European Journal of Combinatorics 33 (2012) 844–852

Contents lists available at SciVerse ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Some new bounds on partition critical hypergraphs

Zoltán Füredi [a,b], Attila Sali [b]

[a] Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street Urbana, IL 61801-2975, USA

[b] Alfréd Rényi Instiute of Mathematics, Hungarian Academy of Sciences Budapest, P.O. Box 127 H-1364, Hungary

ARTICLE INFO

Article history:
Available online 14 October 2011

ABSTRACT

A hypergraph $([n], \mathcal{E})$ is 3-color critical if it is not 2-colorable, but for all $E \in \mathcal{E}$ the hypergraph $([n], \mathcal{E} \setminus \{E\})$ is 2-colorable. Lovász proved in 1976, that $|\mathcal{E}| \le \binom{n}{k-1}$ if $\mathcal{E}$ is k-uniform. Here we give a new algebraic proof and an ordered version that is a sharpening of Lovász' result.

Let $\mathcal{E} \subseteq \binom{[n]}{k}$ be a k-uniform set system on an underlying set $[n]$ of n elements. Let us fix an ordering $E_1, E_2, \dots E_t$ of $\mathcal{E}$ and a prescribed partition $\{A_i, B_i\}$ of each E_i (i.e., $A_i \cup B_i = E_i$ and $A_i \cap B_i = \emptyset$). Assume that for all $i = 1, 2, \dots, t$ there exists a partition $\{C_i, D_i\}$ of $[n]$ such that $E_i \cap C_i = A_i$ and $E_i \cap D_i = B_i$, but $\{E_j \cap C_i, E_j \cap D_i\} \ne \{A_j, B_j\}$ for all $j < i$. That is, the ith partition cuts the ith set as it was prescribed, but it does not cut any earlier set properly. Then

$$t \le f(n, k) := \binom{n-1}{k-1} + \binom{n-1}{k-2} + \dots + \binom{n-1}{0}.$$

This is sharp for $k = 2, 3$. We show that this upper bound is almost the best possible, at least the first three terms are correct; we give constructions of size $f(n, k) - O(n^{k-4})$ (for k fixed and $n \to \infty$). We also give constructions of sizes $\binom{n}{k-1}$ for all n and k.

Furthermore, in the 3-color-critical case (i.e. $\{A_i, B_i\} = \{E_i, \emptyset\}$ for all i), $t \le \binom{n}{k-1}$.

1. Introduction, color critical hypergraphs

Definition 1. A k-uniform hypergraph $\mathcal{H}$ is ℓ-*color critical* if it is not $(\ell - 1)$-colorable, but any proper subhypergraph of $\mathcal{H}$ is $(\ell - 1)$-colorable.

E-mail address: sali@renyi.hu (A. Sali).

0195-6698/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ejc.2011.09.023

We will denote by n the number of vertices of graphs and hypergraphs considered and usually identify their vertex set by $[n] := \{1, 2, \ldots, n\}$.

The only 3-color critical *graphs* are the odd cycles. Dirac [7] showed in 1952 that for $\ell \geq 6$ there exists a $c(\ell) > 0$ such that there are infinitely many ℓ-color critical graphs with at least $(c(\ell)-o(1))n^2$ edges. Later Toft [18] proved that the same holds for all $\ell \geq 4$, and established that $c(4) \geq \frac{1}{16}$.

Minimal 3-color critical hypergraphs were already considered by Bernstein [6] who defined $m(k)$ as the minimum number of edges of a 3-color critical k-uniform hypergraph (on any number of vertices). Erdős and Lovász [9] proved by a random construction that $m(k) < 2k^2 2^k$ and Beck [5] showed $m(k) > ck^{1/3-o(1)}2^k$. The best lower bound up to date was obtained by Radhakrishnan and Srinivasan [15] $m(k) \geq 0.7\sqrt{k/\ln k} \times 2^k$.

There is a similar phenomenon for hypergraphs considering the maximal size of color critical ones as it was observed for graphs. Toft proved [19] that for $k, \ell > 3$ fixed, $n \to \infty$, there exists a k-uniform ℓ-color critical hypergraph on n vertices of size $\Omega(n^k)$. He asked:

Problem 2. What is the maximum size $t_k(n)$ of a 3-color critical k-uniform hypergraph on n vertices?

Toft showed that $\Omega(n^{k-1}) \leq t_k(n) \leq o(n^k)$. Lovász [12] gave a matching upper bound.

Theorem 3.

$$t_k(n) \leq \binom{n}{k-1}.$$

2. Ordered 3-critical hypergraphs

A 3-color critical hypergraph $([n], \mathcal{E})$ has the property that every edge $E \in \mathcal{E}$ has a partition $\{C_E, D_E\}$ of the vertex set $[n]$ (i.e., $C \cup D = [n]$, $C \cap D = \emptyset$) such that both C_E and D_E meet all other edges but E is disjoint to one of them. We generalize this notion as follows.

Definition 4. A hypergraph $\mathcal{H} = ([n], \mathcal{E})$ is called *ordered* 3-*critical* if there exists an ordering $E_1, E_2, \ldots, E_t$ of $\mathcal{E}$ and a partition $\{C_i, D_i\}$ of $[n]$ for each member of $\mathcal{E}$ such that for all $i = 1, 2, \ldots, t$ the restriction of this partition to E_i is the trivial one $\{E_i, \emptyset\}$, but the restriction of $\{C_i, D_i\}$ to E_j is a proper partition, $C_i \cap E_j \neq \emptyset$ and $D_i \cap E_j \neq \emptyset$ for all $j < i$.

The following is a strengthening of Lovász' theorem.

Theorem 5. *Let $\mathcal{E} \subseteq \binom{[n]}{k}$ be an ordered 3-critical k-uniform hypergraph. Then*

$$|\mathcal{E}| \leq \binom{n}{k-1}. \tag{1}$$

The proof is algebraic, and it is postponed to Section 5. We use the tools and methods explained in the book of Babai and Frankl [4], especially some ideas similar to [10].

Unfortunately, using this method one cannot decide if $\limsup t_k(n)/\binom{n}{k-1}$ is less than 1 or not, because one can easily construct an ordered 3-critical hypergraph of size $\binom{n-1}{k-1}$ as follows

$$\mathcal{E} = \{E : 1 \in E \subset [n],\ |E| = k\}. \tag{2}$$

The ordering of the edges can be arbitrary, and the partition belonging to E is $\{E, [n] \setminus E\}$.

3. Partition critical hypergraphs

In this further generalization we only require that each edge has a partition of the vertex set of $[n]$ which cuts it differently than the earlier edges. More precisely we have the following definition.

Definition 6. A hypergraph $\mathcal{H} = ([n], \mathcal{E})$ is called *partition critical* if there exists an ordering $E_1, E_2, \dots, E_t$ of $\mathcal{E}$ and a partition $\{A_i, B_i\}$ of E_i and a partition $\{C_i, D_i\}$ of $[n]$ for all $i = 1, 2, \dots, t$ such that the restriction $\{C_i, D_i\}$ to E_i is exactly $\{A_i, B_i\}$, but the restriction of $\{C_i, D_i\}$ to E_j is not $\{A_j, B_j\}$ (i.e., $C_i \cap E_j \neq A_j$ and $D_i \cap E_j \neq A_j$) for all $j < i$.

In other words, the ith partition cuts the ith set as it is prescribed, but it does not cut any earlier set properly.

A 3-color critical hypergraph is certainly partition critical, as well. Indeed, for an arbitrary ordering of the edges $E_1, E_2, \dots, E_t$ of $\mathcal{E}$, the partition $A_i = E_i, B_i = \emptyset$ works for all edges.

Theorem 7. *For an arbitrary partition critical k-uniform hypergraph we have*

$$|\mathcal{E}| \leq \binom{n-1}{k-1} + \binom{n-1}{k-2} + \cdots + \binom{n-1}{0}. \tag{3}$$

The proof is algebraic, and it is postponed to Section 6.

This theorem improves the earlier upper bound $t \leq \binom{n}{k-1} + \binom{n}{k-2} + \cdots + \binom{n}{0}$ by Anstee, Fleming and the present authors [3]. The partition critical (multi)hypergraphs there came up in the context of forbidden configuration theorems for simple 0–1 matrices. For more about this see the survey [2].

Another remarkable result concerning hypergraphs and partitions is Lovász' k-forest theorem. A hypergraph $(\mathcal{E}, [n])$ is called a *k-forest* if each $E_i \in \mathcal{E}$ has its own k-partition $\pi_i = \{X_1^i, \dots, X_k^i\}$ (here $[n] = X_1^i \cup \dots \cup X_k^i$) such that π_i cuts E_i into k singletons, but it does not cut any other E_j this way. Lovász [13] showed (with an algebraic proof!) that a k-forest on n vertices has at most $\binom{n-1}{k-1}$ edges. This bound is the best possible (see (2)). A new simpler proof was found by Parekh [14].

4. How good is this upper bound?

The bound (3) is sharp for $k = 2, 3$, see Section 7 below. For all $n \geq 2k-1$ and $k \geq 2$ in Section 8 we construct a partition critical k-uniform hypergraphs of size $\binom{n}{k-1}$.

Let $f(n, k) := \sum_{i \leq k-1} \binom{n-1}{i}$, i.e., the right hand side of the inequality (3). Let $p_k(n)$ be the maximum of the left hand side of (3)

$$p_k(n) := \max \left\{ |\mathcal{E}| : \mathcal{E} \subset \binom{[n]}{k} \text{ and it is partition critical} \right\}.$$

Theorem 8. *We have $p_k(n) > f(n, k) - O(n^{k-4})$, in other words*

$$p_k(n) = \binom{n-1}{k-1} + \binom{n-1}{k-2} + \binom{n-1}{k-3} + O(n^{k-4}). \tag{4}$$

We prove this theorem by giving two constructions.

We are convinced that the construction in Section 9 can be developed to an optimal one (for fixed k whenever $n \to \infty$) however the construction in Section 10 is more explicit and gives

$$p_k(n) \geq \binom{n-1}{k-1} + \binom{n-1}{k-2} + \binom{n-1}{k-3} - 7\binom{n-2}{k-4} \tag{5}$$

for all $n \geq 2k$.

We obtain that there exist a partition critical k-uniform hypergraphs whose size is larger than $\binom{n}{k-1}$, the bound (1). This implies that the condition of ordered 3-critical is stronger than that of the partition critical hypergraphs.

5. Algebraic proof of the upper bound, the case of ordered 3-critical hypergraphs

In this section we prove Theorem 5, i.e., the inequality (1). We define n-variable polynomials $P_i(x_1, x_2, \ldots, x_n) \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ for all $E_i \in \mathcal{E}$, furthermore $Q_H(x_1, x_2, \ldots x_n) \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ for all $H \subset [n] = \{1, 2, \ldots, n\}$ with $|H| \le k - 2$. Let P_i be defined by

$$P_i(x_1, x_2, \ldots, x_n) = \prod_{1\le m\le k-1} \left(\left(\sum_{v \in C_i} x_v \right) - m \right),$$

where C_i is one side of the partition $C_i \cup D_i = [n]$ that belongs to edge E_i according to Definition 4. On the other hand, Q_H is defined by

$$Q_H(x_1, x_2, \ldots, x_n) = \prod_{h\in H} x_h \left(\sum_{j=1}^{n} x_i - k \right).$$

Let $\hat{Y}$ denote the characteristic vector of subset $Y \subseteq [n]$. According to Definition 6 $P_j(\widehat{E_i}) = 0$, if $i < j$ but $P_j(\widehat{E_j}) \ne 0$. Indeed, $P_j(\widehat{E_i}) = \prod_{1\le m\le k-1}(|C_j \cap E_i| - m)$. Since the partition $C_j \cup D_j = [n]$ cuts E_i in proper nonempty subsets, $1 \le |C_j \cap E_i| \le k - 1$ for $i < j$. Similarly, $Q_H(\hat{Y}) \ne 0$ iff $H \subseteq Y$ and $|Y| \ne k$. Now let $\tilde{P}_i(x_1, x_2, \ldots, x_n)$ be the polynomial obtained from P_i by expanding the products and the repeatedly replacing higher order factor x_v^2 by x_v for all $1 \le v \le n$. $\tilde{P}_i$ is multilinear of degree at most $k - 1$, furthermore for any subset $Y \subseteq [n]$ we have $\tilde{P}_i(\hat{Y}) = P_i(\hat{Y})$. Let $\tilde{Q}_H$ be obtained from Q_H by the same reduction as above. $\tilde{Q}_H$ is also multilinear of degree at most $k - 1$ and $\tilde{Q}_H(\hat{Y}) = Q_H(\hat{Y})$ for any subset $Y \subseteq [n]$.

Claim 9. *The system of polynomials $\mathcal{P} = \{\tilde{Q}_H : H \subset [n],\ |H| \le k - 2\} \cup \{\tilde{P}_i : 1 \le i \le t\}$ is linearly independent in the space of multilinear polynomials of degree at most $k - 1$ of n variables.*

Proof. Order the polynomials as follows. First put $\tilde{Q}_H$ in decreasing order of the size of H. Then put $\tilde{P}_i$ for $1 \le i \le t$. Suppose on the contrary, that there exists a non-trivial linear combination

$$\sum_{H\subset[n],\ |H|\le k-2} \lambda_H \tilde{Q}_H + \sum_{i=1}^{t} \beta_i \tilde{P}_i = 0 \tag{6}$$

that results in the zero polynomial. Consider the last non-zero coefficient according to the order defined above. If that is λ_H for some H, then evaluate (6) at $\hat{H}$. Since for any $\tilde{Q}_K$ earlier in the order than $\tilde{Q}_H$ we have $\tilde{Q}_K(\hat{H}) = 0$. The value of (6) at $\hat{H}$ is $\lambda_H \tilde{Q}_H(\hat{H}) \ne 0$, a contradiction. Similarly, if the last non-zero coefficient is β_j for some j, then evaluate (6) at $\widehat{E_j}$. $\tilde{Q}_H(\widehat{E_j}) = 0$, since $|E_j| = k$. On the other hand, $\tilde{P}_i(\widehat{E_j}) = 0$ for $j < i$, as it was observed above. Thus, the value of (6) at $\widehat{E_j}$ is $\tilde{P}_j(\widehat{E_j}) \ne 0$, a contradiction again. □

Hence the number of polynomials in $\mathcal{P}$ is at most the dimension of the linear space of multilinear polynomials of degree at most $k - 1$ of n variables. Thus,

$$|\{\tilde{Q}_H : H \subset [n],\ |H| \le k - 2\}| + t = |\mathcal{P}| \le \binom{n}{k-1} + \binom{n}{k-2} + \cdots + \binom{n}{0},$$

which implies (1). □

6. Algebraic proof of the upper bound, the case of partition critical hypergraphs

In this section we prove Theorem 7, i.e., the inequality (3). Define a polynomial $p_i(\underline{x}) \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ for each E_i as follows.

$$p_i(x_1, x_2, \ldots, x_n) = \prod_{a\in A_i}(1 - x_a) \prod_{b\in B_i} x_b + (-1)^{k+1} \prod_{a\in A_i} x_a \prod_{b\in B_i}(1 - x_b). \tag{7}$$

These polynomials are multilinear of degree at most $k-1$, since the product $\prod_{e\in E_i} x_e$ cancels by the coefficient $(-1)^{k+1}$. It can be easily checked that $p_j(\widehat{C_i}) = 0$ if $j < i$ and $p_i(\widehat{C_i}) \neq 0$. Let us assume without loss of generality that the partitions $C_i \cup D_i = [n]$ are so that $n \in D_i$ holds for every $i = 1, 2, \dots, t$. Let the polynomials q_i be defined by

$$q_i(x_1, x_2, \dots, x_{n-1}) = p_i(x_1, x_2, \dots, x_n)|_{x_n=0} \in \mathbb{R}[x_1, x_2, \dots, x_{n-1}].$$

Let $C_i' = C_i|_{\{1,2,\dots,n-1\}}$. Then $q_j(\widehat{C_i'}) = p_j(\widehat{C_i})$ for all $j \leq i$. Thus the polynomials q_j are linearly independent in the space of multilinear polynomials of degree at most $k-1$ with variables $x_1, \dots, x_{n-1}$ and (3) follows.

7. Lower bounds. The cases $k = 2$ and $k = 3$

In this section we show that inequality (3) is sharp for $k = 2, 3$. The case $k = 2$ is trivial, the 3-color critical (hyper)graphs are the odd cycles and they reach equality in (3).

For $k = 3$, consider the following hypergraph $([n], \mathcal{E})$, where $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \{\{2, 4, 5\}\}$. $\mathcal{E}_1$ consists of all triplets that contain 1. $\mathcal{E}_2 = \{\{2, 3, 4\}, \{3, 4, 5\}, \dots, \{n-2, n-1, n\}, \{n-1, n, 2\}, \{n, 2, 3\}\}$. The prescribed partition of $E \in \mathcal{E}_1$ is $\{1\} \cup (E \setminus \{1\})$, while $\{i, i+1, i+2\}$ is decomposed as $\{i\} \cup \{i+1, i+2\} = \{i, i+1, i+2\}$, $i+1$ and $i+2$ should be understood cyclically, that is $n+1 \equiv 2$ and $n+2 \equiv 3$. Finally, $\{2, 4, 5\}$ is cut as $\emptyset \cup \{2, 4, 5\} = \{2, 4, 5\}$. The ordering of the edges in $\mathcal{E}$ is that edges in $\mathcal{E}_1$ are first in arbitrary order, then come edges in $\mathcal{E}_2$ also arbitrarily sorted, finally, $\{2, 4, 5\}$ is the last edge. The partition of the underlying set for $\{1, i, j\} \in \mathcal{E}_1$ is $\{i, j\} \cup ([n] \setminus \{i, j\})$. That for $\{i, i+1, i+2\} \in \mathcal{E}_2$ is $\{i\} \cup ([n] \setminus \{i\})$, finally the last partition (that belongs to $\{2, 4, 5\}$) is $\emptyset \cup [n]$. It is easy to check that this satisfies the conditions of Definition 6.

The following proposition is an easy exercise. We will need a particular proof of it for the second construction later.

Proposition 10. *Suppose $a \leq b$ and $a + b \leq m$. Then there exists a matching from $\binom{[m]}{a}$ to $\binom{[m]}{b}$ so that if $A \in \binom{[m]}{a}$ is matched to $B \in \binom{[m]}{b}$ then $A \subseteq B$.* □

8. Lower bounds. A hypergraph of size $\binom{n}{k-1}$

In this section we construct a k-uniform partition critical hypergraph $([n], \mathcal{E})$ of size $\binom{n}{k-1}$ for all n and k with $n \geq 2k-1$.

The edge set $\mathcal{E}$ is a disjoint union $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \dots \cup \mathcal{E}_k$ where $\mathcal{E}_i$ is on the underlying set $[n]_i = \{i, i+1, \dots, n\}$. Let $\mathcal{E}_i$ consist of the k-sets of $[n]_i$ matched by Proposition 10 to the collection of $k-i+1$-sets of $[n]_i$ that contain the element i. Thus, $|\mathcal{E}_i| = \binom{n-i}{k-i}$. If $F \in \mathcal{E}_i$, then there exists $i \in G_F \subset [n]_i$, such that $|G_F| = k-i+1$ and $G_F \subseteq F$. Let the partition prescribed to F be $F = (G_F \setminus \{i\}) \cup (F \setminus G_F \cup \{i\})$. The partition of the underlying set $[n]$ that belongs to $F \in \mathcal{E}_i$ is $[n] = (G_F \setminus \{i\}) \cup ([n] \setminus G_F \cup \{i\})$. The ordering of edges in $\mathcal{E}$ is that $E \in \mathcal{E}_i$ is before $F \in \mathcal{E}_j$ if $i < j$, within the same $\mathcal{E}_i$ arbitrary.

Claim 11. *The hypergraph $([n], \mathcal{E})$ defined above is partition critical.*

Proof. Let us first consider edges E and F such that $E \in \mathcal{E}_i$ and $F \in \mathcal{E}_j$ with $i < j$. The prescribed partition of E is $E = (G_E \setminus \{i\}) \cup (E \setminus G_E \cup \{i\})$, while the partition of $[n]$ belonging to F is $[n] = (G_F \setminus \{j\}) \cup ([n] \setminus G_F \cup \{j\})$. $k-j = |G_F \setminus \{j\}| < |G_E \setminus \{i\}| = k-i$, hence $(G_F \setminus \{j\}) \cap E \neq G_E \setminus \{i\}$. On the other hand, $i \in E \setminus G_E \cup \{i\}$ but $i \notin G_F \setminus \{j\}$, thus $E \setminus G_E \cup \{i\} \neq E \cap (G_F \setminus \{j\})$.

On the other hand, if E and F belong to the same $\mathcal{E}_i$, then clearly $(G_F \setminus \{i\}) \cap E \neq G_E \setminus \{i\}$ since $G_F \neq G_E$. Furthermore, $E \setminus G_E \cup \{i\} \neq E \cap (G_F \setminus \{i\})$, since i is contained in the left hand side, but not in the right hand side.

Thus, if E is before F in the ordering of the edges, then the partition of $[n]$ belonging to F does not cut E properly. □

The size of $\mathcal{E}$ is

$$|\mathcal{E}| = \sum_{i=1}^{k} |\mathcal{E}_i| = \sum_{i=1}^{k} \binom{n-i}{k-i} = \binom{n}{k-1}$$

as claimed.

9. Lower bounds. A construction with error term $O(n^{k-4})$

Here we give a construction whose size exceeds $\binom{n}{k-1}$ for fixed k and large enough n, a partition critical hypergraph of size $f(n,k) - O(n^{k-4})$.

Let us recall the following elegant symmetric chain decomposition of $\mathcal{B}_n$, the subset lattice of $\{1, 2, \dots, n\}$ given by Greene and Kleitman [11]. A chain $A_1 \subset A_2 \subset \cdots A_t \subseteq \{1, 2, \dots, n\}$ is called *symmetric* if $|A_i| + 1 = |A_{i+1}|$ for $i = 1, 2, \dots, t-1$ and $|A_1| + |A_t| = n$. This symmetric chain decomposition plays an important role in proving *Sperner-type theorems* for $\mathcal{B}_n$. The interested reader is referred to the book of Engel [8]. A sequence of left and right parentheses of length n is assigned to each subset $A \subseteq \{1, 2, \dots, n\}$, as follows. There is a right parenthesis on position i if $i \in A$, while a left parenthesis stands on position i if $i \notin A$. These parentheses are matched in the natural way till there is some unmatched left parenthesis preceding some unmatched right parenthesis. When the matching process finishes, the sequence of unmatched parentheses consists of some right parentheses followed by left ones. Sets with the same pairs of matched parentheses form a symmetric chain. In fact, the smallest element of the chain is the set whose unmatched parentheses are all left ones. Then the other sets of the chain are obtained by successively turning left parentheses to right ones starting with the leftmost unmatched parenthesis. As an example, consider $n = 8, A = \{2, 4, 5\}$.

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \underbrace{(\quad)} & & \underbrace{(\quad)} & &) & (& (& (. \end{array}$$

Thus, the sets in the chain of A are $\{2, 4\}, \{2, 4, 5\}, \{2, 4, 5, 6\}, \{2, 4, 5, 6, 7\}, \{2, 4, 5, 6, 7, 8\}$. Note, that this symmetric chain partition in particular proves Proposition 10.

We will use this construction on the underlying set $\{2, 3, \dots, n\}$ instead of $\{1, 2, \dots, n\}$. For $A \subset \{2, 3, \dots, n\}$, $|A| = k-2$ let $p(A)$ denote the 2-element subset of $\{2, 3, \dots, n\}$ such that $A \cup p(A)$ is the k-element subset in the symmetric chain of A obtained by the parentheses construction. The edge set $\mathcal{E}$ of our partition critical hypergraph is a disjoint union $\mathcal{E} = \mathcal{E}_{k-1} \cup \mathcal{E}_{k-2} \cup \mathcal{E}_{k-3}$. $\mathcal{E}_{k-1} = \{A \cup \{1\} : A \subset \{2, 3, \dots, n\},\ |A| = k-1\}$, while $\mathcal{E}_{k-2} = \{A \cup p(A) : A \subset \{2, 3, \dots, n\},\ |A| = k-2$ and $|A \cap \{2, 3, \dots, k+2\}| \le 1\}$. Finally, $\mathcal{E}_{k-3}$ consists of the k-subsets of $\{5, 6, \dots, n\}$ that are matched by Proposition 10 to those $k-2$-subsets that contain element 5. Thus an edge $E \in \mathcal{E}_{k-i}$ has an own $k-i$-element subset $f(E)$, namely for $E \in \mathcal{E}_{k-1} f(E) = E \setminus \{1\}$, for $E = A \cup p(A) \in \mathcal{E}_{k-2} f(E) = A$, while for $E \in \mathcal{E}_{k-3}$ matched to the $k-2$-set $G, f(E) = G \setminus \{5\}$. The partition of an edge $E \in \mathcal{E}_{k-i}$ is $E = f(E) \cup (E \setminus f(E))$ and the partition of the underlying set $[n]$ that belongs to E is $[n] = ([n] \setminus f(E)) \cup f(E)$. The ordering of the edges is so that $E \in \mathcal{E}_{k-i}$ comes before $F \in \mathcal{E}_{k-j}$ if $i < j$. Edges in the same class are ordered arbitrarily.

Claim 12. *Let* $|A| = k-2$ *and* $|A \cap \{2, 3, \dots, k+2\}| \le 1$. *Then* $p(A) \subset \{2, 3, \dots, k+2\}$. *Furthermore,* $p(A) \cap \{2, 3, 4\} \ne \emptyset$.

Proof. At most one right parenthesis that corresponds to elements of A stands on the first $k+1$ positions. Furthermore, at most $k-2$ left parentheses are matched to the right parentheses of elements of A. That takes up at most $k-1$ parenthesis positions of the first $k+1$ positions. So there are at least 2 unmatched left parentheses there that can be turned to right ones to form $p(A)$. The second statement holds if $A \cap \{2, 3\} = \emptyset$, since it is easy to see that the left parenthesis on the first position (element 2) is not matched, so $2 \in p(A)$. If $2 \in A$, then clearly $3 \in p(A)$. If $3 \in A$, then at most the rightmost $k-3$ left parentheses corresponding to elements $4, 5, \dots, k+1$ are matched, so $4 \in p(A)$. $\square$

Claim 13. *The hypergraph* $([n] = \{1, 2, \dots, n\}, \mathcal{E} = \mathcal{E}_{k-1} \cup \mathcal{E}_{k-2} \cup \mathcal{E}_{k-3})$ *with the partitions defined above is partition critical.*

Proof. Let $E, F \in \mathcal{E}$ such that E comes before F in the ordering. We need to prove that the partition of the underlying set $[n]$ belonging to F does not cut E properly. If both E and F belong to $\mathcal{E}_{k-1}$ or to $\mathcal{E}_{k-3}$, then the corresponding proof of Claim 11 applies. If both E and F belong to $\mathcal{E}_{k-2}$, then let $E = A \cup p(A)$ and $F = B \cup p(B)$. $p(A) \not\subset B$ by definition, so $E \cap f(F) = E \cap B \neq p(A)$. On the other hand, clearly $E \cap f(F) = E \cap B \neq A$, thus the partition that belongs to F does not cut E properly. If $E \in \mathcal{E}_{k-i}$ and $F \in \mathcal{E}_{k-j}$ with $i < j$, then $|f(E)| > |f(F)|$, so $E \cap f(F) \neq f(E)$. On the other hand $([n] \setminus f(F)) \cap E$ contains an element in $E \setminus f(E)$. Indeed, this element is 1 for $i = 1$, and it is one of 2,3,4 if $i = 2$. □

The size of this hypergraph $|\mathcal{E}_{k-1}| + |\mathcal{E}_{k-2}| + |\mathcal{E}_{k-3}|$, where $|\mathcal{E}_{k-1}| = \binom{n-1}{k-1}$, $|\mathcal{E}_{k-2}| = \binom{n-1}{k-2} - \sum_{l=2}^{k-2} \binom{k+1}{l}\binom{n-k-2}{k-2-l}$, $|\mathcal{E}_{k-3}| = \binom{n-5}{k-3}$. Thus

$$f(n,k) - |\mathcal{E}| = \sum_{l=2}^{k-2} \binom{k+1}{l}\binom{n-k-2}{k-2-l} + \binom{n-1}{k-3} - \binom{n-5}{k-3} + \sum_{j=4}^{k} \binom{n-1}{k-j}.$$

Here the right hand side is $O(n^{k-4})$ for fixed k.

Dániel Soltész [17] gave a 4-uniform partition critical hypergraph of size $f(n, 4) - 1$ using some fine tuning of the construction above.

10. Lower bounds. A more explicit construction

Here we give a another partition critical hypergraph of size $f(n, k) - O(n^{k-4})$.

Similarly as above, the k-uniform family $\mathcal{E}$ on the vertex set $[n]$ consists of three parts, $\mathcal{E} = \mathcal{E}_{k-1} \cup \mathcal{E}_{k-2} \cup \mathcal{E}_{k-3}$, and $\mathcal{E}_{k-2}$ again split into three parts. The ordering of the five groups of $\mathcal{E}$ is the same as they are defined, $\mathcal{E}_{k-1}$ precedes all others, the ordering of the members inside the group is arbitrary. The next group is $\mathcal{E}_{k-2}^{23}$, etc. While defining the edges E of $\mathcal{E}_{k-i}$ we also define a subset $f(E) \subset E$ of size $|f(E)| = k - i$. Finally the corresponding partition of E is $\{f(E), E \setminus f(E)\}$, $|f(E)| = k - i$ and the corresponding partition of $[n]$ is $\{f(E), [n] \setminus f(E)\}$.

$\mathcal{E}_{k-1} := \{1 \cup A\}$, where $|A| = k - 1$ and $A \subset \{2, \dots, n\}$. We define $f(E) = A$.

$\mathcal{E}_{k-2}$ consists of three parts $\mathcal{E}_{k-2} = \mathcal{E}_{k-2}^{23} \cup \mathcal{E}_{k-2}^{2} \cup \mathcal{E}_{k-2}^{3}$ where

$\mathcal{E}_{k-2}^{23} := \{\{2, 3\} \cup A\}$, where $|A| = k - 2$ and $A \subset \{4, \dots, n\}$. We define $f(E) = A$.

$\mathcal{E}_{k-2}^{2} := \{\{2, 4, 5\} \cup A\}$, where $|A| = k - 3$ and $A \subset \{6, \dots, n\}$. We define $f(E) = A \cup \{2\}$.

$\mathcal{E}_{k-2}^{3} := \{\{3, 4, 5\} \cup A\}$, where $|A| = k - 3$ and $A \subset \{6, \dots, n\}$. We define $f(E) = A \cup \{3\}$.

$\mathcal{E}_{k-3} := \{\{4, 5, 6\} \cup A\}$, where $|A| = k - 3$ and $A \subset \{7, \dots, n\}$. We define $f(E) = A$.

It is easy the check, that the family is indeed partition critical. We have

$$\begin{aligned} |\mathcal{E}| &= \binom{n-1}{k-1} + \binom{n-3}{k-2} + 2\binom{n-5}{k-3} + \binom{n-6}{k-3} \\ &= \binom{n-1}{k-1} + \binom{n-1}{k-2} + \binom{n-1}{k-3} \\ &\quad - \binom{n-2}{k-4} - 2\binom{n-3}{k-4} - 3\binom{n-4}{k-4} - 3\binom{n-5}{k-4} - \binom{n-6}{k-4}. \quad \square \end{aligned}$$

If one is going to use Proposition 10 then one can further improve this lower bound for $n \geq 2k$ as follows.

The first two groups $\mathcal{E}_{k-1}$ and $\mathcal{E}_{k-2}^{23}$ are unchanged. The next three are modified as follows. Let $q_s(A)$ denote the s-subset of $\{5, \dots, n\}$ matched to A using Proposition 10 for a subset $A \subset \{5, \dots, n\}$.

$\mathcal{E}_{k-2}^{2} := \{\{2, 4\} \cup q_{k-2}(A)\}$, where $|A| = k - 3$ and $A \subset \{5, \dots, n\}$. We define $f(E) = A \cup \{2\}$.

$\mathcal{E}_{k-2}^{3} := \{\{3, 4\} \cup q_{k-2}(A)\}$, where $|A| = k - 3$ and $A \subset \{5, \dots, n\}$. We define $f(E) = A \cup \{3\}$.

$\mathcal{E}_{k-3} := \{\{4\} \cup q_{k-1}(A)\}$, where $|A| = k - 3$ and $A \subset \{5, \dots, n\}$. We define $f(E) = A$.

We have

$$|\mathcal{E}| = \binom{n-1}{k-1} + \binom{n-3}{k-2} + 3\binom{n-4}{k-3}$$
$$= \binom{n-1}{k-1} + \binom{n-1}{k-2} + \binom{n-1}{k-3} - \binom{n-2}{k-4} - 2\binom{n-3}{k-4} - 3\binom{n-4}{k-4}. \quad \square$$

We can prove that $f(n,k) - |\mathcal{E}| \geq \Omega(n^{k-4})$ if the ordering of $\mathcal{E}$ starts with $\mathcal{E}_{k-1}$. Based on this we suggest

Conjecture 14. *For fixed k as $n \to \infty$*

$$f(k,n) - p_k(n) = \Omega(n^{k-4}).$$

11. Further generalizations. Multihypergraphs and smaller edges

Let $p_{\leq k}(n)$ denote the maximum size of an n-vertex partition critical hypergraph with edge sizes *at most* k. The same proof as in Section 6 gives

$$p_{\leq k}(n) \leq f(n,k),$$

the only change one needs is that in (7) in the definition of $p_i(\underline{x})$ the term $(-1)^{k+1}$ has to be replaced by $(-1)^{|E_i|+1}$. In this case it is sharp for all k, as the following construction shows. $\mathcal{E} = \bigcup_{i=1}^{k} \mathcal{E}_i$, where $\mathcal{E}_i$ is the collection of i-element subsets of $[n]$ that contain element 1. The ordering of the edges is so that for $E \in \mathcal{E}_i$ and $F \in \mathcal{E}_j E$ is before F iff $j < i$, while within the same $\mathcal{E}_i$ the ordering is arbitrary. The prescribed partition of $E \in \mathcal{E}$ is $\{1\} \cup (E \setminus \{1\})$. The partition of the underlying set $[n]$ that belongs to E is $(E \setminus \{1\}) \cup [n] \setminus (E \setminus \{1\}) = [n]$.

Let $p_k^{\text{multi}}(n)$ denote the maximum size of an n-vertex, k-uniform partition critical multihypergraph. The same proof as in Section 6 gives

$$p_k^{\text{multi}}(n) \leq f(n,k).$$

This is sharp again for all $n \geq 2k$. One just has to use Proposition 10 for the previous construction and to match the smaller sets to k-sets containing the element 1.

There are many more interesting classes of color critical hypergraphs. For example, recently, Rödl and Siggers [16] extended the results of Toft ($s = k-1$) and Abbott and Liu [1] ($s = 1$) showing that there exists a $c = c(\ell, k, s)$ (where $k > s \geq 2$, $\ell \geq 3$) such that for large enough n one can construct an ℓ-color critical k-uniform system $\mathcal{H}$ on n vertices of size at least $|\mathcal{H}| > cn^s$, such that no s-set of vertices occurs in more than one edge. Obviously, $|\mathcal{H}| \leq \binom{n}{s} / \binom{k}{s}$ holds for every such packing.

Acknowledgments

The second author's research was supported in part by the Hungarian National Science Foundation under grants OTKA NK48826 and NK78439. The first author's research was supported in part by the Hungarian National Science Foundation under grants OTKA 69062 and 60427 and by the National Science Foundation under grant NSF DMS 09-01276 ARRA.

References

[1] H.L. Abbott, A.C. Liu, The existence problem for colour critical linear hypergraphs, Acta Math. Acad. Sci. Hungar. 32 (1978) 273–282.
[2] R.P. Anstee, A survey of forbidden configuration results, (Draft), April 1, 2010. http://www.math.ubc.ca/anstee/FCsurvey10.pdf, As of it was 46 pages.
[3] R.P. Anstee, B. Fleming, Z. Füredi, A. Sali, Color critical hypergraphs and forbidden configurations, in: S. Felsner, (Ed.) Discrete Mathematics and Theoretical Computer Science Proceedings, vol. AE, 2005, pp. 117–122.
[4] L. Babai, P. Frankl, Linear algebra methods in combinatorics, University of Chicago, 1992.
[5] J. Beck, On 3-chromatic hypergraphs, Discrete Math. 24 (1978) 127–137.
[6] F. Bernstein, Zur Theorie der trigonometrische Reihen, Leipz. Ber. 60 (1908) 325–328.

[7] G.A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. 3 (2) (1952) 69–81.
[8] Konrad Engel, Sperner Theory, in: Encyclopedia of Mathematics and its Applications, vol. 65, Cambridge University Press, Cambridge, 1996.
[9] P. Erdős, L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, in: Colloquia Mathematica Societatis János Bolyai, in: Infinite and Finite Sets, Keszthely (Hungary, 1973), János Bolyai Mathematical Society, 1975, pp. 609–627.
[10] Z. Füredi, K.-W. Hwang, P. Weichsel, A proof and generalizations of the Erdős–Ko–Rado theorem using the method of linearly independent polynomials, in: M. Klazar, J. Kratochvil, M. Loebl, J. Matousek, R. Thomas, P. Valtr (Eds.), in: Algorithms Combin, vol. 26, Springer, 2006, pp. 215–224.
[11] C. Greene, D.J. Kleitman, Proof techniques in the theory of finite sets, in: G.-C. Rota (Ed.), Studies in Combinatorics, in: MAA Studies in Math, vol. 17, 1978, pp. 22–79.
[12] L. Lovász, Chromatic number of hypergraphs and linear algebra, Studia Sci. Math. Hungar. 11 (1976) 113–114.
[13] L. Lovász, Topological and algebraic methods in graph theory, in: Graph Theory and Related Topics (Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1977), Academic Press, New York, London, 1979, pp. 1–14.
[14] Ojas Parekh, Forestation in hypergraphs: linear k-trees, Electron. J. Combin. 10 (2003) 6.
[15] J. Radhakrishnan, A. Srinivasan, Improved bounds and algorithms for hypergraph 2-coloring, Random Struct. Algorithms 16 (2000) 4–32.
[16] V. Rödl, M. Siggers, Color critical hypergraphs with many edges, J. Graph Theory 53 (2006) 56–74.
[17] D. Soltész, Personal communication.
[18] B. Toft, On the maximal number of edges of critical k-chromatic graphs, Studia Sci. Math. Hungar. 5 (1970) 461–470.
[19] B. Toft, On colour-critical hypergraphs, Colloquia Mathematica Societatis János Bolyai, in: Infinite and Finite Sets, Keszthely (Hungary), János Bolyai Mathematical Society, 1973, pp. 1445–1457.