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Partition Critical Hypergraphs

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Abstract

A k-uniform hypergraph $(X, \mathcal{E})$ is 3-color critical if it is not 2-colorable, but for all $E \in \mathcal{E}$ the hypergraph $(X, \mathcal{E} \setminus \{E\})$ is 2-colorable. Lovász proved in 1976, that $|\mathcal{E}| \leq \binom{n}{k-1}$ for a 3-color critical k-uniform hypergraph with $|X| = n$. Here we prove an ordered version that is a sharpening of Lovász' result. Let $\mathcal{E} \subseteq \binom{[n]}{k}$ be a k-uniform set system on an underlying set X of n elements. Let us fix an ordering $E_1, E_2, \ldots E_t$ of $\mathcal{E}$ and a prescribed partition $A_i \cup B_i = E_i$ $(A_i \cap B_i = \emptyset)$ for each member of $\mathcal{E}$. Assume that for all $i = 1, 2, \ldots, t$ there exists a partition $C_i \cup D_i = X$ $(C_i \cap D_i = \emptyset)$, such that $E_i \cap C_i = A_i$ and $E_i \cap D_i = B_i$, but $E_j \cap C_i \neq A_j$ and $E_j \cap C_i \neq B_j$ for all $j < i$. (That is, the ith partition cuts the ith set as it is prescribed, but does not cut any earlier set properly.) Then $t \leq \binom{n-1}{k-1} + \binom{n-1}{k-2} + \ldots + \binom{n-1}{0} = \binom{n}{k-1} + O(n^{k-3})$. This is sharp for $k = 2, 3$. Furthermore, if $A_i = E_i$ and $B_i = \emptyset$ for all i, then $t \leq \binom{n}{k-1}$. We also give a construction of size $\binom{n}{k-1}$.

Keywords: color/partition critical hypergraph, linear algebra method

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1 Introduction

Partition critical hypergraphs came up in the context of forbidden configuration theorems for simple $0-1$ matrices [1]. They are generalizations of 3-critical hypergraphs. Our interest here is in the maximum number of edges in a k-uniform ℓ-critical hypergraph.

Definition 1.1 A k-uniform hypergraph $\mathcal{H}$ is *ℓ-critical* if it is not $\ell-1$-colorable, but deleting any edge or vertex results in a $\ell-1$-colorable hypergraph.

Toft proved [4] that for $k, \ell > 3$ fixed, $n \to \infty$, there $\exists k$-uniform ℓ-critical hypergarph on n vertices of size $\Omega(n^k)$. But all 3-critical k-uniform hypergraphs have size $o(n^k)$. He asked: What is the maximum size of a 3-critical k-uniform hypergraph? Lovász [3] gave the following upper bound.

Theorem 1.2 *Let $\mathcal{H}$ be a 3-critical k-uniform hypergraph on an n-element underlying set. Then*

$$|E(\mathcal{H})| \leq \binom{n}{k-1}. \tag{1}$$

Partition critical hypergraphs are generalizations of color critical ones.

Definition 1.3 A k-uniform hypergraph $\mathcal{E} \subseteq \binom{[n]}{k}$ on an underlying set X of n elements is called *partition critical* if the followings hold. There exists an ordering $E_1, E_2, \ldots E_t$ of $\mathcal{E}$ and a prescribed partition $A_i \cup B_i = E_i$ $(A_i \cap B_i = \emptyset)$ for each member of $\mathcal{E}$ such that for all $i = 1, 2, \ldots, t$ there exists a partition $C_i \cup D_i = X$ $(C_i \cap D_i = \emptyset)$, such that $E_i \cap C_i = A_i$ and $E_i \cap D_i = B_i$, but $E_j \cap C_i \neq A_j$ and $E_j \cap C_i \neq B_j$ for all $j < i$. (That is, the ith partition cuts the ith set as it is prescribed, but does not cut any earlier set properly.

A 3-critical hypergraph is certainly partition critical, as well. Indeed, for an arbitrary ordering of the edges $E_1, E_2, \ldots E_t$ of $\mathcal{E}$, the partition $A_i = E_i$, $B_i = \emptyset$ works for all edges. Thus, we may introduce the following definition.

Definition 1.4 A partition critical hypergraph $\mathcal{E} \subseteq \binom{[n]}{k}$ is called *ordered 3-critical* if $A_i = E_i$, $B_i = \emptyset$ for all edges $E_i \in \mathcal{E}$.

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The following theorem is a strenghtening of Lovász' theorem.

Theorem 1.5 *Let $\mathcal{E} \subseteq \binom{[n]}{k}$ be an ordered 3-critical k-uniform hypergraph. Then*

$$|\mathcal{E}| \le \binom{n}{k-1}. \tag{2}$$

For an arbitrary partition critical k-uniform hypergraph we have

$$|\mathcal{E}| \le \binom{n-1}{k-1} + \binom{n-1}{k-2} + \ldots + \binom{n-1}{0}. \tag{3}$$

holds. Bound (3) is sharp for $k = 2, 3$. Furthermore, for all $n \ge 2k - 1$ and $k \ge 2$ there exist partition critical k-uniform hypergraphs of size $\binom{n}{k-1}$.

The proof of the upper bound (2) is based on the polynomial method outlined in [2].

2 Upper bounds

Let us first consider the inequality (2). We define n-variable polynomials $P_i(x_1, x_2, \ldots x_n) \in \mathbb{R}[x_1, x_2, \ldots x_n]$ for all $E_i \in \mathcal{E}$, furthermore $Q_H(x_1, x_2, \ldots x_n) \in \mathbb{R}[x_1, x_2, \ldots x_n]$ for all $H \subset X = \{1, 2, \ldots, n\}$ with $|H| \le k - 2$. Let P_i be defined by

$$P_i(x_1, x_2, \ldots x_n) = \prod_{1 \le m \le k-1} \left(\left(\sum_{v \in C_i} x_v \right) - m \right), \tag{4}$$

where C_i is one side of the partition $C_i \cup D_i = X$ that belongs to edge E_i according to Definition 1.3. On the other hand, Q_H is defined by

$$Q_H(x_1, x_2, \ldots x_n) = \prod_{h \in H} x_h \left(\sum_{j=1}^{n} x_i - k \right). \tag{5}$$

Let $\widehat{Y}$ denote the characteristic vector of subset $Y \subseteq X$. According to Definition 1.3 $P_j(\widehat{E_i}) = 0$, if $i < j$ but $P_j(\widehat{E_j}) \ne 0$. Indeed, $P_j(\widehat{E_i}) = \prod_{1 \le m \le k-1} (|C_j \cap E_i| - m)$. Since the partition $C_j \cup D_j = X$ cuts E_i in proper nonempty subsets, $1 \le |C_j \cap E_i| \le k - 1$ for $i < j$. Similarly, $Q_H(\widehat{Y}) \ne 0$ iff $H \subseteq Y$ and $|Y| \ne k$. Now let $\widetilde{P}_i(x_1, x_2, \ldots x_n)$ be the polynomial obtained from P_i by expanding the products and the repeatedly replacing higher order factor x_v^2 by x_v for all $1 \le v \le n$. $\widetilde{P}_i$ is multilinear of degree at most $k - 1$, furthermore for any subset $Y \subseteq X$ we have $\widetilde{P}_i(\widehat{Y}) = P_i(\widehat{Y})$. Let $\widetilde{Q}_H$ be obtained from Q_H by the same reduction as above. $\widetilde{Q}_H$ is also multilinear of degree at most $k-1$ and $\widetilde{Q}_H(\widehat{Y}) = Q_H(\widehat{Y})$ for any subset $Y \subseteq X$. The system of polynomials $\mathcal{P} = \{\widetilde{Q}_H \colon H \subset X,\ |H| \le k-2\} \cup \{\widetilde{P}_i \colon 1 \le i \le t\}$ is linearly independent in the space of multilinear polynomials of degree at most $k-1$ of n variables. Thus,

$$|\{\widetilde{Q}_H \colon H \subset X,\ |H| \le k-2\}| + t = |\mathcal{P}| \le \binom{n}{k-1} + \binom{n}{k-2} + \ldots + \binom{n}{0}, \quad (6)$$

which implies (2).

For arbitrary prescribed partitions (3) is proved using the polynomials described in [1]. Namely, define a polynomial $p_i(\underline{x}) \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ for each E_i as follows.

$$p_i(x_1, x_2, \ldots, x_n) = \prod_{a \in A_i}(1 - x_a) \prod_{b \in B_i} x_b + (-1)^{k+1} \prod_{a \in A_i} x_a \prod_{b \in B_i}(1 - x_b) \quad (7)$$

Polynomials defined by (7) are multilinear of degree at most $k-1$, since the product $\prod_{e \in E_i} x_e$ cancels by the coefficient $(-1)^{k+1}$. It can be easily checked that $p_j(\widehat{C_i}) = 0$ if $j < i$ and $p_i(\widehat{C_i}) \neq 0$. Let us assume without loss of generality that the partitions $C_i \cup D_i = X$ are so that $n \in D_i$ holds for every $i = 1, 2, \ldots, t$. Let polynomials q_i be defined by

$$q_i(x_1, x_2, \ldots, x_{n-1}) = p_i(x_1, x_2, \ldots, x_n)|_{x_n = 0} \in \mathbb{R}[x_1, x_2, \ldots, x_{n-1}]. \quad (8)$$

Let $C'_i = C_i|_{\{1,2,\ldots,n-1\}}$. Then $q_j(\widehat{C'_i}) = p_j(\widehat{C_i})$ for all $j \le i$. Thus the polynomials defined in (8) are linearly independent, (3) follows.

3 Constructions

In this section we show that inequality (3) is sharp for $k = 2, 3$ and construct a k-uniform partition critical hypergraph $(X, \mathcal{E})$ on the underlying set $X = \{1, 2, \ldots, n\}$ of size $\binom{n}{k-1}$.

For $k = 2$ the 3-critical (hyper)graphs are the odd cycles and they reach equality in (3).

For $k = 3$, consider the following hypergraph $(X, \mathcal{E}$, where $X = \{1, 2, \ldots, n\}$ and $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \{\{2, 4, 5\}\}$. $\mathcal{E}_1$ consists of all triplets that contain 1. $\mathcal{E}_2 = \{\{2, 3, 4\}, \{3, 4, 5\}, \ldots, \{n-2, n-1, n\}, \{n-1, n, 2\}, \{n, 2, 3\}\}$. The prescribed partition of $E \in \mathcal{E}_1$ is $\{1\} \cup (E \setminus \{1\})$, while $\{i, i+1, i+2\}$ is decomposed as $\{i\} \cup \{i+1, i+2\} = \{i, i+1, i+2\}$, $i+1$ and $i+2$ should be understood cyclically, that is $n+1 \equiv 2$ and $n+2 \equiv 3$. Finally, $\{2, 4, 5\}$ is cut as $\emptyset \cup \{2,4,5\} = \{2,4,5\}$. The ordering of the edges in $\mathcal{E}$ is that edges in $\mathcal{E}_1$ are first in arbitrary order, then come edges in $\mathcal{E}_2$ also arbitrarily sorted, finally, $\{2,4,5\}$ is the last edge. The partition of the underlying set for $\{1,i,j\} \in \mathcal{E}_1$ is $\{i,j\} \cup (X \setminus \{i,j\}$. That for $\{i+1,i+2\} \in \mathcal{E}_2$ is $\{i\} \cup (X \setminus \{i\}$, finally the last partition (that belongs to $\{2,4,5\}$) is $\emptyset \cup X$. It is easy to check that this satisfies the conditions of Definition 1.3.

The following proposition is easy exercise.

Proposition 3.1 *Let $a \le b \le \frac{m}{2}$. There exists a matching from $\binom{[m]}{a}$ to $\binom{[m]}{b}$ so that if $A \in \binom{[m]}{a}$ is matched to $B \in \binom{[m]}{b}$ then $A \subseteq B$.* □

The edge set $\mathcal{E}$ is a disjoint union $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \ldots \cup \mathcal{E}_k$ where $\mathcal{E}_i$ is on the underlying set $X_i = \{i, i+1, \ldots, n\} (\subset X)$. Let $\mathcal{E}_i$ consist of the k-sets of X_i matched by Proposition3.1 to the collection of $k-i+1$-sets of X_i that contain the element i. Thus, $|\mathcal{E}_i| = \binom{n-i}{k-i}$. If $F \in \mathcal{E}_i$, then there exists $i \in G_F \subset X_i$, such that $|G_F| = k-i+1$ and $G_F \subseteq F$. Let the partition prescribed to F be $F = (G_F \setminus \{i\}) \cup (F \setminus G_F \cup \{i\})$. The partition of the underlying set X that belongs to $F \in \mathcal{E}_i$ is $X = (G_F \setminus \{i\}) \cup (X \setminus G_F \cup \{i\})$. The ordering of edges in $\mathcal{E}$ is that $E \in \mathcal{E}_i$ is before of $F \in \mathcal{E}_j$ if $i < j$, within the same $\mathcal{E}_i$ arbitrary. The hypergraph $(X, \mathcal{E})$ defined above is partition critical, furthermore. the size of $\mathcal{E}$ is $|\mathcal{E}| = \sum_{i=1}^{k} |\mathcal{E}_i| = \sum_{i=1}^{k} \binom{n-i}{k-i} = \binom{n}{k-1}$.

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