2\big).$$

For the quadrilateral Lindner [13] showed

$$\Lambda(n;C_4) \leqslant 2n + 15.$$

The case $\Lambda(n;D)$ where D denotes the five-vertex graph consisting of a 4-cycle with a pendant edge attached was studied by Jenkins [10], he showed

$$\Lambda(n;D) \leqslant 4n + 22.$$

For a graph K called a *kite*, that is a triangle with a pendant edge attached, it was recently showed by Küçükçifçi, Lindner, and Rodger [12] that

$$\Lambda(n;K) \leqslant 8n + 9.$$

Bryant, Khodkar, and El-Zanati [5] gave explicit upper bounds (linear in n) for an infinite class of bipartite H.

2. Quadrilateral decompositions

In the case of $H = C_4$, i.e., for quadrilateral decompositions, (1) implies that a C_4-admissible integer n must satisfy $n \equiv 1 \pmod 8$. Furthermore, every such integer n is C_4-admissible (see, e.g., in [15]). Indeed, for $n = 9$ and $V = \{0,1,2,\ldots,8\}$, it is clear that the cycles

$$(i, i+2, i+5, i+1) \pmod 9, \quad i = 0,1,\ldots,8$$

form a C_4-decomposition of K_9. Using this construction and the obvious fact that

$$\textit{the complete bipartite graph } K_{2p,2q} \textit{ has a } C_4\textit{-decomposition} \tag{2}$$

we obtain easily that

$$\textit{any } C_4\textit{-decomposition of } K_n \textit{ can be extended into a } C_4\textit{-decomposition of } K_{n+8}. \tag{3}$$

Indeed, choose an arbitrary vertex v of the C_4-decomposition of K_n on V, and add an 8-element disjoint vertex set B. Taking a C_4-decomposition of K_9 on $B \cup \{v\}$ and using that both $a = |V \setminus \{v\}|$ and $b = |B|$ are even, the complete bipartite graph $K_{a,b}$ between $V \setminus \{v\}$ and B has a C_4-decomposition according to (2). Thus we obtain a quadrilateral decomposition of K_{n+8} on $V \cup B$.

Hilton and Lindner [8] achieved a breakthrough recently by showing that the order of magnitude of $\Lambda(n;C_4) - n$ is only $o(n)$. They proved

$$n + n^{1/2} - 1 \leqslant \Lambda(n;C_4) \leqslant n + \sqrt{12}n^{3/4} + o\big(n^{3/4}\big). \tag{4}$$

The aim of this paper is to show that $\Lambda(n;C_4)$ is more like the lower bound which settles a problem proposed decades ago (cf. [16]).

Theorem 1. $\Lambda(n;C_4) = n + \sqrt{n} + o(\sqrt{n})$.

In terms of graphs our result shows that every partial quadrilateral packing on n vertices can be extended into a complete quadrilateral decomposition by the inclusion of at most $\sqrt{n} + o(\sqrt{n})$ additional vertices.

3. Notations, the chromatic index of graphs

For a graph G the set of its vertices and its edges is respectively denoted by $V(G)$ and $E(G)$. All graphs here are simple graphs with no loop or multiple edge. The degree of a vertex $x \in V(G)$ is $\deg(x) = |\{v \in V(G) \mid vx \in E(G)\}|$, the minimum and the maximum degrees of G are $\delta(G) = \min\{\deg(x) \mid x \in V(G)\}$ and $\Delta(G) = \max\{\deg(x) \mid x \in V(G)\}$. A cycle and a path with k vertices are denoted by C_k and P_k, respectively. The 4-cycle C_4 is also called a *quadrilateral*. A *matching* is a set of pairwise disjoint edges, the size of the largest matching in the graph G is called its *matching number* and is denoted by $\nu(G)$.

Let C be an arbitrary set of $k = |C|$ *colors*. A *proper edge k-coloring* of the graph G is a function $c: E(G) \to C$, such that any two incident edges $e, f \in E(G)$ receive distinct colors, that is $c(e) \neq c(f)$. The minimum k such that there is a proper edge k-coloring of G is called the *chromatic index* (or edge chromatic number) of G and it is denoted by $\chi'(G)$. By Vizing's theorem (see e.g. [3]) for every simple graph G one has

$$\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G) + 1.$$

A related parameter, the list chromatic index of a graph G is defined as follows. Given a mapping L from $E(G)$ into the power set of the underlying color set C, a proper edge list coloring c of G from the lists $L(e)$ is a proper edge coloring of G with the additional property that $c(e) \in L(e)$, for every $e \in E(G)$. The *list chromatic index*, $\chi'_\ell(G)$, is the least integer t such that there exists a proper edge list coloring of G for every list assignment L, provided $|L(e)| \geqslant t$, for all $e \in E(G)$.

It is an open problem whether the list chromatic index of a graph equals its edge chromatic number (see [4]). No tight bound comparable to Vizing's theorem is known for the list chromatic index. However, Kahn [11] proved the asymptotic version of the list chromatic index conjecture by showing that $\chi'_\ell = \Delta + o(\Delta)$. A more explicit bound (with a polynomial error term) was proved by Häggkvist and Janssen [7]. The best known upper bound is due to Molloy and Reed [17]: There exists an absolute constant c_0, not depending on Δ, such that for every simple graph of maximum degree Δ one has

$$\chi'_\ell < \Delta + c_0(\log \Delta)^4 \Delta^{1/2}. \tag{5}$$

4. Tightest quadrilateral extensions

Let $f(n; H)$ be the smallest integer t such that, any H-packing on n vertices can be extended to an H-decomposition on at most $n + t$ vertices. By definition,

$$n + f(n; H) \leqslant \Lambda(n; H).$$

On the other hand, (3) implies that every C_4-packing $\mathcal{P}$ on n vertices can be embedded into a C_4-decomposition of order $n + f(n; C_4)$, $n + f(n; C_4) + 8$, $n + f(n; C_4) + 16$, etc., hence $\Lambda(n; C_4) \leqslant n + f(n; C_4)$. Thus we obtain

Proposition 2. *For every integer n, $\Lambda(n; C_4) = n + f(n; C_4)$.*

Construction 3. Define a C_4-packing $\mathcal{P}$ on $V = \{1, 2, \ldots, n\}$ with the quadrilaterals of the form $(2i, 2j, 2i-1, 2j-1)$, for all $1 \leqslant i < j \leqslant \lfloor n/2 \rfloor$. Observe that $\mathcal{P}$ leaves uncovered the matching $M = \{(2i-1, 2i) \mid 1 \leqslant i \leqslant \lfloor n/2 \rfloor\}$, and if n is odd, then all edges incident with vertex n are also uncovered.

Consider any C_4-decomposition $\mathcal{P}^*$ that extends $\mathcal{P}$ of Construction 3 to the vertex set $V \cup B$. Every edge of M must belong to a quadrilateral of $\mathcal{P}^*$ with its opposite edge between a pair of vertices not in the subset of V covered by M. These pairs must be distinct, hence either $\binom{|B|}{2} \geqslant n/2$ or $\binom{|B|+1}{2} \geqslant (n-1)/2$. This implies

$$f(n; C_4) \geqslant |B| \geqslant \sqrt{n} - 1,$$

and the lower bound in (4) follows.

5. Completing a quadrilateral packing

Given a C_4-packing $\mathcal{P}$ on the vertex set V, $|V| = n$, the union of the edge sets of the quadrilaterals in $\mathcal{P}$ define a graph on V, its complement will be called the *graph of the uncovered edges* and it is denoted by $G(\mathcal{P})$, or simply by G. We also use the notation $f(\mathcal{P})$ to denote the smallest integer b such that $\mathcal{P}$ can be extended into a C_4-decomposition of order $n + b$.

Lindner [14] proved that if n is odd then $\mathcal{P}$ can be completed by adding at most $2|E(G)|$ new vertices to it. We include here a short proof of Lindner's result.

Lemma 4. *Let $\mathcal{P}$ be a C_4-packing of order n and let G be the graph of the uncovered edges. If n is odd, then*

$$f(\mathcal{P}) \leqslant 2\big|E(G)\big|. \tag{6}$$

Proof. We shall define an iterative procedure. In each step a set B of new vertices is added to $V(G)$ together with quadrilaterals on $V \cup B$, thus extending $\mathcal{P}$ and reducing the graph of uncovered edges by at least $|B|/2$.

If G contains a C_4, then include it to $\mathcal{P}$.

If G contains a triangle $T = K_3$, then we add a set B of six new vertices. Include a C_4-decomposition of K_9 on the vertex set $V(T) \cup B$ to $\mathcal{P}$, and add a C_4-decomposition of the complete bipartite graph $K_{6,n-3}$ (see (2)) with parts B and $V \setminus V(T)$, each of even cardinality. Observe that in this step $|B| = 6$, $|E(G)|$ reduces by $|B|/2 = 3$, and $|V(G) \cup B|$ remains odd.

If G contains a (not necessarily induced) path $P = (a_1, \dots, a_5)$, then we add $B = \{b_1, b_2\}$. Include the quadrilaterals

$$(a_1, a_2, a_3, b_1), (a_3, a_4, a_5, b_2), (b_1, a_2, b_2, a_4),$$

and add a C_4-decomposition of the complete bipartite graph $K_{2,n-5}$ (see (2)) with parts B and $V \setminus V(P)$, each of even cardinality. In this step we have $|B| = 2$, $|E(G)|$ reduces, and $|V(G) \cup B|$ remains odd.

The procedure consists in repeating these steps above in any order as long as any of the three conditions is satisfied. Since n is odd, the graph of the uncovered edges has only even degrees. Thus, if G contains no quadrilaterals, neither triangles, nor paths of length four, then G cannot have any edge. Therefore, when our procedure stops, the quadrilaterals extending $\mathcal{P}$ form a C_4-decomposition on a vertex set that is not larger than $n + 2|E(G)|$. $\square$

6. Quadrilateral-free graphs, transversals

A set $Q \subset V(G)$ is called a *transversal* (or vertex-cover) of G if Q meets all edges of G. The minimum size of such a Q is called the transversal number and is denoted by $\tau(G)$. The complement of a transversal is an *independent set* of vertices. The size of a largest independent set is denoted by $\alpha(G)$.

Lemma 5. *Let G be a quadrilateral-free graph with a transversal Q. If $\deg(v) \neq 1$ for all $v \in V(G) \setminus Q$, then $|E(G)| \leqslant |Q|^2 - \frac{1}{2}|Q|$.*

Proof. Let M be the set of all isolated edges in the subgraph of G induced by Q, clearly $|M| \leqslant |Q|/2$. Observe that all edges in $E(G) \setminus M$ are covered by a path of the form (q_1, x, q_2) with $q_1, q_2 \in Q$. This is clear for the edges lying in Q, and it follows for any edge incident with a vertex $v \in V(G) \setminus Q$, since $V(G) \setminus Q$ is an independent set and $\deg(v) \geqslant 2$.

Because G is quadrilateral-free, there are at most $\binom{|Q|}{2}$ paths of the form (q_1, x, q_2) with $q_1, q_2 \in Q$. Therefore we have $|E(G) \setminus M| \leqslant 2\binom{|Q|}{2}$, and $|E(G)| = |M| + |E(G) \setminus M| \leqslant \frac{1}{2}|Q| + 2\binom{|Q|}{2} = |Q|^2 - \frac{1}{2}|Q|$ follows. $\square$

Note that the upper bound in Lemma 5 is tight as the following construction shows. Let Q be the vertex set of a complete graph, with $|Q|$ even, then subdivide each of the $\binom{|Q|}{2}$ edges by a vertex and add $|Q|/2$ further edges forming a matching on Q.

Lemma 6. *Let $\mathcal{P}$ be a C_4-packing and let G be the graph of the uncovered edges. If q vertices cover all edges of G, then $\mathcal{P}$ can be completed by the inclusion of at most $2(q+1)^2$ additional vertices, that is*

$$f(\mathcal{P}) \leqslant 2\big(\tau(G)+1\big)^2.$$

Proof. If $\mathcal{P}$ is of even order, then change its parity by adding a further vertex. Consider the (eventually extended) graph G_1 of the uncovered edges. Clearly, G_1 has a transversal Q of size at most $q+1$. Remove a maximal set of edge disjoint quadrilaterals from $E(G_1)$ and include them to $\mathcal{P}$. The remaining graph of the uncovered edges, G_2, is quadrilateral-free. Since $|V(G_2)| = |V(G_1)|$ is odd, the degrees of G_2 are all even. In particular, G_2 has no vertex of degree 1. By Lemmas 4 and 5, applied on G_2, we obtain

$$f(\mathcal{P}) \leqslant 2\big|E(G_2)\big| \leqslant 2\left[|Q|^2 - \frac{1}{2}|Q|\right] = 2\left[(q+1)^2 - \frac{1}{2}(q+1)\right] < 2\big(\tau(G)+1\big)^2. \qquad \square$$

7. Quadrilateral-free graphs, minimum degree

A neat double counting argument yields the particular case of a general estimation used in the Zarankiewicz problems in Ramsey theory, also known as the Johnson's bound for error correcting codes in coding theory (see in [3,19]).

Lemma 7. *If $E_1, \ldots, E_m$ are sets satisfying $|E_i| \geqslant d$, for $i = 1, \ldots, m$, and $|E_i \cap E_j| \leqslant 1$, for $1 \leqslant i < j \leqslant m$, then*

$$\left|\bigcup_{i=1}^{m} E_i\right| \geqslant \frac{d^2 m}{d-1+m}.$$

It is known (see [3]) that if a graph G has n vertices and contains no quadrilateral, then $|E(G)| \leqslant (n + n\sqrt{4n-3})/4$. An obvious corollary is the bound

$$\delta(G) \leqslant \sqrt{n - \frac{3}{4}} + \frac{1}{2}. \tag{7}$$

In the next lemma we use the transversal number of the graph to obtain a better bound on $\delta(G)$ that is independent of the cardinality of $V(G)$.

Lemma 8. *If the edges of a quadrilateral-free graph G are covered by q vertices then*

$$\delta(G) < q^{1/2} + \frac{1}{2}q^{1/4} + \frac{1}{4}. \tag{8}$$

Note that this upper bound cannot be improved significantly without any further restriction. This follows from the fact that the polarity graph G_d is quadrilateral-free, it has d^2+d+1 vertices, its minimum degree is $\delta(G_d) = d$, and $\alpha(G_d) \leqslant d\sqrt{d+1} + d$ (see, e.g., [6]). Even more, Mubayi and Williford [18] recently showed that if G_d is the unitary polarity graph, then $\alpha(G_d) = d^{3/2}+1$. (Such graph exists if d is an even power of a prime.) Hence we should take $q = |V(G_d)| - \alpha(G_d) = d^2 - d^{3/2} + d$, and for these values

$$d \geqslant q^{1/2} + \frac{1}{2}q^{1/4} - \frac{1}{2}.$$

For the proof of the above lemma we will need the following statement.

Proposition 9. *Let $q \geqslant 1$ be fixed and define the polynomial $p(x) := x^3 - x^2 - (x^2 - q)^2$. Then $p(x) < 0$ for $x \geqslant q^{1/2} + \frac{1}{2}q^{1/4} + \frac{1}{4}$.*

Proof. Substitute $q := t^4$ (where $t \geqslant 1$ fixed) and $x := t^2 + \frac{1}{2}t + \frac{1}{4} + y$, where $y \geqslant 0$. After a little algebra (using Maple V) one obtains

$$p(x) = p\left(t^2 + \frac{1}{2}t + \frac{1}{4} + y\right) = -y^4 - y^3\left(4t^2 + 2t\right) - y^2\frac{32t^4 + 48t^3 + 12t^2 + 5}{8}$$
$$- y\frac{32t^5 + 16t^4 + 4t^3 + 10t^2 + 5t + 3}{8} - \frac{144t^4 + 160t^3 + 136t^2 + 48t + 13}{256}.$$

Here each term is negative, hence $p(x) < 0$ follows. □

Proof of Lemma 8. Let Q be a set of q vertices covering the edges of G. Let $V \setminus Q = \{v_1, v_2, \dots, v_m\}$, $E_i := \{y \in Q\colon\ v_i y \in E(G)\}$ and $d := \delta(G)$. Since G is quadrilateral-free, $|E_i \cap E_j| \leqslant 1$, for $1 \leqslant i < j \leqslant m$. Since $q = |Q| \geqslant |\bigcup_{i=1}^m E_i|$, Lemma 7 implies

$$(d - 1 + m)q \geqslant d^2 m. \tag{9}$$

Inequality (7) is equivalent with

$$m + q = \left|V(G)\right| \geqslant d^2 - d + 1. \tag{10}$$

If $d^2 - q \leqslant 0$, then (8) follows immediately. Otherwise, multiply (10) by $(d^2 - q)$ and add it to (9). After rearrangements we obtain

$$d^3 - d^2 - \left(d^2 - q\right)^2 = p(d) \geqslant 0.$$

Thus, by Proposition 9, (8) follows. □

8. Quadrilateral-free graphs, updegree

Let G be a graph with an ordering $\pi = (v_1, \dots, v_m)$ of its vertices. The *updegree* of the vertex v_i with respect to π is defined by

$$\overrightarrow{\deg}_G(v_i) = \left|\left\{v_\ell \;\middle|\; \ell > i,\ v_i v_\ell \in E(G)\right\}\right|,$$

and the *updegree of* G with respect to π is defined by

$$\overrightarrow{\Delta}(G) = \max\left\{\overrightarrow{\deg}_G(v_i) \;\middle|\; 1 \leqslant i \leqslant m\right\}.$$

If G is a graph with $d = \max\{\delta(H) \mid H \subseteq G\}$, then $\overrightarrow{\Delta}(G) \leqslant d$, for some π. Indeed, to obtain an appropriate ordering of $V(G)$ in which the updegree of each vertex becomes d or smaller it is enough to list the vertices by taking greedily the next vertex that has the fewest number of neighbors among the still unlisted vertices. (In fact, we have $d = \min_\pi \overrightarrow{\Delta}(G)$, for every graph G.) Hence (8) of Lemma 8 has the following consequence.

Proposition 10. *If a quadrilateral-free graph G has a transversal of q vertices then*

$$\overrightarrow{\Delta}(G) \leqslant q^{1/2} + q^{1/4}$$

with respect to an appropriate ordering of $V(G)$.

9. Packing paths of length two

Lemma 11. *If G has updegree $d = \overrightarrow{\Delta}(G)$, then its edge set has a decomposition into a matching M and a set $\mathcal{L}$ of edge disjoint copies of paths of length two such that every vertex appears at most $d + 2$ times as an end vertex of a member of $\mathcal{L}$.*

Proof. Let $(v_1, \ldots, v_m)$ the ordering of $V(G)$ that yields the updegree $d = \vec{\Delta}(G)$. The path formed by the two edges $v_i v_\ell$, $v_\ell v_j$ is called a *cherry* in G if $i, j < \ell$. Let $\mathcal{L}_0$ be a maximal set of pairwise edge disjoint cherries of G. Observe that each $x \in V(G)$ can appear in at most $\overrightarrow{\deg}_G(x) \leqslant d$ times as an end vertex of a cherry in $\mathcal{L}_0$.

By the maximality of $\mathcal{L}_0$, the set of edges of G not belonging to $\mathcal{L}_0$ form a forest T such that the updegree of T with respect to the reverse order $(v_m, v_{m-1}, \ldots, v_1)$ satisfies $\vec{\Delta}(T) \leqslant 1$. Let F_1 be a maximal set of pairwise edge disjoint cherries in T. Note that these cherries have no common end vertices, since $\vec{\Delta}(T) \leqslant 1$.

The set of edges of T that are not covered by the cherries in F_1 form a linear forest L, that is the connected components of L are paths. Every path can be decomposed into a matching and copies of P_3 with no common end vertices. Let F_2 be the set of all these copies of P_3, and let M be the matching collected from all components of L. Thus we obtain a decomposition of $E(G)$ into a matching M and a family of cherries $\mathcal{L} = \mathcal{L}_0 \cup F_1 \cup F_2$ as required. $\square$

10. Extending a quadrilateral packing

Let $\mathcal{P}$ be a C_4-packing on the vertex set V, and let G be the graph of the edges uncovered by $\mathcal{P}$. We describe a procedure that extends $\mathcal{P}$ into a packing $\mathcal{P}'$ on $V \cup B$ which covers all edges of G by adding a (small) set B to V. Note that we do not require $\mathcal{P}'$ to be a complete C_4-decomposition.

Let $M \cup \mathcal{L}$ be an arbitrary decomposition of $E(G)$ into a matching M and a set $\mathcal{L}$ of pairwise edge disjoint copies of P_3. Denote $H(\mathcal{L})$ the graph on vertex set V with xy being an edge of $H(\mathcal{L})$ if and only if x, y are the end vertices of some P_3 belonging to $\mathcal{L}$.

Proposition 12. *If B is a set satisfying $B \cap V = \emptyset$ and*

$$|B| \geqslant \max\bigl\{\sqrt{2|M|} + 1,\ \chi'_\ell\bigl(H(\mathcal{L})\bigr) + 2\bigr\}, \tag{11}$$

then $\mathcal{P}$ can be extended into a C_4-packing $\mathcal{P}'$ on $V \cup B$ that covers all edges in G.

Proof. Since $|M| \leqslant \binom{|B|}{2}$ there is a mapping $g : \bigcup\{\{x, y\} \mid xy \in M\} \to B$ such that

$$\bigl\{g(x), g(y)\bigr\} \neq \bigl\{g(x'), g(y')\bigr\} \quad \text{for any two distinct edges } xy, x'y' \in M.$$

Next, assign to every edge uv of $H(\mathcal{L})$ a list $L(uv)$

$$L(uv) := B \setminus \bigl\{g(u), g(v)\bigr\}.$$

By (11), we have $|L(uv)| \geqslant \chi'_\ell(H(\mathcal{L}))$. Hence by the definition of the list chromatic index, $H(\mathcal{L})$ has a list edge coloring $c : E(H(\mathcal{L})) \to B$ from the lists $L(uv)$. For any path $P \in \mathcal{L}$ with end vertices $u, v \in V$, we have $uv \in E(\mathcal{L})$, thus we can define $c(P) := c(uv)$. Then the list edge coloring c of $H(\mathcal{L})$ yields a mapping $c : \mathcal{L} \to B$ such that $c(P) \neq c(P')$ whenever $P, P' \in \mathcal{L}$ share a common end vertex. Therefore $\mathcal{P}$ can be extended with the quadrilaterals of the form

$$\bigl(x, y, g(y), g(x)\bigr), \quad \text{for } xy \in M \quad \text{and} \quad \bigl(u, w, v, c(P)\bigr), \quad \text{for } P = (u, w, v) \in \mathcal{L}.$$

In this way we obtain a packing $\mathcal{P}'$ that covers all edges in $M \cup \{P \mid P \in \mathcal{L}\}$ thus covering all edges of G as required. $\square$

Proposition 13. *Let $\mathcal{P}$ be a C_4-packing on V, let G be the graph of the uncovered edges, and let $\nu = \nu(G)$. Then $\mathcal{P}$ has an extension into $\mathcal{P}'$ on $V \cup B$ saturating G such that*

$$b := |B| \leqslant \sqrt{2\nu} + c_0 \nu^{1/4} (\log \nu)^4 + c_1 \nu^{1/4}. \tag{12}$$

Here c_0 comes from (5) and c_1 is an absolute constant.

Proof. We may suppose that G is quadrilateral-free. The transversal number of a graph is at most 2ν, so that Proposition 10 gives $d := \vec{\Delta}(G) \leqslant \sqrt{2\nu} + (2\nu)^{1/4}$ with respect to an appropriate ordering of $V(G)$. Then Lemma 11 supplies a decomposition of $E(G)$ into a matching M and a set $\mathcal{L}$ of pairwise edge disjoint copies of P_3 such that

$$\Delta\big(H(\mathcal{L})\big) \leqslant d+2 \leqslant \sqrt{2\nu} + O\big(\nu^{1/4}\big),$$

where $H(\mathcal{L})$ is the graph defined by the end vertices of the paths in $\mathcal{L}$ as above. Observe that since G contains no quadrilateral, $H(\mathcal{L})$ is a simple graph. Thus applying the upper bound (5) we obtain

$$\chi'_\ell\big(H(\mathcal{L})\big) \leqslant (d+2) + c_0(d+2)^{1/4}\big(\log(d+2)\big)^{1/4} = \sqrt{2\nu} + c_0\nu^{1/4}(\log\nu)^4 + O\big(\nu^{1/4}\big).$$

So there exists a $c_1 > 0$ such that if we let

$$b := \max\big\{\big\lceil\sqrt{2|M|}\,\big\rceil + 1,\ \chi'_\ell\big(H(\mathcal{L})\big) + 2\big\},$$

then b satisfies both (12) and (11). Finally, Proposition 12 yields the desired packing $\mathcal{P}'$. □

11. Proof of the theorem

In this section we prove Theorem 1. By Proposition 2, we only have to prove $f(\mathcal{P}) \leqslant (1+o(1))\sqrt{n}$ for every C_4-packing on n vertices. Let $\mathcal{P}_0 := \mathcal{P}$ be a quadrilateral packing on $V_0 = V$, and let G_0 be the graph of the edges uncovered by $\mathcal{P}_0$. Observe that $\nu(G_0) \leqslant n/2$.

We define the consecutive extensions $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ as follows. Given the quadrilateral packing $\mathcal{P}_{i-1}$ on V_{i-1}, let G_{i-1} be its graph of uncovered edges. We apply Proposition 13 to obtain $\mathcal{P}_i$ on a set $V(\mathcal{P}_{i-1}) \cup B_i$ that covers all edges of G_{i-1}. Obviously B_i is a transversal set in G_i, in particular, G_i has no matching with more than $b_i := |B_i|$ edges, for $i = 1, 2, 3$. Because $\nu(G_0) \leqslant n/2$, it follows successively by (12) that

$$\begin{aligned} b_1 &\leqslant \big(1+o(1)\big)\sqrt{n},\\ b_2 &\leqslant \big(1+o(1)\big)\sqrt{2b_1} = O\big(n^{1/4}\big),\\ b_3 &\leqslant \big(1+o(1)\big)\sqrt{2b_2} = O\big(n^{1/8}\big). \end{aligned} \tag{13}$$

As the last step of the procedure we apply Lemma 6 to extend $\mathcal{P}_3$ into a C_4-decomposition $\mathcal{P}_4$ by adding a set B_4 to $V(\mathcal{P}_3)$ with $b_4 = |B_4| \leqslant 2(b_3+1)^2$.

By (13), we have $b_4 \leqslant O(n^{1/4})$, and thus we obtain an embedding $\mathcal{P}$ into a C_4-decomposition on $n + b_1 + b_2 + b_3 + b_4 = n + \sqrt{n} + o(\sqrt{n})$ vertices. Hence $f(\mathcal{P}) \leqslant (1+o(1))\sqrt{n}$ follows.

12. Conclusion, conjectures

In the proof of Theorem 1 we determined $f(n; C_4)$ asymptotically. It is tempting to conjecture that the worst example is the one given by Construction 3.

If H is not bipartite, then $f(n; H)$ is very likely linear in n. We conjecture that for every bipartite graph H one has $f(n; H) = o(n)$.

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