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On even-cycle-free subgraphs of the hypercube

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Abstract

It is shown that the size of a subgraph of the hypercube Q_n without a cycle of length $4k+2$ for integer $k \geq 3$ is of order $o(|E(Q_n)|)$.

Keywords: Hypercube, generalized Turan problem.

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1 Introduction

For given two graphs, Q and P, let $\mathrm{ex}(Q,P)$ denote the *generalized Turán number*, i.e., the maximum number of edges in a P-free subgraph of Q. The *n-dimensional hypercube*, Q_n, is the graph with vertex-set $\{0,1\}^n$ and edges assigned between pairs differing in exactly one coordinate. We use $N(G,P)$ for the number of subgraphs of G that are isomorphic to P.

Erdős [5] conjectured that $\mathrm{ex}(Q_n,C_4) = (\frac{1}{2}+o(1))e(Q_n)$. The best upper bound, $(0.6226+o(1))e(Q_n)$, is due to Thomason and Wagner [9], while Brass, Harborth and Nienborg [2] showed $\frac{1}{2}(n+\sqrt{n})2^{n-1} \leq \mathrm{ex}(Q_n,C_4)$, when n is a positive integer power of 4, and $\frac{1}{2}(n+0.9\sqrt{n})2^{n-1} \leq \mathrm{ex}(Q_n,C_4)$ for all $n \geq 9$.

Monotonocity implies that the limit $c_\ell := \lim_{n\to\infty} \mathrm{ex}(Q_n,C_\ell)/e(Q_n)$ exists. It is known that $1/3 \leq c_6 < 0.3941$ (Conder [4] and Lu [8], respectively). Chung [3] showed for $k \geq 2$

$$\mathrm{ex}(Q_n,C_{4k}) = cn^{-\frac{1}{2}+\frac{1}{2k}}e(Q_n) = O(n^{\frac{1}{2}+\frac{1}{2k}}2^n), \tag{1}$$

i.e. $c_{4k} = 0$. Axenovich and Martin [1] proved $c_{4k+2} \leq 1/\sqrt{2}$ for $k \geq 1$.

We prove the following theorem, which shows $c_{4k+2} = 0$ for $k \geq 3$. This generalizes the result in [7]. The problem for C_{10} is still open.

Theorem 1.1 *If G is a subgraph of Q_n containing no cycle of length $4k+2$ and $k \geq 3$, then*

$$|E(G)| = O(n^{\frac{6}{7}}2^n).$$

Hence $|E(G)| = o(|E(Q_n)|)$, i.e., $c_{4k+2} = 0$ for $k \geq 3$.

2 Sketch of the proof

Definition 2.1 There is a partition of $E(Q_n)$ into n matchings M_i, $i \in [n]$, what we call *directions*, where M_i is formed of the edges with endpoints differing in the i'th coordinate.

Definition 2.2 For a subgraph F of Q_n, define $D(F)$ to be the set of directions appearing on some edge of F.

In every $2r$-cycle C in Q_n, each direction must occur an even number of times, so $|D(C)| \leq r$. The proof of Lemma 2.3 follows from this fact.

Lemma 2.3 *Suppose $G \subset Q_n$ with no C_{4k+2} and $b = k-1 \geq 2$. Let C' and C'' be cycles of G sharing an edge, where their lengths are 8 and $4b$, respectively. Then $|D(C') \cap D(C'')| \geq 2$.*

Let G be a C_{4k+2}-free subgraph of Q_n with $k \geq 3$. By applying a counting argument that uses Lemma 2.3 and (1), we obtain an upper bound for $N(G,C_8)$.

To find a lower bound for $N(G, C_8)$, we use the following lemma, that goes back to Erdős (1962) and was published in Erdős and Simonovits [6] in an asymptotic form. As we use it for arbitrary n and e, we revisit the proof.

Lemma 2.4 *Let H be a graph with e edges and n vertices. Then*

$$N(H, C_4) \geq 2\frac{e^3(e-n)}{n^4} - \frac{e^2}{2n} \geq 2\frac{e^4}{n^4} - \frac{3}{4}en. \tag{2}$$

We define a graph $H_x(G)$ for each vertex x in Q_n as it was used by Chung in [3]. In this construction, each 4-cycle in H_x corresponds to an 8-cycle in G, that is represented uniquely. Comparing these bounds give an upper bound on $E(G)$ as $|V(Q_n)|O(n^{6/7})$, which is $o(|E(Q_n)|)$.

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