Journal of Combinatorial Theory, Series A 118 (2011) 1816–1819

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

On even-cycle-free subgraphs of the hypercube

Zoltán Füredi [a,b,1], Lale Özkahya [c]

[a] Rényi Institute of the Hungarian Academy, Budapest, P.O. Box 127, H-1364, Hungary
[b] Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA
[c] Department of Mathematics, Iowa State University, Ames, IA 50011, USA

ARTICLE INFO

Article history:
Received 14 June 2010
Available online 22 March 2011

Keywords:
Cycle
Hypercube
Turán problem

ABSTRACT

It is shown that the size of any C_{4k+2}-free subgraph of the hypercube Q_n, $k \geqslant 3$, is $o(e(Q_n))$.

Published by Elsevier Inc.

The *n-dimensional hypercube*, Q_n, is the graph whose vertex set is $\{0,1\}^n$ and whose edge set is the set of pairs that differ in exactly one coordinate. For graphs Q and P, let $\mathrm{ex}(Q,P)$ denote the *generalized Turán number*, i.e., the maximum number of edges in a P-free subgraph of Q. For a graph G, we use $n(G)$ and $e(G)$ to denote the number of vertices and the number of edges of G, respectively.

Let $c_\ell(n) = \mathrm{ex}(Q_n, C_\ell)/e(Q_n)$ and $c_\ell = \lim_{n\to\infty} c_\ell(n)$. Note that c_ℓ exists, because $c_\ell(n)$ is a non-increasing and bounded function of n. The following conjecture of Erdős is still open.

Conjecture 1. *(See [7].)* $c_4 = \frac{1}{2}$.

Erdős [7] also asked whether $o(n)2^n$ edges in a subgraph of Q_n would imply the existence of a cycle C_{2l} for $l > 2$.

The best upper bound $c_4 \leqslant 0.6226$ was obtained by Thomason and Wagner [11], slightly improving the result of Chung [4]. Brass, Harborth and Nienborg [3] showed that the lower bound for $c_4(n)$ is $\frac{1}{2}(1 + 1/\sqrt{n})$, when $n = 4^r$ for integer r, and $\frac{1}{2}(1 + 0.9/\sqrt{n})$, when $n \geqslant 9$. The problem of deciding the values of c_6 and c_{10} is open as well. The question of Erdős was answered negatively for c_6

E-mail addresses: furedi@renyi.hu, z-furedi@illinois.edu (Z. Füredi), ozkahya@illinoisalumni.org (L. Özkahya).

[1] Research supported in part by the Hungarian National Science Foundation under grants OTKA 69062 and 60427 and by the National Science Foundation under grant NSF DMS 09-01276 ARRA.

0097-3165/$ – see front matter Published by Elsevier Inc.
doi:10.1016/j.jcta.2011.02.009

by Chung [4], showing that $c_6 \geqslant 1/4$. The best known results for c_6 are $1/3 \leqslant c_6 < 0.3941$ due to Conder [5] and Lu [10], respectively. Chung [4] proved for $k \geqslant 2$ that

$$c_{4k}(n) \leqslant cn^{-\frac{1}{2}+\frac{1}{2k}}. \tag{1}$$

Axenovich and Martin [2] gave $c_{4k+2} \leqslant 1/\sqrt{2}$ for $k \geqslant 1$. The present authors [9] recently showed that $c_{14} = 0$. Here, we extend this result to all c_{4k+2} for $k \geqslant 3$ by using similar but simpler methods.

Theorem 2. *For $k \geqslant 3$,*

$$c_{4k+2}(n) = \begin{cases} O(n^{-\frac{1}{2k+1}}), & k \in \{3,5,7\}, \\ O(n^{-\frac{1}{16}+\frac{1}{16(k-1)}}), & \textit{otherwise}, \end{cases}$$

i.e., $c_{4k+2} = 0$.

Recently, Conlon [6] generalized our result by showing $\mathrm{ex}(Q_n, H) = o(e(Q_n))$ for all H that admit a *k-partite representation*, also satisfied by each $H = C_{2\ell}$ except $\ell \in \{2,3,5\}$.

In the rest of the paper, G is assumed to be a C_{4k+2}-free subgraph of Q_n. We fix $a, b \geqslant 2$ such that $4a + 4b = 4k + 4$. This relation between a and b implies that a cycle of length $4a$ cannot intersect a cycle of length $4b$ at a single edge, otherwise their union contains a C_{4k+2}. We define $N(G, P)$ to be the number of subgraphs of G that are isomorphic to P. In the first section, we provide an upper bound on $N(G, C_{4a})$. In the second section, a lower bound on $N(G, C_{4a})$ is obtained via a lower bound on the number of C_{2a}'s in an auxiliary graph obtained from G, which was described by Chung in [4]. Comparing these bounds leads to an upper bound on the average degree of G.

1. An upper bound on $N(G, C_{4a})$

We define the *direction* of an edge uv in $E(Q_n)$, denoted by $d(uv)$, to be the single coordinate from $[n]$ where the 0–1 vectors u and v differ. Similarly,

$$D(F) := \{d(e) \colon e \in E(F)\}$$

where F is any subgraph of Q_n.

Lemma 3. *Let C' and C'' be cycles of length $4a$ and $4b$ of G, respectively, whose intersection contains an edge. Then $|D(C') \cap D(C'')| \geqslant 2$.*

Proof. Let v_1 and v_2 be the endpoints of the edge in the intersection of C' and C''. By previous observation, there must be another vertex v_3 common in C' and C''. Because v_3 differs from either v_1 or v_2 in at least two coordinates, these two coordinates are also contained in the intersection of $D(C')$ and $D(C'')$. □

Observe that, for any cycle C of length $2l$ in Q_n, $D(C) \leqslant l$, because the direction of each edge in C appears an even number of times on $E(C)$. Hence, $N(G, C_{4a}) \leqslant N(Q_n, C_{4a}) = 2^n \times O(n^{2a})$. In the following, we obtain a better bound using Lemma 3.

Claim 4. $N(G, C_{4a}) = e(G)O(n^{2a-2}) + O(2^n n^{2a-\frac{1}{2}+\frac{1}{2b}})$.

Proof. Let $\mathcal{C}$ denote the set of cycles of length $4a$ in G and let $\mathcal{E}$ be the set of edges contained in the cycles in $\mathcal{C}$. We count the cycles of length $4a$ in G over the edges in $\mathcal{E}$. We partition $\mathcal{E} = \mathcal{E}^1 \cup \mathcal{E}^2$ such that $\mathcal{E}^1$ is the collection of edges that are contained in the intersection of a copy of C_{4a} and a copy of C_{4b} in G and $\mathcal{E}^2 := \mathcal{E} \setminus \mathcal{E}^1$. Lemma 3 implies that every edge $e \in \mathcal{E}^1$ is contained in $O(n^{2a-2})$ members of $\mathcal{C}$. The subgraph induced by the edges in $\mathcal{E}^2$ does not contain a copy of C_{4b}, implying that $|\mathcal{E}^2| \leqslant \mathrm{ex}(Q_n, C_{4b})$. By (1), $|\mathcal{E}^2| = O(2^n n^{-\frac{1}{2}+\frac{1}{2b}})$. Using these bounds, we obtain

$$N(G, C_{4a}) = \frac{1}{4a}\left(\sum_{e\in\mathcal{E}^1} O\left(n^{2a-2}\right) + \sum_{e\in\mathcal{E}^2} O\left(n^{2a-1}\right)\right)$$
$$\leqslant e(G)O\left(n^{2a-2}\right) + O\left(2^n n^{2a-\frac{1}{2}+\frac{1}{2b}}\right). \qquad \square \tag{2}$$

2. A lower bound on $N(G, C_{4a})$

For a graph $G \subset Q_n$, we define an auxiliary graph $H_x = H_x(G)$ for each vertex $x \in Q_n$ as it was used by Chung in [4]. The vertex set of H_x consists of the neighbors of x in Q_n. The edge set of H_x is defined as follows. Consider any two vertices y and z in H_x. There is a unique C_4 in Q_n, that contains x, y and z, say $C = yxzw$ and let $w = w(y,z)$. (As vectors over $\mathbf{F}_2$, $w = y + z - x$.) Then yz is an edge of H_x if and only if wz and wy are edges of G. According to the definition of H_x, we have

$$\sum_{x\in V(Q_n)} e(H_x) = \sum_{w\in V(Q_n)} \binom{\deg_G(w)}{2}.$$

By using convexity, we obtain

$$\bar{h} := \sum_{x\in V(Q_n)} e(H_x)/2^n \geqslant \binom{\bar{d}}{2}, \tag{3}$$

where $\bar{d}$ is the average degree of G, i.e. $\bar{d} = 2e(G)/2^n$.

For each cycle of H_x with vertex set $\{y_1, \dots, y_\ell\}$, $\ell \geqslant 3$, there exists a cycle of length 2ℓ in G with vertex set $\{y_1, w(y_1, y_2), \dots, y_\ell, w(y_\ell, y_1)\}$. Since any vertices $x, y \in V(Q_n)$ have at most two common neighbors in Q_n, $V(H_x)$ and $V(H_y)$ intersect in at most two vertices. Therefore

$$N(G, C_{4a}) \geqslant \sum_{x\in V(Q_n)} N(H_x, C_{2a}). \tag{4}$$

By the following theorem of Erdős and Simonovits, we have a lower bound on $N(H_x, C_{2a})$, and therefore on $N(G, C_{4a})$.

Theorem 5. *(See [8].) Let L be a bipartite graph, where there are vertices x and y such that $L - \{x, y\}$ is a tree. Then, for a graph H with n vertices and e edges, there exist constants $c_1, c_2 > 0$ such that if H contains more than $c_1 n^{3/2}$ edges, then*

$$N(H, L) \geqslant c_2 \frac{e^{n(L)}}{n^{2e(L)-n(L)}}.$$

We use this theorem with $L = C_{2a}$ $(n(L) = e(L) = 2a)$ in the following form so that the condition on the minimum number of edges is incorporated.

$$N(H_x, C_{2a}) \geqslant c_2\left(\frac{e(H_x)^{2a}}{n^{2a}} - \frac{(c_1 n^{3/2})^{2a}}{n^{2a}}\right). \tag{5}$$

(4) and (5) imply

$$N(G, C_{4a}) \geqslant \sum_{x\in V(Q_n)} c_2\left(\frac{e(H_x)^{2a}}{n^{2a}} - \frac{(c_1 n^{3/2})^{2a}}{n^{2a}}\right).$$

By using convexity, this inequality implies that

$$N(G, C_{4a}) \geqslant c_2 2^n \frac{\bar{h}^{2a}}{n^{2a}} - O\left(2^n n^a\right).$$

Finally, by (3) and above, we have

$$N(G, C_{4a}) \geqslant c2^n \frac{\bar{d}^{4a}}{n^{2a}} - O\left(2^n n^a\right), \tag{6}$$

for some constant $c > 0$.

3. Conclusion

Claim 4 together with (6) yields

$$\bar{d} = \max\left(O\left(n^{1-\frac{1}{4a-1}}\right), O\left(n^{1-\frac{1}{4a}(\frac{1}{2}-\frac{1}{2b})}\right)\right).$$

This bound is minimized when $a = 2$ and $b = k - 1$ and we obtain

$$\bar{d} = O\left(n^{1-\frac{1}{16}+\frac{1}{16(k-1)}}\right). \tag{7}$$

Note that another approach we could use in Section 1 is to consider $a = b = (k+1)/2$ when k is odd. This changes the counting argument, since $\mathcal{E}^2$ will contain only copies of C_{4a} that are pairwise edge-disjoint and the number of these copies is at most $e(G)/(4a)$. By following the same proof, we obtain for odd k that

$$\bar{d} = O\left(n^{1-\frac{1}{2k+1}}\right).$$

This improves (7) for $k = 3, 5, 7$.

Our proof also implies that $\mathrm{ex}(Q_n, \Theta_{4a-1,1,4b-1})$ is $o(e(Q_n))$ for $a, b \geqslant 2$, where $\Theta_{u,v,w}$ is a Theta-graph consisting of three paths of lengths u, v, and w having the same endpoints and distinct inner vertices. Our result also naturally implies that C_{2l} is Ramsey for odd $l \geqslant 7$, i.e. there is a monochromatic copy of C_{2l} in any r-edge-coloring of Q_n when $n > n(r, l)$. This is also a result of Alon, Radoičić, Sudakov, and Vondrák [1] who showed that C_{2l} is Ramsey for $l \geqslant 5$.

Acknowledgments

The authors thank the referees for their helpful suggestions.

References

[1] N. Alon, R. Radoičić, B. Sudakov, J. Vondrák, A Ramsey-type result for the hypercube, J. Graph Theory 53 (2006) 196–208.
[2] M. Axenovich, R. Martin, A note on short cycles in a hypercube, Discrete Math. 306 (2006) 2212–2218.
[3] P. Brass, H. Harborth, H. Nienborg, On the maximum number of edges in a C_4-free subgraph of Q_n, J. Graph Theory 19 (1995) 17–23.
[4] F. Chung, Subgraphs of a hypercube containing no small even cycles, J. Graph Theory 16 (1992) 273–286.
[5] M. Conder, Hexagon-free subgraphs of hypercubes, J. Graph Theory 17 (1993) 477–479.
[6] D. Conlon, An extremal theorem in the hypercube, Electron. J. Combin. 17 (2010), Research Paper 111.
[7] P. Erdős, On some problems in graph theory, combinatorial analysis and combinatorial number theory, Graph Theory Combin. (1984) 1–17.
[8] P. Erdős, M. Simonovits, Cube-saturated graphs and related problems, in: Progress in Graph Theory, Waterloo, Ont., 1982, Academic Press, 1984, pp. 203–218.
[9] Z. Füredi, L. Özkahya, On 14-cycle-free subgraphs of the hypercube, Combin. Probab. Comput. 18 (2009) 725–729.
[10] Linyuan Lu, Hexagon-free subgraphs in hypercube Q_n, private communication.
[11] A. Thomason, P. Wagner, Bounding the size of square-free subgraphs of the hypercube, Discrete Math. 309 (2009) 1730–1735.